Deriving Distributive Laws for Graded Linear Types (Additional Material)

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This document contains the supplementary material for the paper of the same name which appears in the EPTCS proceedings for TLLA/Linearity 2020. Included are the full proofs for all propositions which appear in the paper, as well as any additional definitions that are required.

A Additional definitions and results

A.1 Typed Equational Theory

$$\frac{\Gamma_1, x : A \vdash t_2 : B \qquad \Gamma_2 \vdash t_1 : A}{\Gamma_1 + \Gamma_2 \vdash (\lambda x.t_2)\, t_1 \equiv t_2[t_1/x] : B}\ \beta \qquad \frac{\Gamma \vdash t : A \multimap B \qquad [x \# t]}{\Gamma \vdash \lambda x.t\, x \equiv t : A \multimap B}\ \eta$$

$$\frac{\Gamma_1, x : A \vdash t_1 : A \qquad \Gamma_2, x : A \vdash t_2 : B}{\Gamma_1 + \Gamma_2 \vdash \textbf{letrec}\ x = t_1\ \textbf{in}\ t_2 \equiv t_2[\textbf{letrec}\ x = t_1\ \textbf{in}\ t_1/x] : B}\ \beta_{letrec}$$

$$\frac{\Gamma_1, x : A \vdash t_1 : A \qquad \Gamma_2 \vdash t_2 : B \qquad \Gamma_3 \vdash f : B \multimap C}{\Gamma_1 + \Gamma_2 + \Gamma_3 \vdash f\,(\textbf{letrec}\ x = t_1\ \textbf{in}\ t_2) \equiv \textbf{letrec}\ x = t_1\ \textbf{in}\ (f\, t_2) : C}\ \textsc{LetRecDistrib}$$

$$\frac{\begin{array}{ll}\Gamma_1 \vdash t : A & \cdot \vdash p_i : A \rhd \Delta_i \\ \Gamma_2, \Delta_i \vdash t_i : B & \end{array}}{\Gamma_1 + \Gamma_2 \vdash \textbf{case}\ t\ \textbf{of}\ \overline{p_i \to t_i} \equiv (t \rhd p_j) t_j : B}\ \beta_{case} \qquad \frac{\begin{array}{ll}\Gamma_1 \vdash t_1 : A & \cdot \vdash p_i : A \rhd \Delta_i \\ \Gamma_2, z : A \vdash t_2 : B & \end{array}}{\Gamma_1 + \Gamma_2 \vdash \textbf{case}\ t_1\ \textbf{of}\ \overline{p_i \to t_2[p_i/z]} \equiv t_2[t_1/z] : B}\ \eta_{case}$$

$$\frac{\Gamma \vdash t : \Box_r A \qquad \cdot \vdash p_i : A \rhd \Delta_i \qquad \Delta_i \vdash p_i : A \qquad 1 \sqsubseteq r}{\Gamma \vdash \textbf{case}\ t\ \textbf{of}\ \overline{[p_i] \to p_i} \equiv \textbf{case}\ t\ \textbf{of}\ [x] \to x : A}\ \textsc{CaseGen}$$

$$\frac{\Gamma \vdash t : A \qquad \cdot \vdash p_i : A \rhd \Delta_i \qquad \Gamma', \Delta'_i \vdash t_i : B \qquad \cdot \vdash p'_i : B \rhd \Delta'_i \qquad \Gamma'', \Delta'_i \vdash t'_i : C}{\Gamma + \Gamma' + \Gamma'' \vdash \textbf{case}\ (\textbf{case}\ t\ \textbf{of}\ \overline{p_i \to t_i})\ \textbf{of}\ \overline{p'_i \to t'_i} \equiv \textbf{case}\ t\ \textbf{of}\ \overline{p_i \to (\textbf{case}\ t_i\ \textbf{of}\ \overline{p'_i \to t'_i})} : C}\ \textsc{CaseAssoc}$$

$$\frac{\begin{array}{lll}\Gamma \vdash t : A & \cdot \vdash p_i : A \rhd \Delta_i & \Gamma', \Delta'_i \vdash t_i : B \\ r \vdash p'_i : B \rhd \Delta'_i & \Gamma'', \Delta'_i \vdash t'_i : C & \mathsf{lin}(p)\end{array}}{r * (\Gamma + \Gamma') + \Gamma'' \vdash \textbf{case}\ [\textbf{case}\ t\ \textbf{of}\ \overline{p_i \to t_i}]\ \textbf{of}\ \overline{[p'_i] \to t'_i} \equiv \textbf{case}\ [t]\ \textbf{of}\ \overline{[p_i] \to \textbf{case}\ [t_i]\ \textbf{of}\ \overline{[p'_i] \to t'_i}} : C}\ [\textsc{CaseAssoc}]$$

$$\frac{\Gamma_1 \vdash t : A \qquad \cdot \vdash p_i : A \rhd \Delta_i \qquad \Gamma_2, \Delta_i \vdash t_i : B \qquad \Gamma_3 \vdash f : B \multimap C}{\Gamma_1 + \Gamma_2 + \Gamma_3 \vdash f\,(\textbf{case}\ t\ \textbf{of}\ \overline{p_i \to t_i}) \equiv \textbf{case}\ t\ \textbf{of}\ \overline{p_i \to (f\, t_i)} : C}\ \textsc{CaseDistrib}$$

In CaseAssoc the predicate $\mathsf{lin}(p)$ classifies those patterns which are *linear*, which are those which are variables or constructor patterns only.

A.1.1 Derived Rules

Proposition A.1 ('Case push' property)**.**

$$\frac{\Gamma \vdash t : A \qquad r \vdash p_i : A \rhd \Delta_i \qquad \Gamma', \Delta_i \vdash t_i : B}{s * r * \Gamma + s * \Gamma' \vdash [\textbf{\textit{case}}\ t\ \textbf{\textit{of}}\ \overline{p_i \to t_i}] \equiv \textbf{\textit{case}}\ [t]\ \textbf{\textit{of}}\ \overline{[p_i] \to [t_i]} : \Box_s B}\ \textsc{CasePush}$$

Proof. Applying β_{case} and congruence over promotion, to the left-hand side of the case push equation yields:

$$[\textbf{case}\ t\ \textbf{of}\ [p_i] \to t_i] = [(t \rhd p_j)t_j]$$

for the smallest j. Applying β_{case} to the right-hand side of the case push equation yields:

$$\textbf{case}\ [t]\ \textbf{of}\ [p_i] \to [t_i] = ([t] \rhd [p_i])[t_j]$$

for the same smallest j (since the patterns pi are the same).

By PATSEMUNBOX, then we have the derivation of pattern matching:

$$\frac{(t \rhd p_i)[t_j] = t''}{([t] \rhd [p_i])[t_j] = t''}\ \textsc{patSemUnbox}$$

therefore **case** $[t]$ **of** $[p_i] \to [t_i] = ([t] \rhd [p_i])[t_j] = (t \rhd p_i)[t_j]$.

Then by Proposition A.2 (below), $(t \rhd p_i)[t_j] = [(t \rhd p_j)t_j]$, yielding case push. □

Proposition A.2 (Pattern matching distributes with promotion)**.** *For all* t, p, t' *then:*

$$(t \rhd p)[t'] = [(t \rhd p)t']$$

Proof. By induction on syntactic pattern matching:

- (wild) $(t \rhd \_)[t'] = [t']$ and $[(t \rhd \_)t'] = [t']$.
- (var) $(t \rhd x)[t'] = [t'][t/x] = [t'[t/x]]$ and $[(t \rhd x)t'] = [t'[t/x]]$
- (unbox)

$$\frac{(t \rhd p)t' = t''}{([t] \rhd [p])t' = t''}\ \textsc{patSemUnbox}$$

  By induction then $(t \rhd p)[t'] = [(t \rhd p)t']$ therefore $([t] \rhd [p])[t'] = [([t] \rhd [p])t']$ since this rule preserves its result in the conclusion.

- (constr)

$$\frac{(t_i \rhd p_i)t_i = t_{i+1}}{(C\, t_0 \mathinner{..} t_n \rhd C\, p_0 \mathinner{..} p_n)t_0 = t_{n+1}}\ \textsc{patSemConstr}$$

  By induction, similarly to the above case, but across multiple terms.

□

A.2 Functor Derivation

Definition A.1 (Deriving functor)**.** Given a function $f : \alpha \multimap \beta$ then there is a function $[\![\mathsf{F}\overline{\alpha}]\!]_{\mathsf{fmap}}(f) : \mathsf{F}\ \alpha \multimap \mathsf{F}\ \beta$ derived from the type $\mathsf{F}\overline{\alpha}$ as follows:

$$
\begin{array}{lll}
[\![1]\!]^{\Sigma}_{\mathsf{fmap}}(f)\ z & = & \textbf{case}\ z\ \textbf{of}\ \mathsf{unit} \to \mathsf{unit} \\
[\![\alpha]\!]^{\Sigma}_{\mathsf{fmap}}(f)\ z & = & f\ z \\
[\![X]\!]^{\Sigma}_{\mathsf{fmap}}(f)\ z & = & (\Sigma(X)\ f)\ z \\
[\![\Box_r A]\!]_{\mathsf{fmap}}(f)\ z & = & \textbf{case}\ z\ \textbf{of}\ [y] \to [[\![A]\!]_{\mathsf{fmap}}(f)\ y] \\
[\![A \oplus B]\!]^{\Sigma}_{\mathsf{fmap}}(f)\ z & = & \textbf{case}\ z\ \textbf{of}\ \ \mathsf{inl}\ x \to \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{fmap}}(f)\ x; \\
 & & \qquad\qquad\ \ \mathsf{inr}\ y \to \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{fmap}}(f)\ y \\
[\![A \otimes B]\!]^{\Sigma}_{\mathsf{fmap}}(f)\ z & = & \textbf{case}\ z\ \textbf{of}\ (x,y) \to ([\![A]\!]^{\Sigma}_{\mathsf{fmap}}(f)\ x, [\![B]\!]^{\Sigma}_{\mathsf{fmap}}(f)\ y) \\
[\![A \multimap B]\!]^{\Sigma}_{\mathsf{fmap}}(f)\ z & = & \lambda x.[\![B]\!]^{\Sigma}_{\mathsf{fmap}}(f)\ (z\ x) \\
[\![\mu X.A]\!]^{\Sigma}_{\mathsf{fmap}}(f)\ z & = & \textbf{letrec}\ g = [\![A]\!]^{\Sigma, X \mapsto g:(\alpha \multimap \beta) \multimap \mu X.A \multimap (\mu X.A)\overrightarrow{[\alpha/\beta]}}_{\mathsf{fmap}}(f)\ \textbf{in}\ g\ z
\end{array}
$$

B Type soundness proofs

The following shows that the calculation of *push* and *pull* distributive laws is well-typed.

Proposition 1. *Type soundness of* $[\![\mathsf{F}\ \overline{\alpha_i}]\!]_{\mathsf{pull}} : \mathsf{F}\ (\overline{\Box_{r_i}\alpha_i}) \to \Box_{\bigwedge_{i=1}^n r_i}(\mathsf{F}\ \overline{\alpha_i})$.

Proof.

- $[\![\mathbf{1}]\!]^{\Sigma}_{\mathsf{pull}} : \mathbf{1} \to \Box_{(\bigwedge_{i=1}^n r_i)}\mathbf{1}$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = \mathbf{1}$).

$$
\dfrac{\dfrac{\dfrac{}{\emptyset \vdash \mathsf{unit} : \mathbf{1}}\ \textsc{Con}}{\emptyset \vdash [\mathsf{unit}] : \Box_{\bigwedge_{i=1}^n r_i}\mathbf{1}}\ \textsc{Pr} \qquad \dfrac{|\mathbf{1} = 1|}{\cdot \vdash \mathsf{unit} : \mathbf{1} \rhd \emptyset}\ \textsc{PCon}}{z : \mathbf{1} \vdash \textbf{case}\ z\ \textbf{of}\ \mathsf{unit} \to [\mathsf{unit}] : \Box_{\bigwedge_{i=1}^n r_i}\mathbf{1}}\ \textsc{Case}
$$

- $[\![X]\!]^{\Sigma}_{\mathsf{pull}} : X \to \Box_{(\bigwedge_{i=1}^n r_i)}X$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = X$).

$$
\dfrac{\dfrac{X : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^n r_i}(\mu X.A) \in \Sigma}{\Sigma \vdash \Sigma(X)\ : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^n r_i}(\mu X.A)}\ \textsc{Lookup} \qquad \dfrac{}{z : (\mu X.A)\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash z : (\mu X.A)\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}}\ \textsc{Var}}{\Sigma, z : (\mu X.A)\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash \Sigma(X)\ z : \Box_{\bigwedge_{i=1}^n r_i}(\mu X.A)}\ \textsc{App}
$$

- $[\![\alpha_j]\!]_{\mathsf{pull}} : \Box_{r_j}\alpha_j \to \Box_{(\bigwedge_{i=1}^n r_i)}\alpha_j$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = \alpha$).

$$
\dfrac{\dfrac{}{\Sigma, z : \Box_{r_j}\alpha_j \vdash z : \Box_{r_j}\alpha_j}\ \textsc{Var}}{\Sigma, z : \Box_{r_j}\alpha_j \vdash z : \Box_{(\bigwedge_{i=1}^n r_i)}\alpha_j}\ \textsc{Approx}
$$

- $[\![(A \oplus B)[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}]]\!]^{\Sigma}_{\mathsf{pull}} : A \oplus B \to \Box_{(\bigwedge_{i=1}^{n} r_i)} A \oplus B$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = A \oplus B$).

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x' : A \vdash x' : A}\,\text{VAR}}{x' : [A]_1 \vdash x' : A}\,\text{DER}}{x' : [A]_1 \vdash \mathsf{inl}\ (x') : A \oplus B}\,\text{CON}}{x' : [A]_{\bigwedge_{i=1}^{n} r_i} \vdash [\mathsf{inl}\ (x')] : \Box_{\bigwedge_{i=1}^{n} r_i} A \oplus B}\,\text{PR} \tag{1}$$

$$\dfrac{\dfrac{\dfrac{}{\emptyset \vdash [\![A]\!]^{\Sigma}_{\mathsf{pull}} : A \multimap \Box_{\bigwedge_{i=1}^{n} r_i} A}\,\text{PULL} \quad \dfrac{}{x : A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \vdash x : A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}]}\,\text{VAR}}{x : A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \vdash [\![A]\!]^{\Sigma}_{\mathsf{pull}}(x) : \Box_{\bigwedge_{i=1}^{n} r_i} A}\,\text{APP} \quad (1) \quad \dfrac{\dfrac{}{\bigwedge_{i=1}^{n} r_i \vdash x' : A \rhd x' : [A]_{\bigwedge_{i=1}^{n} r_i}}\,[\text{PVAR}]}{\cdot \vdash [x'] : \Box_{\bigwedge_{i=1}^{n} r_i} A \rhd x' : [A]_{\bigwedge_{i=1}^{n} r_i}}\,[\text{PBOX}]}{x : A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \vdash \textbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}(x)\ \textbf{of}\ [x'] \to [\mathsf{inl}\ (x')] : \Box_{\bigwedge_{i=1}^{n} r_i} A \oplus B}\,\text{CASE} \tag{2}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y' : B \vdash y' : B}\,\text{VAR}}{y' : [B]_1 \vdash y' : B}\,\text{DER}}{y' : [B]_1 \vdash \mathsf{inr}\ (y') : A \oplus B}\,\text{CON}}{y' : [B]_{\bigwedge_{i=1}^{n} r_i} \vdash [\mathsf{inr}\ (y')] : \Box_{\bigwedge_{i=1}^{n} r_i} A \oplus B}\,\text{PR} \tag{3}$$

$$\dfrac{\dfrac{\dfrac{}{\emptyset \vdash [\![B]\!]^{\Sigma}_{\mathsf{pull}} : B \multimap \Box_{\bigwedge_{i=1}^{n} r_i} B}\,\text{PULL} \quad \dfrac{}{y : B[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \vdash y : B[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}]}\,\text{VAR}}{y : B[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \vdash [\![B]\!]_{\mathsf{pull}}(y) : \Box_{\bigwedge_{i=1}^{n} r_i} B}\,\text{APP} \quad (3) \quad \dfrac{\dfrac{}{\bigwedge_{i=1}^{n} r_i \vdash y' : B \rhd y' : [B]_{\bigwedge_{i=1}^{n} r_i}}\,[\text{PVAR}]}{\cdot \vdash [y'] : \Box_{\bigwedge_{i=1}^{n} r_i} B \rhd y' : [B]_{\bigwedge_{i=1}^{n} r_i}}\,[\text{PBOX}]}{y : B[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \vdash \textbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}(y)\ \textbf{of}\ [y'] \to [\mathsf{inr}\ (x'_2)] : \Box_{\bigwedge_{i=1}^{n} r_i} A \oplus B}\,\text{CASE} \tag{4}$$

$$\dfrac{\dfrac{}{\cdot \vdash x : A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \rhd x : A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}]}\,\text{PVAR}}{\cdot \vdash \mathsf{inl}\ (x) : (A \oplus B)[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \rhd x : A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}]}\,\text{PCON} \tag{5}$$

$$\dfrac{\dfrac{}{\cdot \vdash y : B[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \rhd y : B[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}]}\,\text{PVAR}}{\cdot \vdash \mathsf{inr}\ (x) : (A \oplus B)[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \rhd y : B[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}]}\,\text{PCON} \tag{6}$$

$$\dfrac{(2) \quad (4) \quad (5) \quad (6)}{z : A \oplus B[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \vdash \textbf{case}\ z\ \textbf{of}\ \mathsf{inl}\ (x) \mapsto (\textbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}(x)\ \textbf{of}\ [x'] \to [\mathsf{inl}\ (x')]); \mathsf{inr}\ (y) \to (\textbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}(y)\ \textbf{of}\ [y'] \to [\mathsf{inr}\ (y')]) : \Box_{\bigwedge_{i=1}^{n} r_i} A \oplus B}\,\text{CASE}$$

- $[\![A \otimes B]\!]^{\Sigma}_{\mathsf{pull}} : (A \otimes B)[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \to \Box_{(\bigwedge_{i=1}^{n} r_i)} (A \otimes B)$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = A \otimes B$).

$$\dfrac{\dfrac{\dfrac{}{x' : A \vdash x' : A}\,\text{VAR}}{x' : [A]_1 \vdash x' : A}\,\text{DER} \qquad \dfrac{\dfrac{}{y' : B \vdash y' : B}\,\text{VAR}}{y' : [B]_1 \vdash y' : B}\,\text{DER}}{\dfrac{x' : [A]_1, y' : [B]_1 \vdash (x', y') : A \otimes B}{x' : [A]_{\bigwedge_{i=1}^n r_i}, y' : [B]_{\bigwedge_{i=1}^n r_i} \vdash [(x', y')] : \Box_{\bigwedge_{i=1}^n r_i} A \otimes B}\,\text{PR}}\,\textsc{Con} \tag{7}$$

$$\dfrac{\dfrac{\dfrac{}{\bigwedge_{i=1}^{n} r_i \vdash x' : A \rhd x' : [A]_{\bigwedge_{i=1}^n r_i}}\,[\text{PVAR}]}{\cdot \vdash [x'] : \Box_{\bigwedge_{i=1}^n r_i} A \rhd x' : [A]_{\bigwedge_{i=1}^n r_i}}\,[\text{PBOX}] \qquad \dfrac{\dfrac{}{\bigwedge_{i=1}^{n} r_i \vdash y' : B \rhd y' : [B]_{\bigwedge_{i=1}^n r_i}}\,[\text{PVAR}] \quad |A \otimes B| = 1}{\cdot \vdash [y'] : \Box_{\bigwedge_{i=1}^n r_i} B \rhd y' : [B]_{\bigwedge_{i=1}^n r_i}}\,[\text{PBOX}]}{\cdot \vdash ([x'], [y']) : (\Box_{\bigwedge_{i=1}^n r_i} A) \otimes (\Box_{\bigwedge_{i=1}^n r_i} B) \rhd x' : [A]_{\bigwedge_{i=1}^n r_i}, y' : [B]_{\bigwedge_{i=1}^n r_i}}\,[\textsc{PCon}] \tag{8}$$

$$\dfrac{\dfrac{}{\emptyset \vdash [\![A]\!]^{\Sigma}_{\mathsf{pull}} : A \multimap \Box_{\bigwedge_{i=1}^n r_i} A}\,\text{PULL} \qquad \dfrac{}{x : A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash x : A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}}\,\text{VAR}}{x : A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash [\![A]\!]^{\Sigma}_{\mathsf{pull}}(x) : \Box_{\bigwedge_{i=1}^n r_i} A}\,\text{APP} \tag{9}$$

$$\dfrac{\dfrac{}{\emptyset \vdash [\![B]\!]^{\Sigma}_{\mathsf{pull}} : B \multimap \Box_{\bigwedge_{i=1}^n r_i} B}\,\text{PULL} \qquad \dfrac{}{y : B\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash y : B\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}}\,\text{VAR}}{y : B\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash [\![B]\!]^{\Sigma}_{\mathsf{pull}}(y) : \Box_{\bigwedge_{i=1}^n r_i} B}\,\text{APP} \tag{10}$$

$$\dfrac{\dfrac{(9) \quad (10)}{x : A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}, y : B\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash ([\![A]\!]^{\Sigma}_{\mathsf{pull}}(x), [\![B]\!]^{\Sigma}_{\mathsf{pull}}(y)) : (\Box_{\bigwedge_{i=1}^n r_i} A) \otimes (\Box_{\bigwedge_{i=1}^n r_i} B)}\,\text{PAIR} \quad (7) \quad (8)}{x : A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}, y : B\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash \textbf{case}\ ([\![A]\!]^{\Sigma}_{\mathsf{pull}} x, [\![B]\!]^{\Sigma}_{\mathsf{pull}} y)\ \textbf{of}\ ([x'], [y']) \to [(x', y')] : \Box_{\bigwedge_{i=1}^n r_i} A \otimes B}\,\text{CASE} \tag{11}$$

$$\dfrac{(11) \qquad \dfrac{\dfrac{}{\cdot \vdash x : A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \rhd x : A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}}\,\text{PVAR} \quad \dfrac{}{\cdot \vdash y : B\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \rhd y : B\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}}\,\text{PVAR}}{\cdot \vdash (x, y) : (A \otimes B)\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \rhd x : A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}, y : B\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}}\,\text{PCON}}{z : (A \otimes B)\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash \textbf{case}\ z\ \textbf{of}\ (x, y) \to (\textbf{case}\ ([\![A]\!]^{\Sigma}_{\mathsf{pull}} x, [\![B]\!]^{\Sigma}_{\mathsf{pull}} y)\ \textbf{of}\ ([x'], [y']) \to [(x', y')]) : \Box_{\bigwedge_{i=1}^n r_i} A \otimes B}\,\text{CASE}$$

- $[\![\mu X.A]\!]^{\Sigma}_{\mathsf{pull}} : (\mu X.A)\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \to \Box_{(\bigwedge_{i=1}^n r_i)}(\mu X.A)$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = \mu X.A$).

$$\dfrac{\dfrac{}{\Sigma, f : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^n r_i}(\mu X.A) \vdash f : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^n r_i}(\mu X.A)}\,\text{VAR} \qquad \dfrac{}{\Sigma, z : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash z : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}}\,\text{VAR}}{\Sigma, f : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^n r_i}(\mu X.A), z : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash f\, z : \Box_{\bigwedge_{i=1}^n r_i}(\mu X.A)}\,\text{APP} \tag{12}$$

$$\dfrac{\dfrac{}{\Sigma \vdash [\![A]\!]^{\Sigma, X \mapsto f \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^n r_i}(\mu X.A)}_{\mathsf{pull}} : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^n r_i}(\mu X.A)}\,\text{PULL} \qquad (25)}{\Sigma, z : (\mu X.A)\overrightarrow{[r_i\,\alpha_i/\alpha_i]} \vdash \textbf{letrec}\ f\ =\ [\![A]\!]^{\Sigma, X \mapsto f \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^n r_i}(\mu X.A)}_{\mathsf{pull}}\ \textbf{in}\ f\, z : \Box_{\bigwedge_{i=1}^n r_i} \mu X.A}\,\text{LETREC}$$

$\square$

Proposition 2. *Type soundness of* $[\![\mathsf{F}\ \overline{\alpha_i}]\!]^{\Sigma}_{\mathsf{push}} : [\![\mathsf{F}\overline{\alpha_i}]\!]^{\Sigma}_{\mathsf{push}} : \Box_r \mathsf{F}\overline{\alpha_i} \to \mathsf{F}(\overline{\Box_r \alpha_i})$.

Proof.

- $[\![\mathbf{1}]\!]^{\Sigma}_{\mathsf{push}} : \Box_r \mathbf{1} \to \mathbf{1}$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = \mathbf{1}$).

$$\dfrac{\dfrac{}{\emptyset \vdash \mathsf{unit} : \mathbf{1}}\,\text{CON} \qquad \dfrac{\dfrac{|\mathbf{1}| = 1}{r \vdash \mathsf{unit} : \mathbf{1} \rhd \emptyset}\,[\text{PCON}]}{\cdot \vdash [\mathsf{unit}] : \mathbf{1} \rhd \emptyset}\,[\text{PBOX}]}{z : \Box_r \mathbf{1} \vdash \mathbf{case}\ z\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit} : \mathbf{1}}\,\text{CASE}$$

- $[\![X]\!]^{\Sigma}_{\mathsf{push}} : \Box_r X \to X\overrightarrow{[\Box_r \alpha_i/\alpha_i]}$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = X$).

$$\dfrac{\dfrac{X : \Box_r(\mu X.A) \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]} \in \Sigma}{\Sigma \vdash \Sigma(X)\ : \Box_r(\mu X.A) \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}\,\text{LOOKUP} \qquad \dfrac{}{z : \Box_r(\mu X.A) \vdash z : \Box_r(\mu X.A)}\,\text{VAR}}{\Sigma, z : \Box_r(\mu X.A) \vdash \Sigma(X)\ z : (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}\,\text{APP}$$

- $[\![\alpha_j]\!]^{\Sigma}_{\mathsf{push}} : \Box_{r_j} \alpha_j \to \Box_{r_j} \alpha_j$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = \alpha$).

$$\dfrac{}{z : \Box_{r_j} \alpha_j \vdash z : \Box_{r_j} \alpha_j}\,\text{VAR}$$

- $[\![A \oplus B]\!]^{\Sigma}_{\mathsf{push}} : \Box_r(A \oplus B) \to (A\overrightarrow{[\Box_r \alpha_i/\alpha_i]} \oplus B\overrightarrow{[\Box_r \alpha_i/\alpha_i]})$

$$\dfrac{\dfrac{\dfrac{}{\emptyset \vdash [\![A]\!]^{\Sigma}_{\mathsf{push}} : \Box_r A \multimap A\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}\,\text{PUSH} \qquad \dfrac{\dfrac{\dfrac{}{x : A \vdash x : A}\,\text{VAR}}{x : [A]_1 \vdash x : A}\,\text{DER}}{x : [A]_r \vdash [x] : \Box_r A}\,\text{PR}}{x : [A]_r \vdash [\![A]\!]^{\Sigma}_{\mathsf{push}}([x]) : A\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}\,\text{APP}}{x : [A]_r \vdash \mathsf{inr}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}([x]) : A\overrightarrow{[\Box_r \alpha_i/\alpha_i]} \oplus B\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}\,\text{CON} \tag{13}$$

$$\dfrac{\dfrac{\dfrac{}{\emptyset \vdash [\![B]\!]^{\Sigma}_{\mathsf{push}} : \Box_r B \multimap B\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}\,\text{PUSH} \qquad \dfrac{\dfrac{\dfrac{}{y : B \vdash y : B}\,\text{VAR}}{y : [B]_1 \vdash y : B}\,\text{DER}}{y : [B]_r \vdash [y] : \Box_r B}\,\text{PR}}{y : [B]_r \vdash [\![B]\!]^{\Sigma}_{\mathsf{push}}([y]) : B\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}\,\text{APP}}{y : [B]_r \vdash \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}([y]) : A\overrightarrow{[\Box_r \alpha_i/\alpha_i]} \oplus B\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}\,\text{CON} \tag{14}$$

$$\dfrac{\dfrac{\dfrac{}{r \vdash x : A \rhd x : [A]_r}\,[\text{PVAR}] \qquad |A \oplus B| > 1 \Rightarrow 1 \sqsubseteq r}{r \vdash \mathsf{inl}\ (x) : A \oplus B \rhd x : [A]_r}\,[\text{PCON}]}{\cdot \vdash [\mathsf{inl}\ (x)] : \Box_r A \oplus B \rhd x : [A]_r}\,[\text{PBOX}] \tag{15}$$

$$\dfrac{\dfrac{}{r \vdash y : B \rhd y : [B]_r}\,[\text{PVAR}] \quad |A \oplus B| > 1 \Rightarrow 1 \sqsubseteq r}{\dfrac{r \vdash \mathsf{inr}\ (y) : A \oplus B \rhd y : [B]_r}{\cdot \vdash [\mathsf{inr}\ (y)] : \Box_r A \oplus B \rhd y : [B]_r}\,[\text{PBOX}]}\,[\text{PCON}] \tag{16}$$

$$\dfrac{(13) \quad (14) \quad (15) \quad (16)}{z : \Box_r(A \oplus B) \vdash \textbf{case}\ z\ \textbf{of}\ [\mathsf{inl}\ (x)] \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}([x]); [\mathsf{inr}\ (y)] \to \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}([y]) : A[\overrightarrow{\Box_r \alpha_i / \alpha_i}] \oplus B[\overrightarrow{\Box_r \alpha_i / \alpha_i}]}\,\text{CASE}$$

- $[\![A \otimes B]\!]^{\Sigma}_{\mathsf{push}} : \Box_r(A \otimes B) \to (A[\overrightarrow{\Box_r \alpha_i / \alpha_i}] \otimes B[\overrightarrow{\Box_r \alpha_i / \alpha_i}])$

$$\dfrac{\dfrac{}{\emptyset \vdash [\![A]\!]^{\Sigma}_{\mathsf{push}} : \Box_r A \multimap A[\overrightarrow{\Box_r \alpha_i / \alpha_i}]}\,\text{PUSH} \quad \dfrac{\dfrac{\dfrac{}{x : A \vdash x : A}\,\text{VAR}}{x : [A]_1 \vdash x : A}\,\text{DER}}{x : [A]_r \vdash [x] : \Box_r A}\,\text{PR}}{x : [A]_r \vdash [\![A]\!]^{\Sigma}_{\mathsf{push}}([x]) : A[\overrightarrow{\Box_r \alpha_i / \alpha_i}]}\,\text{APP} \tag{17}$$

$$\dfrac{\dfrac{}{\emptyset \vdash [\![B]\!]^{\Sigma}_{\mathsf{push}} : \Box_r B \multimap B[\overrightarrow{\Box_r \alpha_i / \alpha_i}]}\,\text{PUSH} \quad \dfrac{\dfrac{\dfrac{}{y : B \vdash y : B}\,\text{VAR}}{y : [B]_1 \vdash y : B}\,\text{DER}}{y : [B]_r \vdash [y] : \Box_r B}\,\text{PR}}{y : [B]_r \vdash [\![B]\!]^{\Sigma}_{\mathsf{push}}([y]) : B[\overrightarrow{\Box_r \alpha_i / \alpha_i}]}\,\text{APP} \tag{18}$$

$$\dfrac{(17) \quad (18)}{x : [A]_r, y : [B]_r \vdash ([\![A]\!]^{\Sigma}_{\mathsf{push}}([x]), [\![B]\!]^{\Sigma}_{\mathsf{push}}([y])) : A[\overrightarrow{\Box_r \alpha_i / \alpha_i}] \otimes B[\overrightarrow{\Box_r \alpha_i / \alpha_i}]}\,\text{CON} \tag{19}$$

$$\dfrac{\dfrac{\dfrac{}{r \vdash x : A \rhd x : [A]_r}\,[\text{PVAR}] \quad \dfrac{}{r \vdash y : B \rhd y : [B]_r}\,[\text{PVAR}] \quad |A \otimes B| = 1}{r \vdash (x, y) : A \otimes B \rhd x : [A]_r, y : [B]_r}\,[\text{PCON}]}{\cdot \vdash [(x, y)] : \Box_r A \otimes B \rhd x : [A]_r, y : [B]_r}\,[\text{PBOX}] \tag{20}$$

$$\dfrac{(19) \quad (20)}{z : \Box_r(A \otimes B) \vdash \textbf{case}\ z\ \textbf{of}\ [(x, y)] \to ([\![A]\!]^{\Sigma}_{\mathsf{push}}([x]), [\![B]\!]^{\Sigma}_{\mathsf{push}}([y])) : A[\overrightarrow{\Box_{r_i} \alpha_i / \alpha_i}] \otimes B[\overrightarrow{\Box_{r_i} \alpha_i / \alpha_i}]}\,\text{CASE}$$

- $[\![A \multimap B]\!]^{\Sigma}_{\mathsf{push}} : \Box_r(A \multimap B) \to (A[\overrightarrow{\Box_r \alpha_i / \alpha_i}] \multimap B[\overrightarrow{\Box_{r_i} \alpha_i / \alpha_i}])$

$$\dfrac{\dfrac{}{\emptyset \vdash [\![B]\!]^{\Sigma}_{\mathsf{push}} : \Box_r B \multimap B[\overrightarrow{\Box_r \alpha_i/\alpha_i}]}\;\textsc{Push} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{f : A \multimap B \vdash f : A \multimap B}\;\textsc{Var}}{f : [A \multimap B]_1 \vdash f : A \multimap B}\;\textsc{Der} \qquad \dfrac{\dfrac{}{x : A \vdash x : A}\;\textsc{Var}}{x : [A]_1 \vdash x : A}\;\textsc{Der}}{f : [A \multimap B]_1, x : [A]_1 \vdash f\,x : B}\;\textsc{App}}{f : [A \multimap B]_r, x : [A]_{\bigwedge_{i=1}^{n} r_i} \vdash [f\,x] : \Box_r B}\;\textsc{Pr}}{f : [A \multimap B]_r, x : [A]_{\bigwedge_{i=1}^{n} r_i} \vdash [\![B]\!]^{\Sigma}_{\mathsf{push}} [f\,x] : B[\overrightarrow{\Box_r \alpha_i/\alpha_i}]}\;\textsc{App} \tag{21}$$

$$\dfrac{\dfrac{}{\bigwedge_{i=1}^{n} r_i \vdash x : A \rhd x : [A]_{\bigwedge_{i=1}^{n} r_i}}\;[\textsc{PVar}]}{\cdot \vdash [x] : \Box_r A \rhd x : [A]_{\bigwedge_{i=1}^{n} r_i}}\;[\textsc{PBox}] \tag{22}$$

$$\dfrac{\dfrac{\dfrac{}{\emptyset \vdash [\![A]\!]^{\Sigma}_{\mathsf{pull}} : A \multimap \Box_{\bigwedge_{i=1}^{n} r_i} A}\;\textsc{Pull} \qquad \dfrac{}{y : A \vdash y : A}\;\textsc{Var}}{y : A \vdash [\![A]\!]^{\Sigma}_{\mathsf{pull}}(y) : \Box_{\bigwedge_{i=1}^{n} r_i} A}\;\textsc{App} \qquad (21) \qquad (22)}{y : A, f : [A \multimap B]_r \vdash \textbf{case } [\![A]\!]^{\Sigma}_{\mathsf{pull}}(y) \textbf{ of } [x] \to [\![B]\!]^{\Sigma}_{\mathsf{push}}[f\,x] : B[\overrightarrow{\Box_r \alpha_i/\alpha_i}]}\;\textsc{Case} \tag{23}$$

$$\dfrac{\dfrac{}{r \vdash f : (A \multimap B) \rhd f : [A \multimap B]_r}\;[\textsc{PVar}]}{\cdot \vdash [f] : \Box_r (A \multimap B) \rhd f : [A \multimap B]_r}\;[\textsc{PBox}] \tag{24}$$

$$\dfrac{\dfrac{(23) \qquad (24)}{z : \Box_r (A \multimap B), y : A \vdash \textbf{case } z \textbf{ of } [f] \to \textbf{case } [\![A]\!]^{\Sigma}_{\mathsf{pull}}(y) \textbf{ of } [x] \to [\![B]\!]^{\Sigma}_{\mathsf{push}}[f\,x] : B[\overrightarrow{\Box_r \alpha_i/\alpha_i}]}\;\textsc{Case}}{z : \Box_r (A \multimap B) \vdash \lambda y.\textbf{case } z \textbf{ of } [f] \to \textbf{case } [\![A]\!]^{\Sigma}_{\mathsf{pull}}(y) \textbf{ of } [x] \to [\![B]\!]^{\Sigma}_{\mathsf{push}}[f\,x] : A[\overrightarrow{\Box_r \alpha_i/\alpha_i}] \multimap B[\overrightarrow{\Box_r \alpha_i/\alpha_i}]}\;\textsc{Abs}$$

- $[\![\mu X.A]\!]^{\Sigma}_{\mathsf{push}} : (\mu X.\Box_r A) \to (\mu X.A[\overrightarrow{\Box_{r_i} \alpha_i/\alpha_i}])$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = \mu X.A$).

$$\dfrac{\dfrac{}{\Sigma, f : \mu X.\Box_r A \multimap (\mu X.A[\overrightarrow{\Box_r \alpha_i/\alpha_i}]) \vdash f : \mu X.\Box_r A \multimap (\mu X.A[\overrightarrow{\Box_r \alpha_i/\alpha_i}])}\;\textsc{Var} \qquad \dfrac{}{\Sigma, z : \mu X.\Box_r A \vdash z : \mu X.\Box_r A}\;\textsc{Var}}{\Sigma, f : \mu X.\Box_r A \multimap (\mu X.A[\overrightarrow{\Box_r \alpha_i/\alpha_i}]), z : \mu X.\Box_r A \vdash f\,z : (\mu X.A[\overrightarrow{\Box_r \alpha_i/\alpha_i}])}\;\textsc{App} \tag{25}$$

$$\dfrac{\dfrac{}{\Sigma \vdash [\![A]\!]^{\Sigma, X \mapsto f\,\mu X.\Box_r A \multimap (\mu X.A[\overrightarrow{\Box_r \alpha_i/\alpha_i}])}_{\mathsf{push}} : \mu X.\Box_r A \multimap (\mu X.A[\overrightarrow{\Box_r \alpha_i/\alpha_i}])}\;\textsc{Push} \qquad (25)}{\Sigma, z : (\mu X.\Box_r A) \vdash \textbf{letrec}\, f \;=\; [\![A]\!]^{\Sigma, X \mapsto f\,\mu X.\Box_r A \multimap (\mu X.A[\overrightarrow{\Box_r \alpha_i/\alpha_i}])}_{\mathsf{push}} \textbf{ in } f\,z : \mu X.A[\overrightarrow{\Box_r \alpha_i/\alpha_i}]}\;\textsc{Letrec}$$

$\square$

C Properties of the distributive laws

C.1 Inverse property

Proposition 3.1 (Pull is right inverse to push)**.** *For all n-arity types* F *which do not contain function types, then for type variables* $(\alpha_i)_{i\in[0..n]}$ *and for grades* $r \in \mathscr{R}$ *where* $1 \sqsubseteq r$ *if* $|\mathsf{F}\overline{\alpha_i}| > 1$*, then:*

$$[\![\mathsf{F}\ \overline{\alpha_i}]\!]_{\mathsf{pull}}([\![\mathsf{F}\ \overline{\alpha_i}]\!]_{\mathsf{push}}) = id\ : \Box_r \mathsf{F}\overline{\alpha_i} \multimap \Box_r \mathsf{F}\overline{\alpha_i}$$

Proof. By induction on the syntax of the type $\mathsf{F}\overline{\alpha_i}$ which we denote by T in the following. We first prove a subresult that for $[\![T]\!]^{\Sigma'}_{\mathsf{pull}}([\![T]\!]^{\Sigma}_{\mathsf{push}}\ z) \equiv z$, which by function extensionality then gives us $[\![T]\!]^{\Sigma'}_{\mathsf{pull}}([\![T]\!]^{\Sigma}_{\mathsf{push}}) \equiv id$, under the assumption that for all X, every $f \in \Sigma(X)$ and $g \in \Sigma'(X)$ then $g \circ f = id$, in order to apply the recursive argument.

- $T = 1$

$$\begin{array}{lll}
 & [\![1]\!]^{\Sigma'}_{\mathsf{pull}}([\![1]\!]^{\Sigma}_{\mathsf{push}}\ z) & \\
\equiv & [\![1]\!]^{\Sigma'}_{\mathsf{pull}}(\textbf{case}\ z\ \textbf{of}\ [\mathsf{unit}] \to \mathsf{unit}) & \{\textit{defn.}\ [\![1]\!]^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \textbf{case}\ (\textbf{case}\ z\ \textbf{of}\ [\mathsf{unit}] \to \mathsf{unit})\ \textbf{of}\ \mathsf{unit} \to [\mathsf{unit}] & \{\textit{defn.}\ [\![1]\!]^{\Sigma'}_{\mathsf{pull}}\} \\
\equiv & \textbf{case}\ z\ \textbf{of}\ [\mathsf{unit}] \to \textbf{case}\ \mathsf{unit}\ \textbf{of}\ \mathsf{unit} \to [\mathsf{unit}] & \{\textit{case assoc.}\} \\
\equiv & \textbf{case}\ z\ \textbf{of}\ [\mathsf{unit}] \to [\mathsf{unit}] & \{\beta_{case}\} \\
\equiv & z & \{\eta_{case}\}
\end{array}$$

- $T = \alpha$

$$\begin{array}{lll}
 & [\![\alpha]\!]^{\Sigma'}_{\mathsf{pull}}([\![\alpha]\!]^{\Sigma}_{\mathsf{push}}\ z) & \\
\equiv & [\![\alpha]\!]^{\Sigma'}_{\mathsf{pull}}(z) & \{\textit{defn.}\ [\![\alpha]\!]^{\Sigma}_{\mathsf{push}}\} \\
\equiv & z & \{\textit{defn.}\ [\![\alpha]\!]^{\Sigma'}_{\mathsf{pull}}\}
\end{array}$$

- $T = X$

$$\begin{array}{lll}
 & [\![X]\!]^{\Sigma'}_{\mathsf{pull}}([\![X]\!]^{\Sigma}_{\mathsf{push}}\ z) & \\
\equiv & [\![X]\!]^{\Sigma'}_{\mathsf{pull}}(\Sigma(X)z) & \{\textit{defn.}\ [\![X]\!]^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \Sigma'(X)(\Sigma(X)z) & \{\textit{defn.}\ [\![X]\!]^{\Sigma'}_{\mathsf{pull}}\} \\
\equiv & z & \{\textit{recursion assumption}\}
\end{array}$$

- $T = A \oplus B$:

$$\begin{array}{lll}
 & [\![A\oplus B]\!]^{\Sigma'}_{\mathsf{pull}}([\![A\oplus B]\!]^{\Sigma}_{\mathsf{push}}\ z) & \\
\equiv & [\![A\oplus B]\!]^{\Sigma'}_{\mathsf{pull}}(\textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x];\ \ [\mathsf{inr}\ y] \to \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[y]\ \ ) & \{\textit{defn.}\ [\![A\oplus B]\!]^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \ \textbf{case}\ (\textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x];\ \ [\mathsf{inr}\ y] \to \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[y]\ \ )\ \textbf{of} & \{\textit{defn.}\ [\![A\oplus B]\!]^{\Sigma'}_{\mathsf{pull}}\} \\
 & \quad \mathsf{inl}\ x \to \textbf{case}\ [\![A]\!]^{\Sigma'}_{\mathsf{pull}}\ x\ \textbf{of}\ [u] \to [\mathsf{inl}\ u]; & \\
 & \quad \mathsf{inr}\ y \to \textbf{case}\ [\![B]\!]^{\Sigma'}_{\mathsf{pull}}\ y\ \textbf{of}\ [v] \to [\mathsf{inr}\ v] & \\
\equiv & \textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to \textbf{case}\ \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x]\ \textbf{of}\ \ \mathsf{inl}\ x \to \textbf{case}\ [\![A]\!]^{\Sigma'}_{\mathsf{pull}}\ x\ \textbf{of}\ [u] \to [\mathsf{inl}\ u];\ ; & \{\textit{case assoc.}\} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathsf{inr}\ y \to \textbf{case}\ [\![B]\!]^{\Sigma'}_{\mathsf{pull}}\ y\ \textbf{of}\ [v] \to [\mathsf{inr}\ v] & \\
 & \qquad\qquad\ \ [\mathsf{inr}\ y] \to \textbf{case}\ \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[y]\ \textbf{of}\ \ \mathsf{inl}\ x \to \textbf{case}\ [\![A]\!]^{\Sigma'}_{\mathsf{pull}}\ x\ \textbf{of}\ [u] \to [\mathsf{inl}\ u]; & \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathsf{inr}\ y \to \textbf{case}\ [\![B]\!]^{\Sigma'}_{\mathsf{pull}}\ y\ \textbf{of}\ [v] \to [\mathsf{inr}\ v] & \\
\equiv & \textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to \textbf{case}\ [\![A]\!]^{\Sigma'}_{\mathsf{pull}}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x]\ \textbf{of}\ [u] \to [\mathsf{inl}\ u]; & \{\beta_{case}\} \\
 & \qquad\qquad\ \ [\mathsf{inr}\ y] \to \textbf{case}\ [\![B]\!]^{\Sigma'}_{\mathsf{pull}}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[y]\ \textbf{of}\ [v] \to [\mathsf{inr}\ v] & \\
\equiv & \textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to \textbf{case}\ [x]\ \textbf{of}\ [u] \to [\mathsf{inl}\ u]; & \{\textit{induction}\} \\
 & \qquad\qquad\ \ [\mathsf{inr}\ y] \to \textbf{case}\ [y]\ \textbf{of}\ [v] \to [\mathsf{inr}\ v] & \\
\equiv & \textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to [\mathsf{inl}\ x];\ \ [\mathsf{inr}\ y] \to [\mathsf{inr}\ y] & \{\beta_{case}\} \\
\equiv & z & \{\eta_{case}\}
\end{array}$$

$T = A \otimes B$:

$$
\begin{aligned}
& \llbracket A\otimes B\rrbracket^{\Sigma'}_{\mathsf{pull}}(\llbracket A\otimes B\rrbracket^{\Sigma}_{\mathsf{push}}\ z) && \\
\equiv\ & \llbracket A\otimes B\rrbracket^{\Sigma'}_{\mathsf{pull}}(\textbf{case}\ z\ \textbf{of}\ [(x,y)] \to (\llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}[y])) && \{\textit{defn. } \llbracket A\otimes B\rrbracket^{\Sigma}_{\mathsf{push}}\} \\
\equiv\ & \textbf{case}\ (\textbf{case}\ z\ \textbf{of}\ [(x,y)] \to (\llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}[y]))\ \textbf{of}\ (x,y) \to && \\
& \quad \textbf{case}\ (\llbracket A\rrbracket^{\Sigma'}_{\mathsf{pull}}\ x, \llbracket B\rrbracket^{\Sigma'}_{\mathsf{pull}}\ y)\ \textbf{of}\ ([u],[v]) \to [(u,v)] && \{\textit{defn. } \llbracket A\otimes B\rrbracket^{\Sigma'}_{\mathsf{pull}}\} \\
\equiv\ & \textbf{case}\ z\ \textbf{of}\ [(x,y)] \to \textbf{case}\ (\llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}[y])\ \textbf{of}\ (x,y) \to && \\
& \qquad \textbf{case}\ (\llbracket A\rrbracket^{\Sigma'}_{\mathsf{pull}}\ x, \llbracket B\rrbracket^{\Sigma'}_{\mathsf{pull}}\ y)\ \textbf{of}\ ([u],[v]) \to [(u,v)] && \{\textit{case assoc.}\} \\
\equiv\ & \textbf{case}\ z\ \textbf{of}\ [(x,y)] \to (\textbf{case}\ (\llbracket A\rrbracket^{\Sigma'}_{\mathsf{pull}}\llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}\ [x], \llbracket B\rrbracket^{\Sigma'}_{\mathsf{pull}}\llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}\ [y])\ \textbf{of}\ ([u],[v]) \to [(u,v)]) && \{\beta_{case}\} \\
\equiv\ & \textbf{case}\ z\ \textbf{of}\ [(x,y)] \to (\textbf{case}\ ([x],[y])\ \textbf{of}\ ([u],[v]) \to [(u,v)]) && \{\textit{induction}\} \\
\equiv\ & \textbf{case}\ z\ \textbf{of}\ [(x,y)] \to [(x,y)] && \{\beta_{case}\} \\
\equiv\ & z && \{\eta_{case}\}
\end{aligned}
$$

$T = \mu X.A$:

$$
\begin{aligned}
& \llbracket \mu X.A\rrbracket^{\Sigma'}_{\mathsf{pull}}(\llbracket \mu X.A\rrbracket^{\Sigma}_{\mathsf{push}}\ z) && \\
\equiv\ & \llbracket \mu X.A\rrbracket^{\Sigma'}_{\mathsf{pull}}(\textbf{letrec}\ f = \llbracket A\rrbracket^{\Sigma, X\mapsto f:\mu X.\Box_r A\multimap(\mu X.A)\overrightarrow{[\Box_r\alpha_i/\alpha_i]}}_{\mathsf{push}}\ \textbf{in}\ f\ z) && \{\textit{defn. } \llbracket \mu X.A\rrbracket^{\Sigma}_{\mathsf{push}}\} \\
\equiv\ & \textbf{letrec}\ f' = \llbracket A\rrbracket^{\Sigma', X\mapsto f':\mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}\multimap\Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \textbf{in} && \\
& \qquad f'(\textbf{letrec}\ f = \llbracket A\rrbracket^{\Sigma, X\mapsto f:\mu X.\Box_r A\multimap(\mu X.A)\overrightarrow{[\Box_r\alpha_i/\alpha_i]}}_{\mathsf{push}}\ \textbf{in}\ f\ z) && \{\textit{defn. } \llbracket \mu X.A\rrbracket^{\Sigma'}_{\mathsf{pull}}\} \\
\equiv\ & \textbf{letrec}\ f' = \llbracket A\rrbracket^{\Sigma', X\mapsto f':\mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}\multimap\Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \textbf{in} && \\
& \qquad \textbf{letrec}\ f = \llbracket A\rrbracket^{\Sigma, X\mapsto f:\mu X.\Box_r A\multimap(\mu X.A)\overrightarrow{[\Box_r\alpha_i/\alpha_i]}}_{\mathsf{push}}\ \textbf{in}\ f'(f\ z) && \{\textit{let dist.}\ \} \\
\equiv\ & \textbf{letrec}\ f' = \llbracket A\rrbracket^{\Sigma', X\mapsto f':\mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}\multimap\Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \textbf{in} && \\
& \qquad \textbf{letrec}\ f = \llbracket A\rrbracket^{\Sigma, X\mapsto f:\mu X.\Box_r A\multimap(\mu X.A)\overrightarrow{[\Box_r\alpha_i/\alpha_i]}}_{\mathsf{push}}\ \textbf{in}\ f'(\llbracket A\rrbracket^{\Sigma, X\mapsto f:\mu X.\Box_r A\multimap(\mu X.A)\overrightarrow{[\Box_r\alpha_i/\alpha_i]}}_{\mathsf{push}}\ z) && \{\beta_{letrec}\} \\
\equiv\ & \textbf{letrec}\ f' = \llbracket A\rrbracket^{\Sigma', X\mapsto f'}_{\mathsf{pull}}\ \textbf{in} && \\
& \qquad \textbf{letrec}\ f = \llbracket A\rrbracket^{\Sigma, X\mapsto f:\mu X.\Box_r A\multimap(\mu X.A)\overrightarrow{[\Box_r\alpha_i/\alpha_i]}}_{\mathsf{push}}\ \textbf{in}\ \llbracket A\rrbracket^{\Sigma', X\mapsto f'}_{\mathsf{pull}}(\llbracket A\rrbracket^{\Sigma, X\mapsto f}_{\mathsf{push}}\ z) && \{\beta_{letrec}\} \\
\equiv\ & \textbf{letrec}\ f' = \llbracket A\rrbracket^{\Sigma', X\mapsto f'}_{\mathsf{pull}}\ \textbf{in} && \\
& \qquad \llbracket A\rrbracket^{\Sigma', X\mapsto f'}_{\mathsf{pull}}(\llbracket A\rrbracket^{\Sigma, X\mapsto \textbf{letrec}\ f\ =\ \llbracket A\rrbracket^{\Sigma, X\mapsto f}_{\mathsf{push}}\ \textbf{in}\ f}_{\mathsf{push}}\ z) && \{\beta_{letrec}\} \\
\equiv\ & \llbracket A\rrbracket^{\Sigma', X\mapsto \textbf{letrec}\ f'\ =\ \llbracket A\rrbracket^{\Sigma', X\mapsto f'}_{\mathsf{pull}}\ \textbf{in}\ f'}_{\mathsf{pull}}(\llbracket A\rrbracket^{\Sigma, X\mapsto \textbf{letrec}\ f\ =\ \llbracket A\rrbracket^{\Sigma, X\mapsto f}_{\mathsf{push}}\ \textbf{in}\ f}_{\mathsf{push}}\ z) && \{\beta_{letrec}\} \\
\equiv\ & z && \{\textit{induction}\}
\end{aligned}
$$

$\Box$

Proposition 3.2 (Pull is left inverse to push)**.** *For all n-arity types* F *which do not contain function types, then for type variables* $(\alpha_i)_{i\in[0..n]}$ *and for all grades* $r \in \mathcal{R}$ *where* $1 \sqsubseteq r$ *if* $|\mathsf{F}\overline{\alpha_i}| > 1$*, then:*

$$\llbracket \mathsf{F}\ \overline{\alpha_i}\rrbracket_{\mathsf{push}}(\llbracket \mathsf{F}\ \overline{\alpha_i}\rrbracket_{\mathsf{pull}}) = id\ :\mathsf{F}(\Box_r\overline{\alpha_i}) \multimap \mathsf{F}(\Box_r\overline{\alpha_i})$$

Proof. By induction on the syntax of the type $\mathsf{F}\overline{\alpha_i}$ which we denote by T in the following. The following proof is for $\llbracket T\rrbracket^{\Sigma}_{\mathsf{push}}(\llbracket T\rrbracket^{\Sigma}_{\mathsf{pull}}\ z) \equiv z$, which by function extensionality then gives us $\llbracket T\rrbracket^{\Sigma}_{\mathsf{push}}(\llbracket T\rrbracket^{\Sigma}_{\mathsf{pull}}) \equiv id$, under the assumption that for all X, every $f \in \Sigma(X)$ and $g \in \Sigma'(X)$ then $g \circ f = id$, in order to apply the recursive argument.

- $T = 1$

$$
\begin{array}{lll}
 & [\![1]\!]^{\Sigma}_{\mathsf{push}}([\![1]\!]^{\Sigma'}_{\mathsf{pull}}\ z) & \\
\equiv & [\![1]\!]^{\Sigma}_{\mathsf{push}}(\mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to [\mathsf{unit}]) & \{\mathit{defn.}\ [\![1]\!]^{\Sigma'}_{\mathsf{pull}}\} \\
\equiv & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to [\mathsf{unit}])\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit} & \{\mathit{defn.}\ [\![1]\!]^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to \mathbf{case}\ [\mathsf{unit}]\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit} & \{\mathit{case\ assoc.}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to \mathsf{unit} & \{\beta_{\mathit{case}}\} \\
\equiv & z & \{\eta_{\mathit{case}}\}
\end{array}
$$

- $T = \alpha$

$$
\begin{array}{lll}
 & [\![\alpha]\!]^{\Sigma}_{\mathsf{push}}([\![\alpha]\!]^{\Sigma'}_{\mathsf{pull}}\ z) & \\
\equiv & [\![\alpha]\!]^{\Sigma}_{\mathsf{push}}(z) & \{\mathit{defn.}\ [\![\alpha]\!]^{\Sigma'}_{\mathsf{pull}}\} \\
\equiv & z & \{\mathit{defn.}\ [\![\alpha]\!]^{\Sigma}_{\mathsf{push}}\}
\end{array}
$$

- $T = X$

$$
\begin{array}{lll}
 & [\![X]\!]^{\Sigma}_{\mathsf{push}}([\![X]\!]^{\Sigma'}_{\mathsf{pull}}\ z) & \\
\equiv & [\![X]\!]^{\Sigma}_{\mathsf{push}}(\Sigma'(X)z) & \{\mathit{defn.}\ [\![X]\!]^{\Sigma'}_{\mathsf{pull}}\} \\
\equiv & \Sigma(X)(\Sigma'(X)z) & \{\mathit{defn.}\ [\![X]\!]^{\Sigma}_{\mathsf{push}}\} \\
\equiv & z & \{\mathit{recursion\ assumption}\}
\end{array}
$$

- $T = A \oplus B$

$$
\begin{array}{lll}
 & [\![A \oplus B]\!]^{\Sigma}_{\mathsf{push}}([\![A \oplus B]\!]^{\Sigma'}_{\mathsf{pull}}\ z) & \\
\equiv & [\![A \oplus B]\!]^{\Sigma}_{\mathsf{push}}\mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l} \mathsf{inl}\ x \to \mathbf{case}\ [\![A]\!]^{\Sigma'}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to [\mathsf{inl}\ u]; \\ \mathsf{inr}\ y \to \mathbf{case}\ [\![B]\!]^{\Sigma'}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \to [\mathsf{inr}\ v] \end{array} & \{\mathit{defn.}\ [\![A \oplus B]\!]^{\Sigma'}_{\mathsf{pull}}\} \\
\equiv & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l} \mathsf{inl}\ x \to \mathbf{case}\ [\![A]\!]^{\Sigma'}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to [\mathsf{inl}\ u]; \\ \mathsf{inr}\ y \to \mathbf{case}\ [\![B]\!]^{\Sigma'}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \to [\mathsf{inr}\ v] \end{array})\ \mathbf{of}\ \begin{array}{l} [\mathsf{inl}\ x] \to \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x]; \\ [\mathsf{inr}\ y] \to \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[y] \end{array} & \{\mathit{defn.}\ [\![A \oplus B]\!]^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l} \mathsf{inl}\ x \to \mathbf{case}\ \mathbf{case}\ [\![A]\!]^{\Sigma'}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to [\mathsf{inl}\ u]\ \mathbf{of}\ \begin{array}{l} [\mathsf{inl}\ x] \to \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x];\ ; \\ [\mathsf{inr}\ y] \to \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[y] \end{array} \\ \mathsf{inr}\ y \to \mathbf{case}\ \mathbf{case}\ [\![B]\!]^{\Sigma'}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \to [\mathsf{inr}\ v]\ \mathbf{of}\ \begin{array}{l} [\mathsf{inl}\ x] \to \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x]; \\ [\mathsf{inr}\ y] \to \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[y] \end{array} \end{array} & \{\mathit{case\ assoc.}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l} \mathsf{inl}\ x \to \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[\![A]\!]^{\Sigma'}_{\mathsf{pull}}\ x; \\ \mathsf{inr}\ y \to \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[\![B]\!]^{\Sigma'}_{\mathsf{pull}}\ y \end{array} & \{\beta_{\mathit{case}}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l} \mathsf{inl}\ x \to \mathsf{inl}\ x; \\ \mathsf{inr}\ y \to \mathsf{inr}\ y \end{array} & \{\mathit{induction}\} \\
\equiv & z & \{\eta_{\mathit{case}}\}
\end{array}
$$

- $T = A \otimes B$

$$
\begin{array}{lll}
 & [\![A \otimes B]\!]^{\Sigma}_{\mathsf{push}}([\![A \otimes B]\!]^{\Sigma'}_{\mathsf{pull}}\ z) & \\
\equiv & [\![A \otimes B]\!]^{\Sigma}_{\mathsf{push}}(\mathbf{case}\ z\ \mathbf{of}\ (x,y) \to \mathbf{case}\ ([\![A]\!]^{\Sigma'}_{\mathsf{pull}}\ x, [\![B]\!]^{\Sigma'}_{\mathsf{pull}}\ y)\ \mathbf{of}\ ([u],[v]) \to [(u,v)]) & \{\mathit{defn.}\ [\![A \otimes B]\!]^{\Sigma'}_{\mathsf{pull}}\} \\
\equiv & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ (x,y) \to & \\
 & \quad \mathbf{case}\ ([\![A]\!]^{\Sigma'}_{\mathsf{pull}}\ x, [\![B]\!]^{\Sigma'}_{\mathsf{pull}}\ y)\ \mathbf{of}\ ([u],[v]) \to [(u,v)])\ \mathbf{of}\ [(x,y)] \to ([\![A]\!]^{\Sigma}_{\mathsf{push}}[x], [\![B]\!]^{\Sigma}_{\mathsf{push}}[y]) & \{\mathit{defn.}\ [\![A \otimes B]\!]^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ (x,y) \to & \\
 & \quad \mathbf{case}\ ([\![A]\!]^{\Sigma'}_{\mathsf{pull}}\ x, [\![B]\!]^{\Sigma'}_{\mathsf{pull}}\ y)\ \mathbf{of}\ ([u],[v]) \to \mathbf{case}\ [(u,v)]\ \mathbf{of}\ [(x,y)] \to ([\![A]\!]^{\Sigma}_{\mathsf{push}}[x], [\![B]\!]^{\Sigma}_{\mathsf{push}}[y]) & \{\mathit{case\ assoc.}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ (x,y) \to ([\![A]\!]^{\Sigma}_{\mathsf{push}}[\![A]\!]^{\Sigma'}_{\mathsf{pull}}\ x, [\![B]\!]^{\Sigma}_{\mathsf{push}}[\![B]\!]^{\Sigma'}_{\mathsf{pull}}\ y) & \{\beta_{\mathit{case}}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ (x,y) \to (x,y) & \{\mathit{induction}\} \\
\equiv & z & \{\eta_{\mathit{case}}\}
\end{array}
$$

- $T = \mu X.A$

$$
\begin{array}{lll}
& \llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{push}} (\llbracket \mu X.A \rrbracket^{\Sigma'}_{\mathsf{pull}}\ z) & \\
\equiv & \llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{push}} (\mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma', X \mapsto f' : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ f'\ z) & \{\mathit{defn.}\ \llbracket \mu X.A \rrbracket^{\Sigma'}_{\mathsf{pull}}\} \\
\equiv & \mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.\Box_r A \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}_{\mathsf{push}}\ \mathbf{in} & \\
& \quad f\ (\mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma', X \mapsto f' : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ f'\ z) & \{\mathit{defn.}\ \llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.\Box_r A \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}_{\mathsf{push}}\ \mathbf{in} & \\
& \quad \mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma', X \mapsto f' : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ f\ (f'\ z) & \{\mathit{let\ dist.}\ \} \\
\equiv & \mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.\Box_r A \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}_{\mathsf{push}}\ \mathbf{in} & \\
& \quad \mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma', X \mapsto f' : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ f\ (\llbracket A \rrbracket^{\Sigma', X \mapsto f'}_{\mathsf{pull}}\ z) & \{\beta_{letrec}\} \\
\equiv & \mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.\Box_r A \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}_{\mathsf{push}}\ \mathbf{in} & \\
& \quad \mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma', X \mapsto f'}_{\mathsf{pull}}\ \mathbf{in}\ \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.\Box_r A \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}_{\mathsf{push}}\ (\llbracket A \rrbracket^{\Sigma', X \mapsto f'}_{\mathsf{pull}}\ z) & \{\beta_{letrec}\} \\
\equiv & \mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.\Box_r A \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}_{\mathsf{push}}\ \mathbf{in} & \\
& \quad \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.\Box_r A \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}_{\mathsf{push}}\ (\llbracket A \rrbracket^{\Sigma', X \mapsto \mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma', X \mapsto f'}_{\mathsf{pull}}\ \mathbf{in}\ f'}_{\mathsf{pull}}\ z) & \{\beta_{letrec}\} \\
\equiv & \llbracket A \rrbracket^{\Sigma, X \mapsto \mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f}_{\mathsf{push}}\ \mathbf{in}\ f}_{\mathsf{push}}\ (\llbracket A \rrbracket^{\Sigma', X \mapsto \mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma', X \mapsto f'}_{\mathsf{pull}}\ \mathbf{in}\ f'}_{\mathsf{pull}}\ z) & \{\beta_{letrec}\} \\
\equiv & z & \{\mathit{induction}\}
\end{array}
$$

$\square$

C.2 Naturality of Push and Pull Operations

Proposition 3.3 (Naturality of push)**.** *For all unary type constructors* F *such that* $\llbracket \mathsf{F}\alpha \rrbracket_{\mathsf{push}}$ *is defined, and given a closed function term* $f : \alpha \multimap \beta$*, then:* $\llbracket \Box_r \mathsf{F} \rrbracket_{\mathsf{fmap}} f \circ \llbracket \mathsf{F}\alpha \rrbracket_{\mathsf{push}} = \llbracket \mathsf{F}\beta \rrbracket_{\mathsf{push}} \circ \llbracket \mathsf{F} \rrbracket_{\mathsf{fmap}} \Box_r f$*, i.e.:*

$$
\begin{array}{ccc}
\alpha & \qquad & \Box_r \mathsf{F}\alpha \xrightarrow{\ \llbracket \mathsf{F}\alpha \rrbracket_{\mathsf{push}}\ } \mathsf{F}\Box_r\alpha \\
\Big\downarrow f & & \Box_r \llbracket \mathsf{F}\alpha \rrbracket_{\mathsf{fmap}} f \Big\downarrow \qquad\qquad \Big\downarrow \llbracket \mathsf{F}\alpha \rrbracket_{\mathsf{fmap}} \Box_r f \\
\beta & & \Box_r \mathsf{F}\beta \xrightarrow[\ \llbracket \mathsf{F}\beta \rrbracket_{\mathsf{push}}\ ]{} \mathsf{F}\Box_r\beta
\end{array}
$$

Proof. By induction on the type $T = \mathsf{F}\alpha$, where we consider derivation of the functor action $\llbracket \mathsf{F}\alpha \rrbracket^{\Sigma'}_{\mathsf{fmap}}$ and $\llbracket \mathsf{F}\alpha \rrbracket^{\Sigma}_{\mathsf{push}}$ with respect to open recursion variables Σ' and Σ assuming that the recursive definitions themselves satisfy the naturality property, i.e., for all $X \in \mathsf{dom}(\Sigma), \mathsf{dom}(\Sigma')$ then $\Sigma'(X)\Box_r(f)(\Sigma(X)z) = \Sigma(X)(\Box_r(\Sigma'(X)(f)z)$ (referred to as condition (*) in the proof).

- $T = 1$:

$$
\begin{array}{lll}
 & ([\![1]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f) \circ [\![1]\!]^{\Sigma}_{\mathsf{push}})\ z & \\
\equiv & [\![1]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ ([\![1]\!]^{\Sigma}_{\mathsf{push}}\ z) & \{\beta\} \\
\equiv & [\![1]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ (\textbf{case}\ z\ \textbf{of}\ [\mathsf{unit}] \to \mathsf{unit}) & \{\mathit{defn.}\ [\![1]\!]^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \textbf{case}\ (\textbf{case}\ z\ \textbf{of}\ [\mathsf{unit}] \to \mathsf{unit})\ \textbf{of}\ \mathsf{unit} \to \mathsf{unit} & \{\mathit{defn.}\ [\![1]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\} \\
\equiv & \textbf{case}\ z\ \textbf{of}\ [\mathsf{unit}] \to \textbf{case}\ \mathsf{unit}\ \textbf{of}\ \mathsf{unit} \to \mathsf{unit} & \{\mathit{case\ assoc.}\} \\
\equiv & \textbf{case}\ z\ \textbf{of}\ [\mathsf{unit}] \to \mathsf{unit} & \{\beta_{\mathit{case}}\} \\
 & & \\
\equiv & ([\![1]\!]^{\Sigma}_{\mathsf{push}} \circ [\![\Box_r 1]\!]^{\Sigma'}_{\mathsf{fmap}}(f))\ z & \\
\equiv & [\![1]\!]^{\Sigma}_{\mathsf{push}}\ ([\![\Box_r 1]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ z) & \{\beta\} \\
\equiv & [\![1]\!]^{\Sigma}_{\mathsf{push}}\ (\textbf{case}\ z\ \textbf{of}\ [y] \to [\textbf{case}\ y\ \textbf{of}\ \mathsf{unit} \to \mathsf{unit}]) & \{\mathit{defn.}\ [\![\Box_r 1]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\} \\
\equiv & \textbf{case}\ (\textbf{case}\ z\ \textbf{of}\ [y] \to [\textbf{case}\ y\ \textbf{of}\ \mathsf{unit} \to \mathsf{unit}])\ \textbf{of}\ [\mathsf{unit}] \to \mathsf{unit} & \{\mathit{defn.}\ [\![1]\!]^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \textbf{case}\ z\ \textbf{of}\ [y] \to \textbf{case}\ [\textbf{case}\ y\ \textbf{of}\ \mathsf{unit} \to \mathsf{unit}]\ \textbf{of}\ [\mathsf{unit}] \to \mathsf{unit} & \{\mathit{case\ assoc.}\} \\
\equiv & \textbf{case}\ z\ \textbf{of}\ [y] \to \textbf{case}\ [y]\ \textbf{of}\ [\mathsf{unit}] \to \mathsf{unit} & \{\eta_{\mathit{case}}\} \\
\equiv & \textbf{case}\ z\ \textbf{of}\ [\mathsf{unit}] \to \mathsf{unit} & \{\eta_{\mathit{case}}\}
\end{array}
$$

- $T = \alpha$:

$$
\begin{array}{lll}
 & ([\![\alpha]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f) \circ [\![\alpha]\!]^{\Sigma}_{\mathsf{push}})\ z & \\
\equiv & [\![\alpha]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ ([\![\alpha]\!]^{\Sigma}_{\mathsf{push}}\ z) & \{\beta\} \\
\equiv & [\![\alpha]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f) z & \{\mathit{defn.}\ [\![\alpha]\!]^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \textbf{case}\ z\ \textbf{of}\ [y] \to [f\, y] & \{\mathit{defn.}\ [\![\alpha]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\} \\
 & & \\
\equiv & ([\![\alpha]\!]^{\Sigma}_{\mathsf{push}} \circ [\![\Box_r \alpha]\!]^{\Sigma'}_{\mathsf{fmap}}(f))\ z & \\
\equiv & [\![\alpha]\!]^{\Sigma}_{\mathsf{push}}\ ([\![\Box_r \alpha]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ z) & \{\beta\} \\
\equiv & [\![\alpha]\!]^{\Sigma}_{\mathsf{push}}\ (\textbf{case}\ z\ \textbf{of}\ [y] \to [f\, y]) & \{\mathit{defn.}\ [\![\Box_r \alpha]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\} \\
\equiv & \textbf{case}\ z\ \textbf{of}\ [y] \to [f\, y] & \{\mathit{defn.}\ [\![\alpha]\!]^{\Sigma}_{\mathsf{push}}\}
\end{array}
$$

- $T = X$:

$$
\begin{array}{lll}
 & ([\![X]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f) \circ [\![X]\!]^{\Sigma}_{\mathsf{push}})\ z & \\
\equiv & [\![X]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ ([\![X]\!]^{\Sigma}_{\mathsf{push}}\ z) & \{\beta\} \\
\equiv & [\![X]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ (\Sigma(X)\ z) & \{\mathit{defn.}\ [\![X]\!]^{\Sigma}_{\mathsf{push}}\ \} \\
\equiv & \Sigma'(X)(\Box_r f)\ (\Sigma(X)\ z) & \{\mathit{defn.}\ [\![X]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\} \\
 & & \\
\equiv & ([\![X]\!]^{\Sigma}_{\mathsf{push}} \circ [\![\Box_r X]\!]^{\Sigma'}_{\mathsf{fmap}}(f))\ z & \\
\equiv & [\![X]\!]^{\Sigma}_{\mathsf{push}}\ ([\![\Box_r X]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ z) & \{\beta\} \\
\equiv & [\![X]\!]^{\Sigma}_{\mathsf{push}}\ (\textbf{case}\ z\ \textbf{of}\ [y] \to [(\Sigma'(X)\ f)\ y]) & \{\mathit{defn.}\ [\![X]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\} \\
\equiv & \Sigma(X)\ (\textbf{case}\ z\ \textbf{of}\ [y] \to [(\Sigma'(X) f)\ y]) & \{\mathit{defn.}\ [\![X]\!]^{\Sigma}_{\mathsf{push}}\ \} \\
\equiv & \Sigma(X)\ (\textbf{case}\ \Box_r(\Sigma'(X)(f))\ z\ \textbf{of}\ [y] \to [y]) & \{\mathit{defn.}\ \Box_r(\Sigma'(X)(f))\} \\
\equiv & \Sigma(X)\ (\Box_r(\Sigma'(X)(f))\ z) & \{\eta_{\mathit{case}}\} \\
\equiv & \Sigma'(X)(\Box_r f)\ (\Sigma(X) z) & \{\mathit{condition\ (*)}\}
\end{array}
$$

- $T = A \oplus B$:

$$
\begin{array}{lll}
 & (\llbracket A \oplus B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f) \circ \llbracket A \oplus B \rrbracket^{\Sigma}_{\mathsf{push}})\ z & \\
\equiv & \llbracket A \oplus B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ (\llbracket A \oplus B \rrbracket^{\Sigma}_{\mathsf{push}}\ z) & \{\beta\} \\
\equiv & \llbracket A \oplus B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ (\mathbf{case}\ z\ \mathbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x];\ ) & \{\mathit{defn.}\ \llbracket A \oplus B \rrbracket^{\Sigma}_{\mathsf{push}}\} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad [\mathsf{inr}\ y] \to \mathsf{inr}\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y] & \\
\equiv & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x];\ )\ \mathbf{of}\ \ \mathsf{inl}\ u \to \mathsf{inl}\ (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ u); & \{\mathit{defn.}\ \llbracket A \oplus B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\} \\
 & \qquad\qquad\qquad\quad [\mathsf{inr}\ y] \to \mathsf{inr}\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y] \qquad\quad \mathsf{inr}\ v \to \mathsf{inr}\ \llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ v & \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \ [\mathsf{inl}\ x] \to \mathbf{case}\ \mathsf{inl}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x]\ \mathbf{of}\ \ \mathsf{inl}\ u \to \mathsf{inl}\ (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ u);\ ; & \{\mathit{case\ assoc.}\} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathsf{inr}\ v \to \mathsf{inr}\ (\llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ v) & \\
 & \qquad\qquad\quad [\mathsf{inr}\ y] \to \mathbf{case}\ \mathsf{inr}\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y]\ \mathbf{of}\ \ \mathsf{inl}\ u \to \mathsf{inl}\ (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ u); & \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathsf{inr}\ v \to \mathsf{inr}\ (\llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ v) & \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x]); & \{\beta_{case}\} \\
 & \qquad\qquad\quad [\mathsf{inr}\ y] \to \mathsf{inr}\ (\llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y]) &
\end{array}
$$

$$
\begin{array}{lll}
\equiv & (\llbracket A \oplus B \rrbracket^{\Sigma}_{\mathsf{push}} \circ \llbracket \Box_r(A \oplus B) \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f))\ z & \\
\equiv & \llbracket A \oplus B \rrbracket^{\Sigma}_{\mathsf{push}}\ (\llbracket \Box_r(A \oplus B) \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ z) & \{\beta\} \\
\equiv & \llbracket A \oplus B \rrbracket^{\Sigma}_{\mathsf{push}}\ (\mathbf{case}\ z\ \mathbf{of}\ [u] \to [\mathbf{case}\ u\ \mathbf{of}\ \ \mathsf{inl}\ x \to \mathsf{inl}\ (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ x);\ ]) & \{\mathit{defn.}\ \llbracket \Box_r(A \oplus B) \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathsf{inr}\ y \to \mathsf{inr}\ (\llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ y) & \\
\equiv & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ [u] \to [\mathbf{case}\ u\ \mathbf{of}\ \ \mathsf{inl}\ x \to \mathsf{inl}\ (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ x);\ ])\ \mathbf{of}\ \ [\mathsf{inl}\ v] \to \mathsf{inl}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[v]; & \{\mathit{defn.}\ \llbracket A \oplus B \rrbracket^{\Sigma}_{\mathsf{push}}\} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathsf{inr}\ y \to \mathsf{inr}\ (\llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ y) \qquad [\mathsf{inr}\ w] \to \mathsf{inr}\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[w] & \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ [u] \to \mathbf{case}\ [\mathbf{case}\ u\ \mathbf{of}\ \ \mathsf{inl}\ x \to \mathsf{inl}\ (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ x);\ ]\ \mathbf{of}\ \ [\mathsf{inl}\ v] \to \mathsf{inl}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[v]; & \{\mathit{case\ assoc.}\} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\quad \mathsf{inr}\ y \to \mathsf{inr}\ (\llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ y) \qquad [\mathsf{inr}\ w] \to \mathsf{inr}\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[w] & \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ [u] \to \mathbf{case}\ [u]\ \mathbf{of}\ \ [\mathsf{inl}\ x] \to \mathbf{case}\ [\mathsf{inl}\ (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ x)]\ \mathbf{of}\ \ [\mathsf{inl}\ v] \to \mathsf{inl}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[v];\ \ ; & \{\mathit{box\ case\ assoc.}\} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\mathsf{inr}\ w] \to \mathsf{inr}\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[w] & \\
 & \qquad\qquad\qquad\qquad\qquad [\mathsf{inr}\ y] \to \mathbf{case}\ [\mathsf{inl}\ (\llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ y)]\ \mathbf{of}\ \ [\mathsf{inl}\ v] \to \mathsf{inl}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[v]; & \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\mathsf{inr}\ w] \to \mathsf{inr}\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[w] & \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ [u] \to \mathbf{case}\ [u]\ \mathbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}\ [\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ x]; & \{\beta_{case}\} \\
 & \qquad\qquad\qquad\qquad\qquad [\mathsf{inr}\ y] \to \mathsf{inr}\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[\llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ y] & \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ [u] \to \mathbf{case}\ [u]\ \mathbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}\ [x]); & \{\mathit{induction}\} \\
 & \qquad\qquad\qquad\qquad\qquad [\mathsf{inr}\ y] \to \mathsf{inr}\ (\llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}\ [y]) & \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}\ [x]); & \{\eta_{case}\} \\
 & \qquad\qquad\quad [\mathsf{inr}\ y] \to \mathsf{inr}\ (\llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}\ [y]) &
\end{array}
$$

- $T = A \otimes B$:

$$
\begin{array}{lll}
 & (\llbracket A \otimes B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f) \circ \llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{push}})\ z & \\
\equiv & \llbracket A \otimes B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ (\llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{push}}\ z) & \{\beta\} \\
\equiv & \llbracket A \otimes B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ (\mathbf{case}\ z\ \mathbf{of}\ [(x,y)] \to (\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y])) & \{\mathit{defn.}\ \llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ [(x,y)] \to (\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y]))\ \mathbf{of}\ (u,\ v) \to (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ u,\ \llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ v) & \{\mathit{defn.}\ \llbracket A \otimes B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ [(x,y)] \to \mathbf{case}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y])\ \mathbf{of}\ (u,\ v) \to (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ u,\ \llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ v) & \{\mathit{case\ assoc.}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ [(x,y)] \to (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x],\ \llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y]) & \{\beta_{case}\}
\end{array}
$$

$$
\begin{array}{lll}
\equiv & (\llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{push}} \circ \llbracket \Box_r(A \otimes B) \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f))\ z & \\
\equiv & \llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{push}}\ (\llbracket \Box_r(A \otimes B) \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ z) & \{\beta\} \\
\equiv & \llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{push}}\ (\mathbf{case}\ z\ \mathbf{of}\ [g] \to [\mathbf{case}\ g\ \mathbf{of}\ (x,\ y) \to (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ x,\ \llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ y)]) & \{\mathit{defn.}\ \llbracket \Box_r(A \otimes B) \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\} \\
\equiv & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ [g] \to & \\
 & \qquad [\mathbf{case}\ g\ \mathbf{of}\ (x,\ y) \to (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ x,\ \llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ y)])\ \mathbf{of}\ [(v,\ w)] \to (\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[v], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[w]) & \{\mathit{defn.}\ \llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ [g] \to & \\
 & \qquad \mathbf{case}\ [\mathbf{case}\ g\ \mathbf{of}\ (x,\ y) \to (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ x,\ \llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ y)]\ \mathbf{of}\ [(v,\ w)] \to (\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[v], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[w]) & \{\mathit{case\ assoc.}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ [g] \to & \\
 & \qquad \mathbf{case}\ [g]\ \mathbf{of}\ [(x,\ y)] \to \mathbf{case}\ [(\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ x,\ \llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ y)]\ \mathbf{of}\ [(v,\ w)] \to (\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[v], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[w]) & \{\mathit{box\ case\ assoc.}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ [g] \to & \\
 & \qquad \mathbf{case}\ [g]\ \mathbf{of}\ [(x,\ y)] \to (\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ x], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[\llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ y]) & \{\beta_{case}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ [g] \to & \\
 & \qquad \mathbf{case}\ [g]\ \mathbf{of}\ [(x,\ y)] \to (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x],\ \llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y]) & \{\mathit{induction}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ [(x,\ y)] \to (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x],\ \llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y]) & \{\eta_{case}\}
\end{array}
$$

- $T = A \multimap B$.

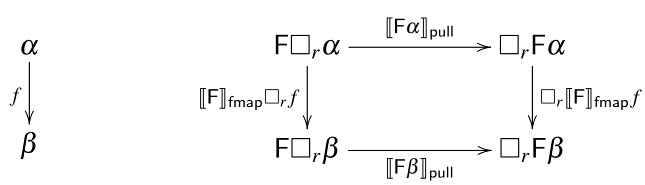
$$
\begin{array}{lll}
 & (\llbracket A \multimap B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f) \circ \llbracket A \multimap B \rrbracket^{\Sigma}_{\mathsf{push}})\ z & \\
\equiv & \llbracket A \multimap B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ (\llbracket A \multimap B \rrbracket^{\Sigma}_{\mathsf{push}}\ z) & \{\beta\} \\
\equiv & \llbracket A \multimap B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ (\lambda y.\mathbf{case}\ z\ \mathbf{of}\ [g] \to \mathbf{case}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ y\ \mathbf{of}\ [u] \to \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[(g\ u)]) & \{\mathit{defn.}\ \llbracket A \multimap B \rrbracket^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \lambda x.\llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ (\lambda y.\mathbf{case}\ z\ \mathbf{of}\ [g] \to \mathbf{case}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ y\ \mathbf{of}\ [u] \to \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[(g\ u)])\ x & \{\mathit{defn.}\ \llbracket A \multimap B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\} \\
\equiv & \lambda x.\llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ (\mathbf{case}\ z\ \mathbf{of}\ [g] \to \mathbf{case}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[(g\ u)]) & \{\beta\} \\
\equiv & \lambda x.\mathbf{case}\ z\ \mathbf{of}\ [g] \to \mathbf{case}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to \llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ (\llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[(g\ u)]) & \{\mathit{case\ distrib.}\} \\
 & & \\
\equiv & (\llbracket A \multimap B \rrbracket^{\Sigma}_{\mathsf{push}} \circ \llbracket \Box_r(A \multimap B) \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f))\ z & \\
\equiv & \llbracket A \multimap B \rrbracket^{\Sigma}_{\mathsf{push}}\ (\llbracket \Box_r(A \multimap B) \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ z) & \{\beta\} \\
\equiv & \llbracket A \multimap B \rrbracket^{\Sigma}_{\mathsf{push}}\ (\mathbf{case}\ z\ \mathbf{of}\ [g] \to [\lambda v.\llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ (g\ v)]) & \{\mathit{defn.}\ \llbracket \Box_r(A \multimap B) \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\} \\
\equiv & \lambda x.\mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ [g] \to [\lambda v.\llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ (g\ v)])\ \mathbf{of}\ [y] \to \mathbf{case}\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}\ [(y\ u)] & \{\mathit{defn.}\ \llbracket A \multimap B \rrbracket^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \lambda x.\mathbf{case}\ z\ \mathbf{of}\ [g] \to \mathbf{case}\ [\lambda v.\llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ (g\ v)]\ \mathbf{of}\ [y] \to \mathbf{case}\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}\ [(y\ u)] & \{\mathit{case\ assoc.}\} \\
\equiv & \lambda x.\mathbf{case}\ z\ \mathbf{of}\ [g] \to \mathbf{case}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}\ [(\lambda v.\llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ (g\ v))\ u] & \{\beta_{case}\} \\
\equiv & \lambda x.\mathbf{case}\ z\ \mathbf{of}\ [g] \to \mathbf{case}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}\ [\llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ (g\ u)] & \{\beta\} \\
\equiv & \lambda x.\mathbf{case}\ z\ \mathbf{of}\ [g] \to \mathbf{case}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to \llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ (\llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}\ [(g\ u)]) & \{\mathit{induction}\}
\end{array}
$$

- $T = \mu X.A$

$$
\begin{array}{lll}
 & (\llbracket \mu X.A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f) \circ \llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{push}})\ z & \\
\equiv & \llbracket \mu X.A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ (\llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{push}}\ z) & \{\beta\} \\
\equiv & \llbracket \mu X.A \rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\,(\mathbf{letrec}\ h\ =\ \llbracket A \rrbracket^{\Sigma,X\mapsto h:\mu X.\Box_r A\multimap(\mu X.A)[\Box_r\alpha/\alpha]}_{\mathsf{push}}\ \mathbf{in}\ h\,z) & \{\mathit{defn.}\ \llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \mathbf{letrec}\ g\ =\ \llbracket A \rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}}(\Box_r f)\ \mathbf{in}\ g\,(\mathbf{letrec}\ h\ =\ \llbracket A \rrbracket^{\Sigma,X\mapsto h:\mu X.\Box_r A\multimap(\mu X.A)[\Box_r\alpha/\alpha]}_{\mathsf{push}}\ \mathbf{in}\ h\,z) & \{\mathit{defn.}\ \llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{fmap}}\} \\
\equiv & \mathbf{letrec}\ g\ =\ \llbracket A \rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}}(\Box_r f)\ \mathbf{in}\ (\mathbf{letrec}\ h\ =\ \llbracket A \rrbracket^{\Sigma,X\mapsto h:\mu X.\Box_r A\multimap(\mu X.A)[\Box_r\alpha/\alpha]}_{\mathsf{push}}\ \mathbf{in}\ g\,(h\,z)) & \{\mathit{letrec\ distrib.}\} \\
\equiv & \mathbf{letrec}\ g\ =\ \llbracket A \rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}}(\Box_r f)\ \mathbf{in}\ (\mathbf{letrec}\ h\ =\ \llbracket A \rrbracket^{\Sigma,X\mapsto h}_{\mathsf{push}}\ \mathbf{in}\ \llbracket A \rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}}(\Box_r f)\,(\llbracket A \rrbracket^{\Sigma,X\mapsto h}_{\mathsf{push}}\,z)) & \{\beta_{letrec}\} \\
\equiv & \llbracket A \rrbracket^{\Sigma',X\mapsto \mathbf{letrec}\ g\,=\,\llbracket A \rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}}(\Box_r f)\ \mathbf{in}\ g}_{\mathsf{fmap}}(\Box_r f)\,(\llbracket A \rrbracket^{\Sigma,X\mapsto \mathbf{letrec}\ h\,=\,\llbracket A \rrbracket^{\Sigma,X\mapsto h}_{\mathsf{push}}\ \mathbf{in}\ h}_{\mathsf{push}}\,z) & \{\beta_{letrec}\} \\
\equiv & (\llbracket A \rrbracket^{\Sigma,X\mapsto \mathbf{letrec}\ h\,=\,\llbracket A \rrbracket^{\Sigma,X\mapsto h}_{\mathsf{push}}\ \mathbf{in}\ h}_{\mathsf{push}}\,(\Box_r \llbracket A \rrbracket^{\Sigma',X\mapsto \mathbf{letrec}\ g\,=\,\llbracket A \rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}} f\ \mathbf{in}\ g}_{\mathsf{fmap}} f\,z)) & \{\mathit{induction}\} \\
 & & \\
\equiv & (\llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{push}} \circ \llbracket \Box_r(\mu X.A) \rrbracket_{\mathsf{fmap}}(f))\ z & \\
\equiv & \llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{push}}\ \llbracket \Box_r(\mu X.A) \rrbracket_{\mathsf{fmap}}(f)\ z) & \{\beta\} \\
\equiv & \llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{push}}(\mathbf{case}\ z\ \mathbf{of}\ [y] \to [\mathbf{letrec}\ g\ =\ \llbracket A \rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}} f\ \mathbf{in}\ g\,y]) & \{\mathit{defn.}\ \llbracket \Box_r(\mu X.A) \rrbracket_{\mathsf{fmap}}(f)\} \\
\equiv & \mathbf{letrec}\ h\ =\ \llbracket A \rrbracket^{\Sigma,X\mapsto h}_{\mathsf{push}}\ \mathbf{in}\ h\,(\mathbf{case}\ z\ \mathbf{of}\ [y] \to [\mathbf{letrec}\ g\ =\ \llbracket A \rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}} f\ \mathbf{in}\ g\,y]) & \{\mathit{defn.}\ \llbracket (\mu X.A) \rrbracket_{\mathsf{push}}\} \\
\equiv & \llbracket A \rrbracket^{\Sigma,X\mapsto \mathbf{letrec}\ h\,=\,\llbracket A \rrbracket^{\Sigma,X\mapsto h}_{\mathsf{push}}\ \mathbf{in}\ h}_{\mathsf{push}}(\mathbf{case}\ z\ \mathbf{of}\ [y] \to [\mathbf{letrec}\ g\ =\ \llbracket A \rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}} f\ \mathbf{in}\ g\,y]) & \{\beta_{letrec}\} \\
\equiv & \llbracket A \rrbracket^{\Sigma,X\mapsto \mathbf{letrec}\ h\,=\,\llbracket A \rrbracket^{\Sigma,X\mapsto h}_{\mathsf{push}}\ \mathbf{in}\ h}_{\mathsf{push}}(\mathbf{case}\ z\ \mathbf{of}\ [y] \to [\llbracket A \rrbracket^{\Sigma',X\mapsto \mathbf{letrec}\ g\,=\,\llbracket A \rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}} f\ \mathbf{in}\ g}_{\mathsf{fmap}} f\,y]) & \{\beta_{letrec}\} \\
\equiv & \llbracket A \rrbracket^{\Sigma,X\mapsto \mathbf{letrec}\ h\,=\,\llbracket A \rrbracket^{\Sigma,X\mapsto h}_{\mathsf{push}}\ \mathbf{in}\ h}_{\mathsf{push}}(\Box_r(\llbracket A \rrbracket^{\Sigma',X\mapsto \mathbf{letrec}\ g\,=\,\llbracket A \rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}} f\ \mathbf{in}\ g}_{\mathsf{fmap}} f)\,z) & \{\mathit{defn.}\ \Box_r + \eta\}
\end{array}
$$

$\Box$

Proposition 3.4 (Naturality of pull)**.** *For all unary type constructors* F *such that* $\llbracket \mathsf{F}\alpha \rrbracket_{\mathsf{pull}}$ *is defined, and given a closed function term* $f : \alpha \multimap \beta$*, then:* $\llbracket \Box_r \mathsf{F} \rrbracket_{\mathsf{fmap}} f \circ \llbracket \mathsf{F}\alpha \rrbracket_{\mathsf{pull}} = \llbracket \mathsf{F}\beta \rrbracket_{\mathsf{pull}} \circ \llbracket \mathsf{F} \rrbracket_{\mathsf{fmap}} \Box_r f$*, i.e.:*



Proof. By induction on the type $T = \mathsf{F}\alpha$, where we consider derivation of the functor action $\llbracket \mathsf{F}\alpha \rrbracket^{\Sigma'}_{\mathsf{fmap}}$ and $\llbracket \mathsf{F}\alpha \rrbracket^{\Sigma}_{\mathsf{pull}}$ with respect to open recursion variables Σ' and Σ assuming that the recursive definitions themselves satisfy the naturality property, i.e., for all $X \in \mathsf{dom}(\Sigma), \mathsf{dom}(\Sigma')$ then $\Sigma'(X)\Box_r(f)(\Sigma(X)z) = \Sigma(X)(\Box_r(\Sigma'(X)(f)z)$ (referred to as condition (*) in the proof).

- $T = 1$:

$$\begin{array}{lll}
& ([\![\Box_r 1]\!]^{\Sigma'}_{\mathsf{fmap}}(f) \circ [\![1]\!]^{\Sigma}_{\mathsf{pull}})\ z & \\
\equiv & [\![\Box_r 1]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ ([\![1]\!]^{\Sigma}_{\mathsf{pull}}\ z) & \{\beta\} \\
\equiv & [\![\Box_r 1]\!]_{\mathsf{fmap}}(f)\ (\mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to [\mathsf{unit}]) & \{\mathit{defn.}\ [\![1]\!]^{\Sigma}_{\mathsf{pull}}\} \\
\equiv & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to [\mathsf{unit}])\ \mathbf{of}\ [y] \to [\mathbf{case}\ y\ \mathbf{of}\ \mathsf{unit} \to \mathsf{unit}] & \{\mathit{defn.}\ [\![\Box_r 1]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to \mathbf{case}\ [\mathsf{unit}]\ \mathbf{of}\ [y] \to [\mathbf{case}\ y\ \mathbf{of}\ \mathsf{unit} \to \mathsf{unit}] & \{\mathit{case\ assoc.}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to [\mathbf{case}\ \mathsf{unit}\ \mathbf{of}\ \mathsf{unit} \to \mathsf{unit}] & \{\beta_{case}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to [\mathsf{unit}] & \{\beta_{case}\} \\
& & \\
\equiv & ([\![1]\!]^{\Sigma}_{\mathsf{pull}} \circ [\![1]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f))\ z & \\
\equiv & [\![1]\!]^{\Sigma}_{\mathsf{pull}}\ ([\![1]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ z) & \{\beta\} \\
\equiv & [\![1]\!]^{\Sigma}_{\mathsf{pull}}\ (\mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to \mathsf{unit}) & \{\mathit{defn.}\ [\![1]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\} \\
\equiv & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to \mathsf{unit})\ \mathbf{of}\ \mathsf{unit} \to [\mathsf{unit}] & \{\mathit{defn.}\ [\![1]\!]^{\Sigma}_{\mathsf{pull}}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to \mathbf{case}\ \mathsf{unit}\ \mathbf{of}\ \mathsf{unit} \to [\mathsf{unit}] & \{\mathit{case\ assoc.}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to [\mathsf{unit}] & \{\beta_{case}\}
\end{array}$$

- $T = \alpha$:

$$\begin{array}{lll}
& ([\![\Box_r \alpha]\!]^{\Sigma'}_{\mathsf{fmap}}(f) \circ [\![\alpha]\!]^{\Sigma}_{\mathsf{pull}})\ z & \\
\equiv & [\![\Box_r \alpha]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ ([\![\alpha]\!]^{\Sigma}_{\mathsf{pull}}\ z) & \{\beta\} \\
\equiv & [\![\Box_r \alpha]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ z & \{\mathit{defn.}\ [\![\alpha]\!]^{\Sigma}_{\mathsf{pull}}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ [y] \to [f\,y] & \{\mathit{defn.}\ [\![\Box_r \alpha]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\} \\
& & \\
\equiv & ([\![\alpha]\!]^{\Sigma}_{\mathsf{pull}} \circ [\![\alpha]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f))\ z & \\
\equiv & [\![\alpha]\!]^{\Sigma}_{\mathsf{pull}}\ ([\![\alpha]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ z) & \{\beta\} \\
\equiv & [\![\alpha]\!]^{\Sigma}_{\mathsf{pull}}\ (\mathbf{case}\ z\ \mathbf{of}\ [y] \to [f\,y]) & \{\mathit{defn.}\ [\![\alpha]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ [y] \to [f\,y] & \{\mathit{defn.}\ [\![\alpha]\!]^{\Sigma}_{\mathsf{pull}}\}
\end{array}$$

- $T = X$:

$$\begin{array}{lll}
& ([\![\Box_r X]\!]^{\Sigma'}_{\mathsf{fmap}}(f) \circ [\![X]\!]^{\Sigma}_{\mathsf{pull}})\ z & \\
\equiv & [\![\Box_r X]\!]^{\Sigma'}_{\mathsf{fmap}}(f)([\![X]\!]^{\Sigma}_{\mathsf{pull}}\ z) & \{\beta\} \\
\equiv & [\![\Box_r X]\!]^{\Sigma'}_{\mathsf{fmap}}(f)(\Sigma(X)\ z) & \{\mathit{defn.}\ [\![X]\!]^{\Sigma}_{\mathsf{pull}}\} \\
\equiv & \mathbf{case}\ (\Sigma(X)\ z)\ \mathbf{of}\ [y] \to [\Sigma'(X)(f)\ y] & \{\mathit{defn.}\ [\![X]\!]^{\Sigma'}_{\mathsf{fmap}}\} \\
\equiv & \mathbf{case}\ \Box_r(\Sigma'(X)(f))\ (\Sigma(X)\ z)\ \mathbf{of}\ [y] \to [y] & \{\mathit{defn.}\ [(\Sigma'(X)(f))]\} \\
\equiv & \Box_r(\Sigma'(X)(f))\ (\Sigma(X)\ z) & \{\eta_{case}\} \\
\equiv & \Sigma(X)\ (\Sigma'(X)(\Box_r f)\ z) & \{\mathit{condition\ (*)}\} \\
& & \\
\equiv & ([\![X]\!]^{\Sigma}_{\mathsf{pull}} \circ [\![X]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f))\ z & \\
\equiv & [\![X]\!]^{\Sigma}_{\mathsf{pull}}\ ([\![X]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ z) & \{\beta\} \\
\equiv & [\![X]\!]^{\Sigma}_{\mathsf{pull}}\ (\Sigma'(X)(\Box_r f)\ z) & \{\mathit{defn.}\ [\![X]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\} \\
\equiv & \Sigma(X)\ (\Sigma'(X)(\Box_r f)\ z) & \{\mathit{defn.}\ [\![X]\!]^{\Sigma}_{\mathsf{pull}}\}
\end{array}$$

- $T = A \oplus B$:

$$
\begin{array}{lll}
 & ([\![\Box_r(A\oplus B)]\!]^{\Sigma'}_{\mathsf{fmap}}(f) \circ [\![A\oplus B]\!]^{\Sigma}_{\mathsf{pull}})\ z & \\
\equiv & [\![\Box_r(A\oplus B)]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ ([\![A\oplus B]\!]^{\Sigma}_{\mathsf{pull}}\ z) & \{\beta\} \\
\equiv & [\![\Box_r(A\oplus B)]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ (\mathbf{case}\ z\ \mathbf{of}\ \ \mathsf{inl}\ x \to \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to [\mathsf{inl}\ u];\ ) & \{\mathit{defn.}\ [\![A\oplus B]\!]^{\Sigma}_{\mathsf{pull}}\} \\
 & \qquad\qquad\qquad\qquad\qquad\quad \mathsf{inr}\ y \to \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \to [\mathsf{inr}\ v] & \\
\equiv & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ \ \mathsf{inl}\ x \to \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to [\mathsf{inl}\ u];\ )\ \mathbf{of}\ [w] \to [\mathbf{case}\ w\ \mathbf{of}\ \ \mathsf{inl}\ m \to \mathsf{inl}\ ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ m);\ ] & \{\mathit{defn.}\ [\![\Box_r(A\oplus B)]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\} \\
 & \qquad\qquad\qquad\quad \mathsf{inr}\ y \to \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \to [\mathsf{inr}\ v] \qquad\qquad\qquad\qquad \mathsf{inr}\ n \to \mathsf{inr}\ (f\ n) & \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \ \mathsf{inl}\ x \to \mathbf{case}\ (\mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to [\mathsf{inl}\ u])\ \mathbf{of}\ [w] \to [\mathbf{case}\ w\ \mathbf{of}\ \ \mathsf{inl}\ m \to \mathsf{inl}\ ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ m);\ ]; & \{\mathit{case\ assoc.}\} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathsf{inr}\ n \to \mathsf{inr}\ ([\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ n) & \\
 & \qquad\qquad\ \mathsf{inr}\ y \to \mathbf{case}\ (\mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \to [\mathsf{inr}\ v])\ \mathbf{of}\ [w] \to [\mathbf{case}\ w\ \mathbf{of}\ \ \mathsf{inl}\ m \to \mathsf{inl}\ ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ m);\ ] & \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathsf{inr}\ n \to \mathsf{inr}\ ([\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ n) & \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \ \mathsf{inl}\ x \to \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to \mathbf{case}\ [\mathsf{inl}\ u]\ \mathbf{of}\ [w] \to [\mathbf{case}\ w\ \mathbf{of}\ \ \mathsf{inl}\ m \to \mathsf{inl}\ ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ m);\ ]; & \{\mathit{case\ assoc.}\} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathsf{inr}\ n \to \mathsf{inr}\ ([\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ n) & \\
 & \qquad\qquad\ \mathsf{inr}\ y \to \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \to \mathbf{case}\ [\mathsf{inr}\ v]\ \mathbf{of}\ [w] \to [\mathbf{case}\ w\ \mathbf{of}\ \ \mathsf{inl}\ m \to \mathsf{inl}\ ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ m);\ ] & \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathsf{inr}\ n \to \mathsf{inr}\ ([\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ n) & \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \ \mathsf{inl}\ x \to \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to [\mathbf{case}\ \mathsf{inl}\ u\ \mathbf{of}\ \ \mathsf{inl}\ m \to \mathsf{inl}\ ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ m);\ ]; & \{\beta_{case}\} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathsf{inr}\ n \to \mathsf{inr}\ ([\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ n) & \\
 & \qquad\qquad\ \mathsf{inr}\ y \to \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \to [\mathbf{case}\ \mathsf{inr}\ v\ \mathbf{of}\ \ \mathsf{inl}\ m \to \mathsf{inl}\ ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ m);\ ] & \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathsf{inr}\ n \to \mathsf{inr}\ ([\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ n) & \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \ \mathsf{inl}\ x \to \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to [\mathsf{inl}\ ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ u)]; & \{\beta_{case}\} \\
 & \qquad\qquad\ \mathsf{inr}\ y \to \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \to [\mathsf{inr}\ ([\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ v)] & \\
 & & \\
\equiv & ([\![A\oplus B]\!]^{\Sigma}_{\mathsf{pull}} \circ [\![A\oplus B]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f))\ z & \\
\equiv & [\![A\oplus B]\!]^{\Sigma}_{\mathsf{pull}}\ ([\![A\oplus B]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ z) & \{\beta\} \\
\equiv & [\![A\oplus B]\!]^{\Sigma}_{\mathsf{pull}}\ (\mathbf{case}\ z\ \mathbf{of}\ \ \mathsf{inl}\ x \to \mathsf{inl}\ ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ x);\ ) & \{\mathit{defn.}\ [\![A\oplus B]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\} \\
 & \qquad\qquad\qquad\qquad\quad \mathsf{inr}\ y \to \mathsf{inr}\ ([\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ y) & \\
\equiv & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ \ \mathsf{inl}\ x \to \mathsf{inl}\ ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ x);\ )\ \mathbf{of}\ \ \mathsf{inl}\ m \to \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ m\ \mathbf{of}\ [u] \to [\mathsf{inl}\ u]; & \{\mathit{defn.}\ [\![A\oplus B]\!]^{\Sigma}_{\mathsf{pull}}\} \\
 & \qquad\qquad\qquad\quad \mathsf{inr}\ y \to \mathsf{inr}\ ([\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ y) \qquad\ \ \mathsf{inr}\ n \to \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ n\ \mathbf{of}\ [v] \to [\mathsf{inr}\ v] & \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \ \mathsf{inl}\ x \to \mathbf{case}\ \mathsf{inl}\ ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ x)\ \mathbf{of}\ \ \mathsf{inl}\ m \to \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ m\ \mathbf{of}\ [u] \to [\mathsf{inl}\ u];\ ; & \{\mathit{case\ assoc.}\} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathsf{inr}\ n \to \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ n\ \mathbf{of}\ [v] \to [\mathsf{inr}\ v] & \\
 & \qquad\qquad\ \mathsf{inr}\ y \to \mathbf{case}\ \mathsf{inr}\ ([\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ y)\ \mathbf{of}\ \ \mathsf{inl}\ m \to \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ m\ \mathbf{of}\ [u] \to [\mathsf{inl}\ u]; & \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathsf{inr}\ n \to \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ n\ \mathbf{of}\ [v] \to [\mathsf{inr}\ v] & \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \ \mathsf{inl}\ x \to \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ x)\ \mathbf{of}\ [u] \to [\mathsf{inl}\ u]; & \{\beta_{case}\} \\
 & \qquad\qquad\ \mathsf{inr}\ y \to \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ ([\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ y)\ \mathbf{of}\ [v] \to [\mathsf{inr}\ v] & \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \ \mathsf{inl}\ x \to \mathbf{case}\ \Box_r [\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ ([\![A]\!]^{\Sigma}_{\mathsf{pull}}\ x)\ \mathbf{of}\ [u] \to [\mathsf{inl}\ u]; & \{\mathit{induction}\ \} \\
 & \qquad\qquad\ \mathsf{inr}\ y \to \mathbf{case}\ \Box_r [\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ ([\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y)\ \mathbf{of}\ [v] \to [\mathsf{inr}\ v] & \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \ \mathsf{inl}\ x \to \mathbf{case}\ \Box_r \mathsf{inl}\ (\Box_r [\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(f))\ ([\![A]\!]^{\Sigma}_{\mathsf{pull}}\ x)\ \mathbf{of}\ [u] \to [u]; & \{\mathit{defns.}\} \\
 & \qquad\qquad\ \mathsf{inr}\ y \to \mathbf{case}\ \Box_r \mathsf{inr}\ (\Box_r [\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(f))\ ([\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y)\ \mathbf{of}\ [v] \to [v] & \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \ \mathsf{inl}\ x \to \mathbf{case}\ \Box_r \mathsf{inl}\ [\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ ([\![A]\!]^{\Sigma}_{\mathsf{pull}}\ x)\ \mathbf{of}\ [u] \to [u]; & \{\mathit{functoriality\ of}\ \Box_r\} \\
 & \qquad\qquad\ \mathsf{inr}\ y \to \mathbf{case}\ \Box_r \mathsf{inr}\ [\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ ([\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y)\ \mathbf{of}\ [v] \to [v] & \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ \ \mathsf{inl}\ x \to \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to [\mathsf{inl}\ [\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ u]; & \{\mathit{defns.}\} \\
 & \qquad\qquad\ \mathsf{inr}\ y \to \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \to [\mathsf{inr}\ [\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(f)\ v] &
\end{array}
$$

- $T = A \otimes B$:

$$
\begin{aligned}
& (\llbracket \Box_r(A\otimes B)\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\circ\llbracket A\otimes B\rrbracket^{\Sigma}_{\mathsf{pull}})\ z \\
\equiv\ & \llbracket \Box_r(A\otimes B)\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ (\llbracket A\otimes B\rrbracket^{\Sigma}_{\mathsf{pull}}\ z) && \{\beta\}\\
\equiv\ & \llbracket \Box_r(A\otimes B)\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ (\mathbf{case}\ z\ \mathbf{of}\ (x,\ y)\to\mathbf{case}\ (\llbracket A\rrbracket^{\Sigma}_{\mathsf{pull}}\ x,\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{pull}}\ y)\ \mathbf{of}\ ([u],\ [v])\to[(u,\ v)]) && \{\mathit{defn.}\ \llbracket A\otimes B\rrbracket^{\Sigma}_{\mathsf{pull}}\}\\
\equiv\ & \llbracket \Box_r(A\otimes B)\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ (\mathbf{case}\ z\ \mathbf{of}\ (x,\ y)\to\mathbf{case}\ (x,\ y)\ \mathbf{of}\ ([u],\ [v])\to[(u,\ v)]) && \{\mathit{induction}\}\\
\equiv\ & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ (\llbracket A\rrbracket^{\Sigma}_{\mathsf{pull}}\ x,\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{pull}}\ y)\to\mathbf{case}\ (x,\ y)\ \mathbf{of}\ ([u],\ [v])\to[(u,\ v)])\ \mathbf{of}\ [l]\to\\
& \qquad [\mathbf{case}\ l\ \mathbf{of}\ (m,\ n)\to(\llbracket A\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ m,\ \llbracket B\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ n)] && \{\mathit{defn.}\ \llbracket \Box_r(A\otimes B)\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\}\\
\equiv\ & \mathbf{case}\ z\ \mathbf{of}\ (x,\ y)\to\mathbf{case}\ (\mathbf{case}\ (\llbracket A\rrbracket^{\Sigma}_{\mathsf{pull}}\ x,\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{pull}}\ y)\ \mathbf{of}\ ([u],\ [v])\to[(u,\ v)])\ \mathbf{of}\ [l]\to\\
& \qquad [\mathbf{case}\ l\ \mathbf{of}\ (m,\ n)\to(\llbracket A\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ m,\ \llbracket B\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ n)] && \{\mathit{case\ assoc.}\}\\
\equiv\ & \mathbf{case}\ z\ \mathbf{of}\ (x,\ y)\to\mathbf{case}\ (\llbracket A\rrbracket^{\Sigma}_{\mathsf{pull}}\ x,\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{pull}}\ y)\ \mathbf{of}\ ([u],\ [v])\to\\
& \qquad \mathbf{case}\ [(u,\ v)]\ \mathbf{of}\ [l]\to[\mathbf{case}\ l\ \mathbf{of}\ (m,\ n)\to(\llbracket A\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ m,\ \llbracket B\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ n)] && \{\mathit{case\ assoc.}\}\\
\equiv\ & \mathbf{case}\ z\ \mathbf{of}\ (x,\ y)\to\mathbf{case}\ (\llbracket A\rrbracket^{\Sigma}_{\mathsf{pull}}\ x,\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{pull}}\ y)\ \mathbf{of}\ ([u],\ [v])\to\\
& \qquad [\mathbf{case}\ (u,\ v)\ \mathbf{of}\ (m,n)\to(\llbracket A\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ m,\ \llbracket B\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ n)] && \{\beta_{case}\}\\
\equiv\ & \mathbf{case}\ z\ \mathbf{of}\ (x,\ y)\to\mathbf{case}\ (\llbracket A\rrbracket^{\Sigma}_{\mathsf{pull}}\ x,\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{pull}}\ y)\ \mathbf{of}\ ([u],\ [v])\to[(\llbracket A\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ u,\ \llbracket B\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ v)] && \{\beta_{case}\}\\
\\
\equiv\ & (\llbracket A\otimes B\rrbracket^{\Sigma}_{\mathsf{pull}}\circ\llbracket A\otimes B\rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f))\ z\\
\equiv\ & \llbracket A\otimes B\rrbracket^{\Sigma}_{\mathsf{pull}}\ \llbracket A\otimes B\rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ z && \{\beta\}\\
\equiv\ & \llbracket A\otimes B\rrbracket^{\Sigma}_{\mathsf{pull}}\ (\mathbf{case}\ z\ \mathbf{of}\ (x,\ y)\to(\llbracket A\rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ x,\llbracket A\rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ y)) && \{\mathit{def.\ of}\ \llbracket A\otimes B\rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\}\\
\equiv\ & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ (x,\ y)\to(\llbracket A\rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ x,\llbracket B\rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ y))\ \mathbf{of}\ (m,\ n)\to\\
& \qquad \mathbf{case}\ (\llbracket A\rrbracket^{\Sigma}_{\mathsf{pull}}\ m,\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{pull}}\ n)\ \mathbf{of}\ ([u],\ [v])\to[(u,\ v)] && \{\mathit{defn.}\ \llbracket A\otimes B\rrbracket^{\Sigma}_{\mathsf{pull}}\}\\
\equiv\ & \mathbf{case}\ z\ \mathbf{of}\ (x,\ y)\to\mathbf{case}\ (\llbracket A\rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ x,\llbracket B\rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ y)\ \mathbf{of}\ (m,\ n)\to\\
& \qquad \mathbf{case}\ (\llbracket A\rrbracket^{\Sigma}_{\mathsf{pull}}\ m,\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{pull}}\ n)\ \mathbf{of}\ ([u],\ [v])\to[(u,\ v)] && \{\mathit{case\ assoc.}\}\\
\equiv\ & \mathbf{case}\ z\ \mathbf{of}\ (x,\ y)\to\mathbf{case}\ (\llbracket A\rrbracket^{\Sigma}_{\mathsf{pull}}\ \llbracket A\rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ x,\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{pull}}\ \llbracket B\rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ y)\ \mathbf{of}\ ([u],\ [v])\to[(u,\ v)] && \{\beta_{case}\}\\
\equiv\ & \mathbf{case}\ z\ \mathbf{of}\ (x,\ y)\to\mathbf{case}\ (\Box_r\llbracket A\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ \llbracket A\rrbracket^{\Sigma}_{\mathsf{pull}}\ x,\ \Box_r\llbracket B\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{pull}}\ y)\ \mathbf{of}\ ([u],\ [v])\to[(u,\ v)] && \{\mathit{induction}\}\\
\equiv\ & \mathbf{case}\ z\ \mathbf{of}\ (x,\ y)\to\mathbf{case}\ (\llbracket A\rrbracket^{\Sigma}_{\mathsf{pull}}\ x,\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{pull}}\ y)\ \mathbf{of}\ ([u],\ [v])\to[(\llbracket A\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ u,\ \llbracket B\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ v)] && \{\mathit{defns.}\}
\end{aligned}
$$

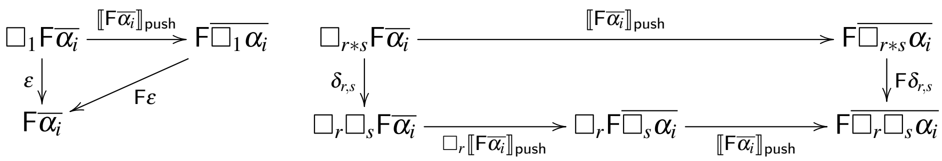
- $T = \mu X.A$:

$$
\begin{aligned}
& (\llbracket \Box_r(\mu X.A)\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\circ\llbracket \mu X.A\rrbracket^{\Sigma}_{\mathsf{pull}})\ z\\
\equiv\ & \llbracket \Box_r(\mu X.A)\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ (\llbracket \mu X.A\rrbracket^{\Sigma}_{\mathsf{pull}}\ z) && \{\beta\}\\
\equiv\ & \llbracket \Box_r(\mu X.A)\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ (\mathbf{letrec}\ h=\llbracket A\rrbracket^{\Sigma,X\mapsto h:\mu X.A[\Box_r\alpha/\alpha]\multimap\Box_r(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ h\ z) && \{\mathit{defn.}\ \llbracket \mu X.A\rrbracket^{\Sigma}_{\mathsf{pull}}\}\\
\equiv\ & \llbracket \Box_r(\mu X.A)\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ (\mathbf{letrec}\ h=\llbracket A\rrbracket^{\Sigma,X\mapsto h:\mu X.A[\Box_r\alpha/\alpha]\multimap\Box_r(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ \llbracket A\rrbracket^{\Sigma,X\mapsto h:\mu X.A[\Box_r\alpha/\alpha]\multimap\Box_r(\mu X.A)}_{\mathsf{pull}}\ z) && \{\beta_{letrec}\}\\
\equiv\ & \llbracket \Box_r(\mu X.A)\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\ (\llbracket A\rrbracket^{\Sigma,X\mapsto\mathbf{letrec}\ h\,=\,\llbracket A\rrbracket^{\Sigma,X\mapsto h}_{\mathsf{pull}}\ \mathbf{in}\ h}_{\mathsf{pull}}\ z) && \{\beta_{letrec}\}\\
\equiv\ & \mathbf{case}\ (\llbracket A\rrbracket^{\Sigma,X\mapsto\mathbf{letrec}\ h\,=\,\llbracket A\rrbracket^{\Sigma,X\mapsto h}_{\mathsf{pull}}\ \mathbf{in}\ h}_{\mathsf{pull}}\ z)\ \mathbf{of}\ [y]\to\\
& \qquad [\mathbf{letrec}\ g=\llbracket A\rrbracket^{\Sigma',X\mapsto g:(\alpha\multimap\beta)\multimap\mu X.A\multimap(\mu X.A)[\alpha/\beta]}_{\mathsf{fmap}}\ \mathbf{in}\ (g\ f)\ y] && \{\mathit{defn.}\ \llbracket \Box_r(\mu X.A)\rrbracket^{\Sigma'}_{\mathsf{fmap}}(f)\}\\
\equiv\ & \mathbf{case}\ (\llbracket A\rrbracket^{\Sigma,X\mapsto\mathbf{letrec}\ h\,=\,\llbracket A\rrbracket^{\Sigma,X\mapsto h}_{\mathsf{pull}}\ \mathbf{in}\ h}_{\mathsf{pull}}\ z)\ \mathbf{of}\ [y]\to\\
& \qquad [\mathbf{letrec}\ g=\llbracket A\rrbracket^{\Sigma',X\mapsto g:(\alpha\multimap\beta)\multimap\mu X.A\multimap(\mu X.A)[\alpha/\beta]}_{\mathsf{fmap}}\ \mathbf{in}\ \llbracket A\rrbracket^{\Sigma',X\mapsto g:(\alpha\multimap\beta)\multimap\mu X.A\multimap(\mu X.A)[\alpha/\beta]}_{\mathsf{fmap}}\ (f)\ y] && \{\beta_{letrec}\}\\
\equiv\ & \mathbf{case}\ (\llbracket A\rrbracket^{\Sigma,X\mapsto\mathbf{letrec}\ h\,=\,\llbracket A\rrbracket^{\Sigma,X\mapsto h}_{\mathsf{pull}}\ \mathbf{in}\ h}_{\mathsf{pull}}\ z)\ \mathbf{of}\ [y]\to[\llbracket A\rrbracket^{\Sigma',X\mapsto\mathbf{letrec}\ g\,=\,\llbracket A\rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}}(f)\ \mathbf{in}\ g}_{\mathsf{fmap}}\ y] && \{\beta_{letrec}\}\\
\\
\equiv\ & (\llbracket \mu X.A\rrbracket^{\Sigma}_{\mathsf{pull}}\circ\llbracket \mu X.A\rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f))\ z\\
\equiv\ & \llbracket \mu X.A\rrbracket^{\Sigma}_{\mathsf{pull}}\ (\llbracket \mu X.A\rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\ z) && \{\beta\}\\
\equiv\ & \llbracket \mu X.A\rrbracket^{\Sigma}_{\mathsf{pull}}\ (\mathbf{letrec}\ g=\llbracket A\rrbracket^{\Sigma',X\mapsto g:(\alpha\multimap\beta)\multimap\mu X.A\multimap(\mu X.A)[\alpha/\beta]}_{\mathsf{fmap}}\ \mathbf{in}\ (g\ (\Box_r f)\ z)) && \{\mathit{defn.}\ \llbracket \mu X.A\rrbracket^{\Sigma'}_{\mathsf{fmap}}(\Box_r f)\}\\
\equiv\ & \llbracket \mu X.A\rrbracket^{\Sigma}_{\mathsf{pull}}\ (\mathbf{letrec}\ g=\llbracket A\rrbracket^{\Sigma',X\mapsto g:(\alpha\multimap\beta)\multimap\mu X.A\multimap(\mu X.A)[\alpha/\beta]}_{\mathsf{fmap}}\ \mathbf{in}\ \llbracket A\rrbracket^{\Sigma',X\mapsto g:(\alpha\multimap\beta)\multimap\mu X.A\multimap(\mu X.A)[\alpha/\beta]}_{\mathsf{fmap}}\ (\Box_r f)\ z) && \{\beta_{letrec}\}\\
\equiv\ & \llbracket \mu X.A\rrbracket^{\Sigma}_{\mathsf{pull}}\ (\llbracket A\rrbracket^{\Sigma',X\mapsto\mathbf{letrec}\ g\,=\,\llbracket A\rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}}(\Box_r f)\ \mathbf{in}\ g}_{\mathsf{fmap}}\ (\Box_r f)\ z) && \{\beta_{letrec}\}\\
\equiv\ & \mathbf{letrec}\ h=\llbracket A\rrbracket^{\Sigma,X\mapsto h:\mu X.A[\Box_r\alpha/\alpha]\multimap\Box_r(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ h\ (\llbracket A\rrbracket^{\Sigma',X\mapsto\mathbf{letrec}\ g\,=\,\llbracket A\rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}}(\Box_r f)\ \mathbf{in}\ g}_{\mathsf{fmap}}\ (\Box_r f)\ z) && \{\mathit{defn.}\ \llbracket \mu X.A\rrbracket^{\Sigma}_{\mathsf{pull}}\}\\
\equiv\ & \mathbf{letrec}\ h=\llbracket A\rrbracket^{\Sigma,X\mapsto h:\mu X.A[\Box_r\alpha/\alpha]\multimap\Box_r(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ \llbracket A\rrbracket^{\Sigma,X\mapsto h:\mu X.A[\Box_r\alpha/\alpha]\multimap\Box_r(\mu X.A)}_{\mathsf{pull}}\ (\llbracket A\rrbracket^{\Sigma',X\mapsto\mathbf{letrec}\ g\,=\,\llbracket A\rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}}(\Box_r f)\ \mathbf{in}\ g}_{\mathsf{fmap}}\ (\Box_r f)\ z) && \{\beta_{letrec}\}\\
\equiv\ & \llbracket A\rrbracket^{\Sigma,X\mapsto\mathbf{letrec}\ h\,=\,\llbracket A\rrbracket^{\Sigma,X\mapsto h}_{\mathsf{pull}}\ \mathbf{in}\ h}_{\mathsf{pull}}\ (\llbracket A\rrbracket^{\Sigma',X\mapsto\mathbf{letrec}\ g\,=\,\llbracket A\rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}}(\Box_r f)\ \mathbf{in}\ g}_{\mathsf{fmap}}\ (\Box_r f)\ z) && \{\beta_{letrec}\}\\
\equiv\ & \Box_r\llbracket A\rrbracket^{\Sigma',X\mapsto\mathbf{letrec}\ g\,=\,\llbracket A\rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}}(\Box_r f)\ \mathbf{in}\ g}_{\mathsf{fmap}}(f)\ (\llbracket A\rrbracket^{\Sigma,X\mapsto\mathbf{letrec}\ h\,=\,\llbracket A\rrbracket^{\Sigma,X\mapsto h}_{\mathsf{pull}}\ \mathbf{in}\ h}_{\mathsf{pull}}\ z) && \{\mathit{induction}\}\\
\equiv\ & \lambda x.\mathbf{case}\ x\ \mathbf{of}\ [y]\to[\llbracket A\rrbracket^{\Sigma',X\mapsto\mathbf{letrec}\ g\,=\,\llbracket A\rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}}(f)\ \mathbf{in}\ g}_{\mathsf{fmap}}(f)\ y](\llbracket A\rrbracket^{\Sigma,X\mapsto\mathbf{letrec}\ h\,=\,\llbracket A\rrbracket^{\Sigma,X\mapsto h}_{\mathsf{pull}}\ \mathbf{in}\ h}_{\mathsf{pull}}\ z) && \{\mathit{defn.}\ \Box_r\}\\
\equiv\ & \mathbf{case}\ (\llbracket A\rrbracket^{\Sigma,X\mapsto\mathbf{letrec}\ h\,=\,\llbracket A\rrbracket^{\Sigma,X\mapsto h}_{\mathsf{pull}}\ \mathbf{in}\ h}_{\mathsf{pull}}\ z)\ \mathbf{of}\ [y]\to[\llbracket A\rrbracket^{\Sigma',X\mapsto\mathbf{letrec}\ g\,=\,\llbracket A\rrbracket^{\Sigma',X\mapsto g}_{\mathsf{fmap}}(f)\ \mathbf{in}\ g}_{\mathsf{fmap}}(f)\ y] && \{\beta\}
\end{aligned}
$$

□

C.3 Preservation of Graded Comonad Operations

Proposition 3.5 (Push preserves graded comonads). *For all* F *such that* $[\![\mathsf{F}\overline{\alpha_i}]\!]_{\mathsf{push}}$ *is defined and* F *does not contain* $\multimap$ *(to avoid issues of contravariance in* F*) then:*



Proof. We consider first the property involving ε, by induction on the type $T = \mathsf{F}\overline{\alpha_i}$, for open recursion Σ, Σ' where we assume that for all $f \in \Sigma(X)$ and $\mathsf{F} \in \Sigma'(X)$ then $(\mathsf{F}\varepsilon) \circ f = \varepsilon$ (called, condition (*))

- $T = 1$

$$\begin{array}{lll} & (([\![1]\!]^{\Sigma'}_{\mathsf{fmap}}\varepsilon) \circ [\![1]\!]^{\Sigma}_{\mathsf{push}})\ z & \\ \equiv & (\lambda x.\mathbf{case}\ x\ \mathbf{of}\ \mathsf{unit} \to \mathsf{unit})(\mathbf{case}\ z\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit}) & \{\textit{defns.} + \beta\} \\ \equiv & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit})\ \mathbf{of}\ \mathsf{unit} \to \mathsf{unit} & \{\beta\} \\ \equiv & \mathbf{case}\ z\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit} & \{\eta_{case}\} \\ \equiv & \mathbf{case}\ z\ \mathbf{of}\ [x] \to x & \{\textit{case gen.}\} \\ \equiv & \varepsilon\ z & \{\textit{def.}\} \end{array}$$

- $T = \alpha$:

$$\begin{array}{lll} & ([\![\alpha]\!]^{\Sigma'}_{\mathsf{fmap}}\varepsilon)([\![\alpha]\!]^{\Sigma}_{\mathsf{push}}\ z) & \\ \equiv & (\lambda z.\mathbf{case}\ z\ \mathbf{of}\ [x] \to x)[\![\alpha]\!]^{\Sigma}_{\mathsf{push}}\ z & \{\textit{def.}\} \\ \equiv & (\lambda z.\mathbf{case}\ z\ \mathbf{of}\ [x] \to x)z & \{\textit{def.}\} \\ \equiv & \mathbf{case}\ z\ \mathbf{of}\ [x] \to x & \{\beta\} \\ \equiv & \varepsilon\ z & \{\textit{def.}\} \end{array}$$

- $T = X$:

$$\begin{array}{lll} & ([\![X]\!]^{\Sigma'}_{\mathsf{fmap}}\varepsilon)([\![X]\!]^{\Sigma}_{\mathsf{push}}\ z) & \\ \equiv & \Sigma'(X)\varepsilon(\Sigma(X)z) & \{\textit{def.}\} \\ \equiv & \varepsilon & \{\textit{assumption (*)}\} \end{array}$$

- $T = A \oplus B$:

$$
\begin{array}{lll}
& [\![A \oplus B]\!]^{\Sigma'}_{\mathsf{fmap}} \varepsilon ([\![A \oplus B]\!]^{\Sigma}_{\mathsf{push}}\ z) & \\
\equiv & (\lambda z.\textbf{case}\ z\ \textbf{of}\ \ \mathsf{inl}\ x \to \mathsf{inl}\ ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}\ \varepsilon\, x);\ \ \mathsf{inr}\ y \to \mathsf{inr}\ ([\![B]\!]^{\Sigma'}_{\mathsf{fmap}}\ \varepsilon\, y)) & \\
& (\textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x];\ \ [\mathsf{inr}\ y] \to \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[y]\ ) & \{\mathit{defns.}\} \\
\equiv & \textbf{case}\ (\textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x];\ \ [\mathsf{inr}\ y] \to \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[y]\ )\ \textbf{of} & \{\beta\} \\
& \quad \mathsf{inl}\ x \to \mathsf{inl}\ ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}\ \varepsilon\, x);\ \ \mathsf{inr}\ y \to \mathsf{inr}\ ([\![B]\!]^{\Sigma'}_{\mathsf{fmap}}\ \varepsilon\, y) & \\
\equiv & \textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to \textbf{case}\ \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x]\ \textbf{of}\ \ \mathsf{inl}\ x \to \mathsf{inl}\ ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}\ \varepsilon\, x);\ \mathsf{inr}\ y \to \mathsf{inr}\ ([\![B]\!]^{\Sigma'}_{\mathsf{fmap}}\ \varepsilon\, y) & \\
& \quad ;\ [\mathsf{inr}\ y] \to \textbf{case}\ \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x]\ \textbf{of}\ \ \mathsf{inl}\ x \to \mathsf{inl}\ ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}\ \varepsilon\, x);\ \mathsf{inr}\ y \to \mathsf{inr}\ ([\![B]\!]^{\Sigma'}_{\mathsf{fmap}}\ \varepsilon\, y) & \{\mathit{case\ assoc.}\} \\
\equiv & \textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}\ \varepsilon\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x]);\ \ [\mathsf{inr}\ y] \to \mathsf{inr}\ ([\![B]\!]^{\Sigma'}_{\mathsf{fmap}}\ \varepsilon\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[y]) & \{\beta\} \\
\equiv & \textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ (\varepsilon\ [x]);\ \ [\mathsf{inr}\ y] \to \varepsilon\ [y]) & \{\mathit{induction}\} \\
\equiv & \textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ x;\ \ [\mathsf{inr}\ y] \to \mathsf{inr}\ y & \{\mathit{def.}\ \varepsilon + \beta\} \\
\equiv & \textbf{case}\ z\ \textbf{of}\ [x] \to x & \{\mathit{case\ gen}\} \\
\equiv & \varepsilon\ z & \{\mathit{def.}\}
\end{array}
$$

- $T = A \otimes B$:

$$
\begin{array}{lll}
& [\![A \otimes B]\!]^{\Sigma'}_{\mathsf{fmap}} \varepsilon ([\![A \otimes B]\!]^{\Sigma}_{\mathsf{push}}\ z) & \\
\equiv & (\lambda z.\textbf{case}\ z\ \textbf{of}\ (x,y) \to ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}\ \varepsilon\ x, [\![B]\!]^{\Sigma'}_{\mathsf{fmap}} \varepsilon\ y))(\textbf{case}\ z\ \textbf{of}\ [(x,y)] \to ([\![A]\!]^{\Sigma}_{\mathsf{push}}[x], [\![B]\!]^{\Sigma}_{\mathsf{push}}[y])) & \{\mathit{def.}\} \\
\equiv & \textbf{case}\ (\textbf{case}\ z\ \textbf{of}\ [(x,y)] \to ([\![A]\!]^{\Sigma}_{\mathsf{push}}[x], [\![B]\!]^{\Sigma}_{\mathsf{push}}[y]))\ \textbf{of}\ (x,y) \to ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}\ \varepsilon\ x, [\![B]\!]^{\Sigma'}_{\mathsf{fmap}} \varepsilon\ y) & \{\beta\} \\
\equiv & \textbf{case}\ z\ \textbf{of}\ [(x,y)] \to \textbf{case}\ ([\![A]\!]^{\Sigma}_{\mathsf{push}}[x], [\![B]\!]^{\Sigma}_{\mathsf{push}}[y])\ \textbf{of}\ (x,y) \to ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}\ \varepsilon\ x, [\![B]\!]^{\Sigma'}_{\mathsf{fmap}} \varepsilon\ y) & \{\mathit{case\ assoc.}\} \\
\equiv & \textbf{case}\ z\ \textbf{of}\ [(x,y)] \to ([\![A]\!]^{\Sigma'}_{\mathsf{fmap}} \varepsilon\ ([\![A]\!]^{\Sigma}_{\mathsf{push}}[x]), [\![B]\!]^{\Sigma'}_{\mathsf{fmap}} \varepsilon\ ([\![B]\!]^{\Sigma}_{\mathsf{push}}[y])) & \{\beta_{case}\} \\
\equiv & \textbf{case}\ z\ \textbf{of}\ [(x,y)] \to (\varepsilon\ [x], \varepsilon\ [y]) & \{\mathit{induction}\} \\
\equiv & \textbf{case}\ z\ \textbf{of}\ [(x,y)] \to (x,y) & \{\varepsilon\ \mathit{defn.}\} \\
\equiv & \textbf{case}\ z\ \textbf{of}\ [x] \to x & \{\mathit{case\ gen.}\} \\
\equiv & \varepsilon\ z & \{\mathit{defn.}\}
\end{array}
$$

- $T = A \multimap B$.

  Cannot be handled, as per the restriction of the proposition, and indeed is not derivable because the types in the commuting diagram do not match for this case due to contravariance in A.

- $T = \mu X.A$

$$
\begin{array}{lll}
& [\![\mu X.A]\!]^{\Sigma'}_{\mathsf{fmap}} \varepsilon ([\![\mu X.A]\!]^{\Sigma}_{\mathsf{push}}\ z) & \\
\equiv & (\lambda z.\textbf{letrec}\ g = [\![A]\!]^{\Sigma,X \mapsto g}_{\mathsf{fmap}}(\varepsilon)\ \textbf{in}\ g\ z)(\textbf{letrec}\ f = [\![A]\!]^{\Sigma,X \mapsto f:\mu X.\Box_r A \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}_{\mathsf{push}}\ \textbf{in}\ f\ z) & \{\mathit{defn.}\} \\
\equiv & \textbf{letrec}\ g = [\![A]\!]^{\Sigma,X \mapsto g}_{\mathsf{fmap}}(\varepsilon)\ \textbf{in}\ g\ (\textbf{letrec}\ f = [\![A]\!]^{\Sigma,X \mapsto f:\mu X.\Box_r A \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}_{\mathsf{push}}\ \textbf{in}\ f\ z) & \{\beta\} \\
\equiv & \textbf{letrec}\ g = [\![A]\!]^{\Sigma,X \mapsto g}_{\mathsf{fmap}}(\varepsilon)\ \textbf{in}\ (\textbf{letrec}\ f = [\![A]\!]^{\Sigma,X \mapsto f:\mu X.\Box_r A \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}_{\mathsf{push}}\ \textbf{in}\ g\ (f\ z)) & \{\mathit{letrec\ distrib.}\} \\
\equiv & \textbf{letrec}\ g = [\![A]\!]^{\Sigma,X \mapsto g}_{\mathsf{fmap}}(\varepsilon)\ \textbf{in}\ (\textbf{letrec}\ f = [\![A]\!]^{\Sigma,X \mapsto f:\mu X.\Box_r A \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}_{\mathsf{push}}\ \textbf{in}\ \varepsilon\ z) & \{\mathit{induction\ (subst\ in\ g\ and\ f)}\} \\
\equiv & \varepsilon\ z & \{\beta_{letrec}\}
\end{array}
$$

  Note that in the recursive case here we are assuming by induction on A that f itself satisfies $\mathsf{F}\varepsilon \circ f = \varepsilon$ which enables the rewrite on the right-hand side of in here as well.

Thus from the above we have that $[\![\mathsf{F}\overline{\alpha_i}]\!]_{\mathsf{fmap}}\varepsilon \circ [\![\mathsf{F}\overline{\alpha_i}]\!]_{\mathsf{push}} = \varepsilon$ by function extensionality and since the initial environemnt $\Sigma' = \Sigma = \emptyset$ trivially satisfies condition (*).

Next we consider the second property involving δ, by induction on the type $T = \mathsf{F}\overline{\alpha_i}$, for open recursion Σ, Σ' where we assume that for all $f \in \Sigma(X)$ and $\mathsf{F} \in \Sigma'(X)$ then $f \circ \Box f \circ \delta = \mathsf{F}\delta \circ f$ (called, condition (**)).

- $T = 1$:

$$
\begin{array}{lll}
& [\![1]\!]^{\Sigma}_{\mathsf{push}}(\Box_r [\![1]\!]^{\Sigma}_{\mathsf{push}}(\delta_{r,s}\ z)) & \\
= & ((\lambda y.\mathbf{case}\ y\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit}) & \\
& \quad \circ(\lambda x'.\mathbf{case}\ x'\ \mathbf{of}\ [x] \to [\mathbf{case}\ x\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit}]) \circ (\lambda z.\mathbf{case}\ z\ \mathbf{of}\ [x] \to [[x]]))z & \{\mathit{defns.}\} \\
= & \mathbf{case}\ (\mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ [x] \to [[x]])\ \mathbf{of}\ [x] \to [\mathbf{case}\ x\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit}])\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit} & \{\beta\ (\times 3)\} \\
= & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ [x] \to \mathbf{case}\ [[x]]\ \mathbf{of}\ [x] \to [\mathbf{case}\ x\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit}])\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit} & \{\mathit{case\ assoc.}\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [x] \to \mathbf{case}\ (\mathbf{case}\ [[x]]\ \mathbf{of}\ [x] \to [\mathbf{case}\ x\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit}])\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit} & \{\mathit{case\ assoc.}\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [x] \to \mathbf{case}\ [[x]]\ \mathbf{of}\ [x] \to \mathbf{case}\ ([\mathbf{case}\ x\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit}])\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit} & \{\mathit{case\ assoc.}\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [x] \to \mathbf{case}\ [\mathbf{case}\ [x]\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit}]\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit} & \{\beta_{case}\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [x] \to \mathbf{case}\ (\mathbf{case}\ [x]\ \mathbf{of}\ [\mathsf{unit}] \to [\mathsf{unit}])\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit} & \{\mathit{case\ push}\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [x] \to \mathbf{case}\ [x]\ \mathbf{of}\ [\mathsf{unit}] \to \mathbf{case}\ [\mathsf{unit}]\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit} & \{\mathit{case\ assoc}\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [x] \to \mathbf{case}\ [x]\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit} & \{\beta_{case}\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit} & \{\eta_{case}\}
\end{array}
$$

$$
\begin{array}{lll}
& [\![1]\!]^{\Sigma'}_{\mathsf{fmap}}\delta([\![1]\!]^{\Sigma}_{\mathsf{push}} z) & \\
\equiv & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit})\ \mathbf{of}\ \mathsf{unit} \to \mathsf{unit} & \{\mathit{defns.} + \beta\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ [\mathsf{unit}] \to \mathbf{case}\ \mathsf{unit}\ \mathbf{of}\ \mathsf{unit} \to \mathsf{unit} & \{\mathit{case\ assoc.}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ [\mathsf{unit}] \to \mathsf{unit} & \{\beta_{case}\}
\end{array}
$$

- $T = \alpha$

$$
\begin{array}{lll}
& [\![\alpha]\!]^{\Sigma}_{\mathsf{push}}(\Box_r [\![\alpha]\!]^{\Sigma}_{\mathsf{push}}(\delta_{r,s}\ z)) & \\
= & ((\lambda x'.\mathbf{case}\ x'\ \mathbf{of}\ [x] \to [x]) \circ (\lambda z.\mathbf{case}\ z\ \mathbf{of}\ [x] \to [[x]]))z & \{\mathit{defn.} + \beta\} \\
= & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ [x] \to [[x]])\ \mathbf{of}\ [x] \to [x] & \{\beta\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [x] \to [[x]] & \{\eta_{case}\}
\end{array}
$$

$$
\begin{array}{lll}
& [\![\alpha]\!]^{\Sigma'}_{\mathsf{fmap}}\delta([\![\alpha]\!]^{\Sigma}_{\mathsf{push}} z) & \\
= & (\lambda z.\mathbf{case}\ z\ \mathbf{of}\ [x] \to [[x]])([\![\alpha]\!]^{\Sigma}_{\mathsf{push}} z) & \{\mathit{defns.} + \beta\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [x] \to [[x]] &
\end{array}
$$

- $T = X$

$$
\begin{array}{lll}
& [\![X]\!]^{\Sigma}_{\mathsf{push}}(\Box_r [\![X]\!]^{\Sigma}_{\mathsf{push}}(\delta_{r,s}\ z)) & \\
= & ((\lambda z.\Sigma(X)z) \circ (\lambda z.\mathbf{case}\ z\ \mathbf{of}\ [x] \to [\Sigma(X)x]) \circ (\lambda z.\mathbf{case}\ z\ \mathbf{of}\ [x] \to [[x]]))z & \{\mathit{defn.}\} \\
= & [\![X]\!]^{\Sigma'}_{\mathsf{fmap}} w\delta_{r,s}([\![X]\!]^{\Sigma}_{\mathsf{push}}\ z) & \{\mathit{condition\ (**)}\}
\end{array}
$$

where in the binding of X to f we recursively apply the inductive evidence on A itself that it satisfies the property and thus provides that $X \mapsto f$ satisfies the condition of (**) in the induction step.

- $T = A \oplus B$

$$
\begin{array}{lll}
& [\![A \oplus B]\!]^{\Sigma}_{\mathsf{push}}(\Box_r [\![A \oplus B]\!]^{\Sigma}_{\mathsf{push}}(\delta_{r,s}\ z)) & \\
\equiv & [\![A \oplus B]\!]^{\Sigma}_{\mathsf{push}}\textbf{case}\ (\textbf{case}\ z\ \textbf{of}\ [x] \to [[x]])\ \textbf{of}\ [x] \to [[\![A \oplus B]\!]^{\Sigma}_{\mathsf{push}} x] & \{\mathit{defn.} + \beta\} \\
\equiv & [\![A \oplus B]\!]^{\Sigma}_{\mathsf{push}}\textbf{case}\ z\ \textbf{of}\ [x] \to \textbf{case}\ [[x]]\ \textbf{of}\ [x] \to [[\![A \oplus B]\!]^{\Sigma}_{\mathsf{push}} x] & \{\mathit{case\ assoc.}\} \\
\equiv & [\![A \oplus B]\!]^{\Sigma}_{\mathsf{push}}\textbf{case}\ z\ \textbf{of}\ [x] \to [[\![A \oplus B]\!]^{\Sigma}_{\mathsf{push}} [x]] & \{\beta\} \\
\equiv & [\![A \oplus B]\!]^{\Sigma}_{\mathsf{push}}\textbf{case}\ z\ \textbf{of}\ [x] \to [\textbf{case}\ [x]\ \textbf{of}\ [\mathsf{inl}\ x] \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x]; [\mathsf{inr}\ y] \to \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[y]] & \{\mathit{defn.}\} \\
\equiv & \begin{array}{l} \textbf{case}\ (\textbf{case}\ z\ \textbf{of}\ [x] \to [\textbf{case}\ [x]\ \textbf{of}\ [\mathsf{inl}\ x] \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x]; [\mathsf{inr}\ y] \to \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[y]])\ \textbf{of} \\ \quad [\mathsf{inl}\ x] \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x] \\ \quad [\mathsf{inr}\ y] \mapsto \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[y] \end{array} & \{\mathit{defn.}\} \\
\equiv & \textbf{case}\ z\ \textbf{of} \begin{array}{l} [x] \mapsto \begin{array}{l} \textbf{case}\ [\textbf{case}\ [x]\ \textbf{of}\ [\mathsf{inl}\ x] \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x]; [\mathsf{inr}\ y] \to \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[y]]\ \textbf{of} \\ \quad [\mathsf{inl}\ x] \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x] \end{array} \\ [\mathsf{inr}\ y] \to \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[y] \end{array} & \{\mathit{case\ assoc.}\} \\
\equiv & \textbf{case}\ z\ \textbf{of} \begin{array}{l} [x] \mapsto \begin{array}{l} \textbf{case}\ (\textbf{case}\ [x]\ \textbf{of}\ [\mathsf{inl}\ x] \mapsto [\mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x]]; [\mathsf{inr}\ y] \to [\mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[y]])\ \textbf{of} \\ \quad [\mathsf{inl}\ x] \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x] \end{array} \\ [\mathsf{inr}\ y] \to \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[y] \end{array} & \{\mathit{case\ push}\} \\
\equiv & \textbf{case}\ z\ \textbf{of} \begin{array}{l} [x] \mapsto \begin{array}{l} \textbf{case}\ [x]\ \textbf{of} \\ \quad [\mathsf{inl}\ x] \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[[\![A]\!]^{\Sigma}_{\mathsf{push}}[x]] \end{array} \\ [\mathsf{inr}\ y] \to \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[[\![B]\!]^{\Sigma}_{\mathsf{push}}[y]] \end{array} & \{\mathit{case\ assoc} + \beta\} \\
\equiv & \begin{array}{l} \textbf{case}\ z\ \textbf{of} \\ \quad [\mathsf{inl}\ x] \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[[\![A]\!]^{\Sigma}_{\mathsf{push}}[x]] \\ \quad [\mathsf{inr}\ y] \mapsto \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[[\![B]\!]^{\Sigma}_{\mathsf{push}}[y]] \end{array} & \{\eta_{case}\}
\end{array}
$$

$$
\begin{array}{lll}
& [\![A \oplus B]\!]^{\Sigma'}_{\mathsf{fmap}}\delta([\![A \oplus B]\!]^{\Sigma}_{\mathsf{push}} z) & \\
\equiv & \textbf{case}\ [\![A \oplus B]\!]^{\Sigma}_{\mathsf{push}} z\ \textbf{of}\ \mathsf{inl}\ x \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(\delta\, x); \mathsf{inr}\ y \to \mathsf{inr}\ [\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(\delta\, y) & \{\mathit{defn.}\} \\
\equiv & \begin{array}{l} \textbf{case}\ (\textbf{case}\ z\ \textbf{of}\ [\mathsf{inl}\ x] \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}} x; [\mathsf{inr}\ y] \to \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}} y)\ \textbf{of} \\ \quad \mathsf{inl}\ x \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(\delta\, x) \\ \quad \mathsf{inr}\ y \mapsto \mathsf{inr}\ [\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(\delta\, y) \end{array} & \{\mathit{defn.}\} \\
\equiv & \begin{array}{l} \textbf{case}\ z\ \textbf{of} \\ \quad [\mathsf{inl}\ x] \mapsto (\begin{array}{l} \textbf{case}\ (\mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}} x)\ \textbf{of} \\ \quad \mathsf{inl}\ x \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(\delta\, x) \\ \quad \mathsf{inr}\ y \mapsto \mathsf{inr}\ [\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(\delta\, y) \end{array}) \\ \quad [\mathsf{inr}\ y] \mapsto (\begin{array}{l} \textbf{case}\ (\mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}} y)\ \textbf{of} \\ \quad \mathsf{inl}\ x \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(\delta\, x) \\ \quad \mathsf{inr}\ y \mapsto \mathsf{inr}\ [\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(\delta\, y) \end{array}) \end{array} & \{\mathit{case\ assoc.}\} \\
\equiv & \begin{array}{l} \textbf{case}\ z\ \textbf{of} \\ \quad [\mathsf{inl}\ x] \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma'}_{\mathsf{fmap}}(\delta\, [\![A]\!]^{\Sigma}_{\mathsf{push}} x) \\ \quad [\mathsf{inr}\ y] \mapsto \mathsf{inr}\ [\![B]\!]^{\Sigma'}_{\mathsf{fmap}}(\delta\, [\![B]\!]^{\Sigma}_{\mathsf{push}} y) \end{array} & \{\beta\} \\
\equiv & \begin{array}{l} \textbf{case}\ z\ \textbf{of} \\ \quad [\mathsf{inl}\ x] \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}(\Box_r [\![A]\!]^{\Sigma}_{\mathsf{push}} \delta\, x) \\ \quad [\mathsf{inr}\ y] \mapsto \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}(\Box_r [\![B]\!]^{\Sigma}_{\mathsf{push}} \delta\, y) \end{array} & \{\mathit{induction}\} \\
\equiv & \begin{array}{l} \textbf{case}\ z\ \textbf{of} \\ \quad [\mathsf{inl}\ x] \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}(\textbf{case}\ (\textbf{case}\ x\ \textbf{of}\ [x] \to [[x]])\ \textbf{of}\ [x] \to [[\![A]\!]^{\Sigma}_{\mathsf{push}} x]) \\ \quad [\mathsf{inr}\ y] \mapsto \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}(\textbf{case}\ (\textbf{case}\ y\ \textbf{of}\ [x] \to [[x]])\ \textbf{of}\ [x] \to [[\![B]\!]^{\Sigma}_{\mathsf{push}} x]) \end{array} & \{\mathit{defns.} + \beta\} \\
\equiv & \begin{array}{l} \textbf{case}\ z\ \textbf{of} \\ \quad [\mathsf{inl}\ x] \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}(\textbf{case}\ x\ \textbf{of}\ [x] \to \textbf{case}\ [[x]]\ \textbf{of}\ [x] \to [[\![A]\!]^{\Sigma}_{\mathsf{push}} x]) \\ \quad [\mathsf{inr}\ y] \mapsto \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}(\textbf{case}\ y\ \textbf{of}\ [x] \to \textbf{case}\ [[x]]\ \textbf{of}\ [x] \to [[\![B]\!]^{\Sigma}_{\mathsf{push}} x]) \end{array} & \{\mathit{case\ assoc.}\} \\
\equiv & \begin{array}{l} \textbf{case}\ z\ \textbf{of} \\ \quad [\mathsf{inl}\ x] \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}(\textbf{case}\ x\ \textbf{of}\ [x] \to [[\![A]\!]^{\Sigma}_{\mathsf{push}}[x]]) \\ \quad [\mathsf{inr}\ y] \mapsto \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}(\textbf{case}\ y\ \textbf{of}\ [x] \to [[\![B]\!]^{\Sigma}_{\mathsf{push}}[x]]) \end{array} & \{\mathit{case\ assoc.}\} \\
\equiv & \begin{array}{l} \textbf{case}\ z\ \textbf{of} \\ \quad [\mathsf{inl}\ x] \mapsto \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[[\![A]\!]^{\Sigma}_{\mathsf{push}}[x]] \\ \quad [\mathsf{inr}\ y] \mapsto \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[[\![B]\!]^{\Sigma}_{\mathsf{push}}[y]] \end{array} & \{\eta_{case}\}
\end{array}
$$

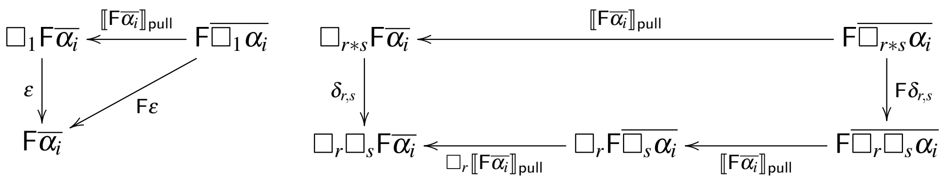
- $T = A \otimes B$

$$
\begin{array}{rll}
& \llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{push}}(\Box_r \llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{push}}(\delta_{r,s}\ z)) & \\
= & \llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{push}}((\lambda y.\mathbf{case}\ y\ \mathbf{of}\ [z] \to [\mathbf{case}\ z\ \mathbf{of}\ [(x,y)] \to (\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y])]) & \\
& \quad \mathbf{case}\ z\ \mathbf{of}\ [z'] \to [[z']]) & \{\mathit{defns.}\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [z'] \to \llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{push}}(\mathbf{case}\ [[z']]\ \mathbf{of}\ [z] \to [\mathbf{case}\ z\ \mathbf{of}\ [(x,y)] \to (\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y])]) & \{\beta\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [z'] \to \llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{push}}([\mathbf{case}\ [z']\ \mathbf{of}\ [(x,y)] \to (\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y])]) & \{\beta\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [z'] \to \mathbf{case}\ ([\mathbf{case}\ [z']\ \mathbf{of}\ [(x,y)] \to & \\
& \quad (\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y])])\ \mathbf{of}\ [(x,y)] \to (\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y]) & \{\mathit{defns.} + \beta\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [z'] \to \mathbf{case}\ (\mathbf{case}\ [z']\ \mathbf{of}\ [(x,y)] \to & \\
& \quad [(\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y])])\ \mathbf{of}\ [(x,y)] \to (\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y]) & \{\mathit{case\ push}\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [z'] \to \mathbf{case}\ [z']\ \mathbf{of}\ [(x,y)] \to \begin{array}{l} \mathbf{case}\ [(\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y])]\ \mathbf{of} \\ \quad [(x,y)] \mapsto (\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y]) \end{array} & \{\mathit{case\ assoc.}\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [(x,y)] \to \mathbf{case}\ [(\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y])]\ \mathbf{of}\ [(x,y)] \to (\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y]) & \{\eta_{case}\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [(x,y)] \to (\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x]], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[\llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y]]) & \{\beta\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [(x,y)] \to (\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}(\Box_r \llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}(\delta[x])), (\llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}(\Box_r \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}(\delta[y])))) & \{\eta\text{-}\mathit{expand} + \mathit{defns.}\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [(x,y)] \to (\llbracket A \rrbracket^{\Sigma'}_{\mathsf{fmap}}\delta(\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x]), \llbracket B \rrbracket^{\Sigma'}_{\mathsf{fmap}}\delta(\llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y])) & \{\mathit{induction}\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [(x,y)] \to \llbracket A \otimes B \rrbracket^{\Sigma'}_{\mathsf{fmap}}\delta(\mathbf{case}\ z\ \mathbf{of}\ [(x,y)] \to (\llbracket A \rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[y])) & \{\mathit{defns.}\} \\
= & \llbracket A \otimes B \rrbracket^{\Sigma'}_{\mathsf{fmap}}\delta(\llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{push}} z) & \{\mathit{defns.}\}
\end{array}
$$

- $T = A \multimap B$ Cannot be handled, as per the restriction of the proposition.

□

Proposition 3.6 (Pull preserves graded comonads)**.** *For all* F *such that* $\llbracket \mathsf{F}\overline{\alpha_i} \rrbracket_{\mathsf{pull}}$ *is defined then:*



Proof. We consider first the property involving ε, by induction on the type $T = \mathsf{F}\overline{\alpha_i}$, for open recursion Σ where we assume that for all $f \in \Sigma(X)$ then $\mathsf{F}\varepsilon = \varepsilon \circ f$ (called, condition (*))

- $T = 1$:

$$
\begin{array}{rll}
& \varepsilon(\llbracket 1 \rrbracket^{\Sigma}_{\mathsf{pull}}\ z) & \\
= & (\lambda x.\mathbf{case}\ x\ \mathbf{of}\ [z] \to z)\ (\mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to [\mathsf{unit}]) & \{\mathit{defns.}\} \\
= & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to [\mathsf{unit}])\ \mathbf{of}\ [z] \to z & \{\beta\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to \mathsf{unit} & \{\eta_{case}\} \\
= & (\llbracket 1 \rrbracket^{\Sigma}_{\mathsf{fmap}}\varepsilon)\ z & \{\mathit{defns.}\}
\end{array}
$$

- $T = \alpha$:

$$
\begin{array}{rll}
& \varepsilon(\llbracket \alpha \rrbracket^{\Sigma}_{\mathsf{pull}} z) & \\
= & (\lambda x.\mathbf{case}\ x\ \mathbf{of}\ [z] \to z)\ (\llbracket \alpha \rrbracket^{\Sigma}_{\mathsf{pull}} z) & \{\mathit{def.}\} \\
= & (\lambda x.\mathbf{case}\ x\ \mathbf{of}\ [z] \to z)\ z & \{\mathit{def.}\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [z'] \to z' & \{\beta\} \\
= & (\llbracket \alpha \rrbracket^{\Sigma}_{\mathsf{fmap}}\varepsilon)\ z & \{\mathit{def.}\}
\end{array}
$$

- $T = X$:

$$
\begin{aligned}
& \varepsilon(\llbracket X \rrbracket^{\Sigma}_{\mathsf{pull}} z) \\
= \;& (\lambda x.\mathbf{case}\ x\ \mathbf{of}\ [z] \to z)(\Sigma(X)\ z) && \{\mathit{defns.}\} \\
= \;& (\llbracket X \rrbracket^{\Sigma}_{\mathsf{fmap}} \varepsilon)\ z && \{\mathit{condition\ (*)}\}
\end{aligned}
$$

- $T = A \oplus B$:

$$
\begin{aligned}
& \varepsilon(\llbracket A \oplus B \rrbracket^{\Sigma}_{\mathsf{pull}} z) \\
= \;& (\lambda x.\mathbf{case}\ x\ \mathbf{of}\ [z] \to z)(\mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l} \mathsf{inl}\ x \to \mathbf{case}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to [\mathsf{inl}\ u]; \\ \mathsf{inr}\ y \to \mathbf{case}\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \to [\mathsf{inr}\ v] \end{array}) && \{\mathit{defns.}\} \\
= \;& \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l} \mathsf{inl}\ x \to \mathbf{case}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to [\mathsf{inl}\ u]; \\ \mathsf{inr}\ y \to \mathbf{case}\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \to [\mathsf{inr}\ v] \end{array})\ \mathbf{of}\ [z] \to z && \{\beta\} \\
= \;& \begin{array}{l} \mathbf{case}\ z\ \mathbf{of} \\ \quad \mathsf{inl}\ x \mapsto \mathbf{case}\ (\mathbf{case}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}} x\ \mathbf{of}\ [u] \to [\mathsf{inl}\ u])\ \mathbf{of}\ [z] \to z \\ \quad \mathsf{inr}\ y \mapsto \mathbf{case}\ (\mathbf{case}\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}} y\ \mathbf{of}\ [v] \to [\mathsf{inr}\ v])\ \mathbf{of}\ [z] \to z \end{array} && \{\mathit{case\ assoc.}\} \\
= \;& \begin{array}{l} \mathbf{case}\ z\ \mathbf{of} \\ \quad \mathsf{inl}\ x \mapsto \mathbf{case}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}} x\ \mathbf{of}\ [u] \to \mathbf{case}\ [\mathsf{inl}\ u]\ \mathbf{of}\ [z] \to z \\ \quad \mathsf{inr}\ y \mapsto \mathbf{case}\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}} y\ \mathbf{of}\ [v] \to \mathbf{case}\ [\mathsf{inr}\ v]\ \mathbf{of}\ [z] \to z \end{array} && \{\mathit{case\ assoc.}\} \\
= \;& \mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l} \mathsf{inl}\ x \to \mathbf{case}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to \mathsf{inl}\ u; \\ \mathsf{inr}\ y \to \mathbf{case}\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \to \mathsf{inr}\ v \end{array} && \{\beta_{case}\} \\
= \;& \mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l} \mathsf{inl}\ x \to \mathsf{inl}\ \varepsilon \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ x; \\ \mathsf{inr}\ y \to \mathsf{inr}\ \varepsilon \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ y \end{array} && \{\beta_{case} + \mathit{defns.}\} \\
= \;& \mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l} \mathsf{inl}\ x \to \mathsf{inl}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{fmap}} \varepsilon)\ x; \\ \mathsf{inr}\ y \to \mathsf{inr}\ (\llbracket B \rrbracket^{\Sigma}_{\mathsf{fmap}} \varepsilon)\ y \end{array} && \{\mathit{induction.}\} \\
= \;& (\llbracket A \oplus B \rrbracket^{\Sigma}_{\mathsf{fmap}} \varepsilon)\ z && \{\mathit{defns.}\}
\end{aligned}
$$

- $T = A \otimes B$:

$$
\begin{aligned}
& \varepsilon(\llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{pull}} z) \\
= \;& (\lambda x.\mathbf{case}\ x\ \mathbf{of}\ [z] \to z)\ (\mathbf{case}\ z\ \mathbf{of}\ (x,y) \to \mathbf{case}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ x, \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ y)\ \mathbf{of}\ ([u],[v]) \to [(u,v)]) && \{\mathit{defns.}\} \\
= \;& \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ (x,y) \to \mathbf{case}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ x, \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ y)\ \mathbf{of}\ ([u],[v]) \to [(u,v)])\ \mathbf{of}\ [z] \to z && \{\beta\} \\
= \;& \mathbf{case}\ z\ \mathbf{of}\ (x,y) \to \mathbf{case}\ (\mathbf{case}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}} x, \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}} y)\ \mathbf{of}\ ([u],[v]) \to [(u,v)])\ \mathbf{of}\ [z] \to z && \{\mathit{case\ assoc.}\} \\
= \;& \mathbf{case}\ z\ \mathbf{of}\ (x,y) \to \mathbf{case}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}} x, \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}} y)\ \mathbf{of}\ ([u],[v]) \to \mathbf{case}\ [(u,v)]\ \mathbf{of}\ [z] \to z && \{\mathit{case\ assoc.}\} \\
= \;& \mathbf{case}\ z\ \mathbf{of}\ (x,y) \to \mathbf{case}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ x, \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ y)\ \mathbf{of}\ ([u],[v]) \to (u,v) && \{\beta_{case}\} \\
= \;& \mathbf{case}\ z\ \mathbf{of}\ (x,y) \to (\varepsilon\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ x), \varepsilon\ (\llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ y)) && \{\mathit{defns.} + \beta_{case}\} \\
= \;& \mathbf{case}\ z\ \mathbf{of}\ (x,y) \to (\llbracket A \rrbracket^{\Sigma}_{\mathsf{fmap}} \varepsilon\ x, \llbracket B \rrbracket^{\Sigma}_{\mathsf{fmap}}\ y) && \{\mathit{induction.}\} \\
= \;& (\llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{fmap}} \varepsilon)\ z && \{\mathit{defns.}\}
\end{aligned}
$$

- $T = \mu X.A$

$$
\begin{aligned}
& \varepsilon(\llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{pull}} z) \\
= \;& (\lambda x.\mathbf{case}\ x\ \mathbf{of}\ [z] \to z)\ (\mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.A[\overrightarrow{\Box_{r_i} \alpha_i / \alpha_i}] \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ f\ z) && \{\mathit{defns.}\} \\
= \;& \mathbf{case}\ (\mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.A[\overrightarrow{\Box_{r_i} \alpha_i / \alpha_i}] \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ f\ z)\ \mathbf{of}\ [z'] \to z' && \{\beta\} \\
= \;& \mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.A[\overrightarrow{\Box_{r_i} \alpha_i / \alpha_i}] \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ \varepsilon(f\ z) && \{\mathit{defn.}\} \\
= \;& \mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.A[\overrightarrow{\Box_{r_i} \alpha_i / \alpha_i}] \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{fmap}} \varepsilon)\ z && \{\mathit{induction}\} \\
= \;& (\llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{fmap}} \varepsilon)\ z && \{\mathit{simplify} + \mathit{defn.}\}
\end{aligned}
$$

where in the binding of X to f we recursively apply the inductive evidence on A itself that it satisfies the property and thus provides that $X \mapsto f$ satisfies the condition of (*) in the induction step.

Thus from the above we have that $\mathsf{F}\varepsilon = \varepsilon \circ [\![\mathsf{F}\overline{\alpha_i}]\!]_{\mathsf{push}}$ by function extensionality and since the initial environemnt $\Sigma = \emptyset$ trivially satisfies condition (*).

Next we consider the second property involving δ, by induction on the type $T = \mathsf{F}\overline{\alpha_i}$, for open recursion Σ where we assume that for all $f \in \Sigma(X)$ then $\Box_r f \circ f \circ \mathsf{F}\delta = \delta \circ f$ (called, condition (**)).

- $T = 1$:

$$
\begin{array}{lll}
& \Box_r [\![1]\!]^{\Sigma}_{\mathsf{pull}}([\![1]\!]^{\Sigma}_{\mathsf{pull}}\ ([\![1]\!]^{\Sigma}_{\mathsf{fmap}}\ \delta_{r,s} z)) & \\
= & (\lambda x'.\mathbf{case}\ x'\ \mathbf{of}\ [x] \to [[\![1]\!]^{\Sigma}_{\mathsf{pull}} x])(([\![1]\!]^{\Sigma}_{\mathsf{pull}}([\![1]\!]^{\Sigma}_{\mathsf{fmap}}\ \lambda z.\mathbf{case}\ z\ \mathbf{of}\ [x] \to [[x]]))) & \{\mathit{defn.}\} \\
= & (\lambda x'.\mathbf{case}\ x'\ \mathbf{of}\ [x] \to [\mathbf{case}\ x\ \mathbf{of}\ \mathsf{unit} \to [\mathsf{unit}]])(\mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to [\mathsf{unit}]) & \{\mathit{defn.} + \beta\} \\
= & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to [\mathsf{unit}])\ \mathbf{of}\ [x] \to [\mathbf{case}\ x\ \mathbf{of}\ \mathsf{unit} \to [\mathsf{unit}]] & \{\beta\} \\
= & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to [\mathsf{unit}])\ \mathbf{of}\ [x] \to [[x]] & \{\eta_{case}\} \\
= & (\lambda z.\mathbf{case}\ z\ \mathbf{of}\ [x] \to [[x]])(\mathbf{case}\ z\ \mathbf{of}\ \mathsf{unit} \to [\mathsf{unit}]) & \{\beta\} \\
= & \delta_{r,s}([\![1]\!]^{\Sigma}_{\mathsf{pull}}\ z) & \{\mathit{defn.}\}
\end{array}
$$

- $T = \alpha$

$$
\begin{array}{lll}
& \Box_r [\![\alpha]\!]^{\Sigma}_{\mathsf{pull}}([\![\alpha]\!]^{\Sigma}_{\mathsf{pull}}\ ([\![\alpha]\!]^{\Sigma}_{\mathsf{fmap}}\ \delta_{r,s} z)) & \\
= & (\lambda x'.\mathbf{case}\ x'\ \mathbf{of}\ [x] \to [[\![\alpha]\!]^{\Sigma}_{\mathsf{pull}} x])(([\![\alpha]\!]^{\Sigma}_{\mathsf{pull}}([\![\alpha]\!]^{\Sigma}_{\mathsf{fmap}}\ \lambda z.\mathbf{case}\ z\ \mathbf{of}\ [x] \to [[x]]))) & \{\mathit{defn.}\} \\
= & (\lambda x'.\mathbf{case}\ x'\ \mathbf{of}\ [x] \to [x])(\mathbf{case}\ z\ \mathbf{of}\ [x] \to [[x]]) & \{\mathit{defn.} + \beta\} \\
= & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ [x] \to [[x]])\ \mathbf{of}\ [x] \to [x] & \{\beta\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [x] \to \mathbf{case}\ [[x]]\ \mathbf{of}\ [x] \to [x] & \{\mathit{case\ assoc.}\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ [x] \to [[x]] & \{\beta_{case}\} \\
= & \delta_{r,s} z & \{\mathit{defn.}\} \\
= & \delta_{r,s}([\![\alpha]\!]^{\Sigma}_{\mathsf{pull}}\ z) & \{\mathit{defn.}\}
\end{array}
$$

- $T = X$

$$
\begin{array}{lll}
& \Box_r [\![X]\!]^{\Sigma}_{\mathsf{pull}}([\![X]\!]^{\Sigma}_{\mathsf{pull}}\ ([\![X]\!]^{\Sigma}_{\mathsf{fmap}}\ \delta_{r,s} z)) & \\
= & (\lambda x'.\mathbf{case}\ x'\ \mathbf{of}\ [x] \to [[\![X]\!]^{\Sigma}_{\mathsf{pull}} x])(([\![X]\!]^{\Sigma}_{\mathsf{pull}}([\![X]\!]^{\Sigma}_{\mathsf{fmap}}\ \lambda z.\mathbf{case}\ z\ \mathbf{of}\ [x] \to [[x]]))) & \{\mathit{defn.}\} \\
= & (\lambda x'.\mathbf{case}\ x'\ \mathbf{of}\ [x] \to [\Sigma(X)\ x])((\Sigma(X)\ ([\![X]\!]^{\Sigma}_{\mathsf{fmap}}\ \lambda z.\mathbf{case}\ z\ \mathbf{of}\ [x] \to [[x]]))) & \{\mathit{defn.}\} \\
= & \delta_{r,s}\ (\Sigma(X)\ z) & \{\mathit{condition\ (**)}\} \\
= & \delta_{r,s}([\![X]\!]^{\Sigma}_{\mathsf{pull}}\ z) & \{\mathit{defn.}\}
\end{array}
$$

- $T = A \oplus B$

$$
\begin{array}{ll}
& \Box_r[\![A\oplus B]\!]^{\Sigma}_{\mathsf{pull}}([\![A\oplus B]\!]^{\Sigma}_{\mathsf{pull}}\ ([\![A\oplus B]\!]^{\Sigma}_{\mathsf{fmap}}\ \delta_{r,s} z)) \\
= & (\lambda x'.\mathbf{case}\ x'\ \mathbf{of}\ [x] \rightarrow [[\![A\oplus B]\!]^{\Sigma}_{\mathsf{pull}} x])(([\![A\oplus B]\!]^{\Sigma}_{\mathsf{pull}}([\![A\oplus B]\!]^{\Sigma}_{\mathsf{fmap}}\ (\lambda z.\mathbf{case}\ z\ \mathbf{of}\ [x] \rightarrow [[x]])\ z))) \quad \{\mathit{defn.}\} \\
= & (\lambda x'.\mathbf{case}\ x'\ \mathbf{of}\ [x] \rightarrow [[\![A\oplus B]\!]^{\Sigma}_{\mathsf{pull}} x])((\\
& \quad \mathbf{case}\ ([\![A\oplus B]\!]^{\Sigma}_{\mathsf{fmap}}\ (\lambda z.\mathbf{case}\ z\ \mathbf{of}\ [x] \rightarrow [[x]])\ z)\ \mathbf{of}\ \begin{array}{l}\mathsf{inl}\ x \rightarrow \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \rightarrow [\mathsf{inl}\ u];\ )) \\ \mathsf{inr}\ y \rightarrow \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \rightarrow [\mathsf{inr}\ v]\end{array} \quad \{\mathit{defn.}\} \\
= & (\lambda x'.\mathbf{case}\ x'\ \mathbf{of}\ [x] \rightarrow [[\![A\oplus B]\!]^{\Sigma}_{\mathsf{pull}} x])((\\
& \quad \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l}\mathsf{inl} x \rightarrow \mathsf{inl}(([\![A]\!]^{\Sigma}_{\mathsf{fmap}}\ \delta)x);\ ) \\ \mathsf{inr} y \rightarrow \mathsf{inr}(([\![B]\!]^{\Sigma}_{\mathsf{fmap}}\ \delta)y)\end{array}\ \mathbf{of}\ \begin{array}{l}\mathsf{inl}\ x \rightarrow \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \rightarrow [\mathsf{inl}\ u];\ )) \\ \mathsf{inr}\ y \rightarrow \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \rightarrow [\mathsf{inr}\ v]\end{array} \quad \{\mathit{defn.}\} \\
= & (\lambda x'.\mathbf{case}\ x'\ \mathbf{of}\ [x] \rightarrow [[\![A\oplus B]\!]^{\Sigma}_{\mathsf{pull}} x]) \\
& (\mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l}\mathsf{inl} x \rightarrow \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ (([\![A]\!]^{\Sigma}_{\mathsf{fmap}}\ \delta)x)\ \mathbf{of}\ [u] \rightarrow [\mathsf{inl}\ u];\ ) \\ \mathsf{inr}\ y \rightarrow \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ (([\![B]\!]^{\Sigma}_{\mathsf{fmap}}\ \delta)y)\ \mathbf{of}\ [v] \rightarrow [\mathsf{inr}\ v]\end{array} \quad \{\mathit{case\ assoc.} + \beta\} \\
= & (\lambda x'.\mathbf{case}\ x'\ \mathbf{of}\ [x] \rightarrow [\mathbf{case}\ x\ \mathbf{of}\ \begin{array}{l}\mathsf{inl}\ x \rightarrow \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \rightarrow [\mathsf{inl}\ u];\ ]) \\ \mathsf{inr}\ y \rightarrow \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \rightarrow [\mathsf{inr}\ v]\end{array} \\
& (\mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l}\mathsf{inl} x \rightarrow \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ (([\![A]\!]^{\Sigma}_{\mathsf{fmap}}\ \delta)x)\ \mathbf{of}\ [u] \rightarrow [\mathsf{inl}\ u];\ ) \\ \mathsf{inr}\ y \rightarrow \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ (([\![B]\!]^{\Sigma}_{\mathsf{fmap}}\ \delta)y)\ \mathbf{of}\ [v] \rightarrow [\mathsf{inr}\ v]\end{array} \quad \{\mathit{defn}\} \\
= & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l}\mathsf{inl} x \rightarrow \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ (([\![A]\!]^{\Sigma}_{\mathsf{fmap}}\ \delta)x)\ \mathbf{of}\ [u] \rightarrow [\mathsf{inl}\ u];\ )\ \mathbf{of} \\ \mathsf{inr}\ y \rightarrow \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ (([\![B]\!]^{\Sigma}_{\mathsf{fmap}}\ \delta)y)\ \mathbf{of}\ [v] \rightarrow [\mathsf{inr}\ v]\end{array} \\
& \qquad [x] \rightarrow [\mathbf{case}\ x\ \mathbf{of}\ \begin{array}{l}\mathsf{inl}\ x \rightarrow \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \rightarrow [\mathsf{inl}\ u];\ ] \\ \mathsf{inr}\ y \rightarrow \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \rightarrow [\mathsf{inr}\ v]\end{array} \quad \{\beta\}
\end{array}
$$

$$
\begin{array}{ll}
& \mathbf{case}\ z\ \mathbf{of} \\
= & \begin{array}{l}
\mathsf{inl}\ x \mapsto \mathbf{case}\ (\begin{array}{l}\mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}([\![A]\!]^{\Sigma'}_{\mathsf{fmap}}\delta)\,x\ \mathbf{of} \\ [u] \mapsto [\mathsf{inl}\ u]\end{array})\ \mathbf{of}\ [x] \rightarrow [\begin{array}{l}\mathbf{case}\ x\ \mathbf{of} \\ \mathsf{inl}\ x \mapsto \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}} x\ \mathbf{of}\ [u] \rightarrow [\mathsf{inl}\ u] \\ \mathsf{inr}\ y \mapsto \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}} y\ \mathbf{of}\ [v] \rightarrow [\mathsf{inr}\ v]\end{array}] \\
\mathsf{inr}\ y \mapsto \mathbf{case}\ (\begin{array}{l}\mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}([\![B]\!]^{\Sigma'}_{\mathsf{fmap}}\delta)\,y\ \mathbf{of} \\ [v] \mapsto [\mathsf{inr}\ u]\end{array})\ \mathbf{of}\ [x] \rightarrow [\begin{array}{l}\mathbf{case}\ x\ \mathbf{of} \\ \mathsf{inl}\ x \mapsto \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}} x\ \mathbf{of}\ [u] \rightarrow [\mathsf{inl}\ u] \\ \mathsf{inr}\ y \mapsto \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}} y\ \mathbf{of}\ [v] \rightarrow [\mathsf{inr}\ v]\end{array}]
\end{array} \quad \{\mathit{case\ assoc.}\}
\end{array}
$$

$$
\begin{array}{ll}
= & \mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l}\mathsf{inl} x \rightarrow \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ (([\![A]\!]^{\Sigma}_{\mathsf{fmap}}\ \delta)x)\ \mathbf{of}\ [u] \rightarrow [\mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ u\ \mathbf{of}\ [u'] \rightarrow [\mathsf{inl}\ u']]; \\ \mathsf{inr}\ y \rightarrow \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ (([\![B]\!]^{\Sigma}_{\mathsf{fmap}}\ \delta)y)\ \mathbf{of}\ [v] \rightarrow [\mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ v\ \mathbf{of}\ [v'] \rightarrow [\mathsf{inl}\ v']]\end{array} \quad \{\mathit{case\ assoc.} + \beta\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l}\mathsf{inl} x \rightarrow \mathbf{case}\ \Box_r[\![A]\!]^{\Sigma}_{\mathsf{pull}}([\![A]\!]^{\Sigma}_{\mathsf{pull}}\ (([\![A]\!]^{\Sigma}_{\mathsf{fmap}}\ \delta)x))\ \mathbf{of}\ [[x]] \rightarrow [[\mathsf{inl}\ x]]; \\ \mathsf{inr}\ y \rightarrow \mathbf{case}\ \Box_r[\![B]\!]^{\Sigma}_{\mathsf{pull}}([\![B]\!]^{\Sigma}_{\mathsf{pull}}\ (([\![B]\!]^{\Sigma}_{\mathsf{fmap}}\delta)y))\ \mathbf{of}\ [[x]] \rightarrow [[\mathsf{inr}\ x]]\end{array} \quad \{\mathit{defn.} + \beta\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l}\mathsf{inl} x \rightarrow \mathbf{case}\ \delta_{r,s}([\![A]\!]^{\Sigma}_{\mathsf{pull}} x)\ \mathbf{of}\ [[x]] \rightarrow [[\mathsf{inl}\ x]]; \\ \mathsf{inr}\ y \rightarrow \mathbf{case}\ \delta_{r,s}([\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y)\ \mathbf{of}\ [[x]] \rightarrow [[\mathsf{inr}\ x]]\end{array} \quad \{\mathit{induction}\} \\
= & \mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l}\mathsf{inl} x \rightarrow \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}} x\ \mathbf{of}\ [x] \rightarrow \delta_{r,s}[\mathsf{inl}\ x]; \\ \mathsf{inr}\ y \rightarrow \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y\ \mathbf{of}\ [x] \rightarrow \delta_{r,s}[\mathsf{inr}\ x]\end{array} \quad \{\eta + \beta\} \\
= & \delta_{r,s}(\mathbf{case}\ z\ \mathbf{of}\ \begin{array}{l}\mathsf{inl} x \rightarrow \mathbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}} x\ \mathbf{of}\ [x] \rightarrow [\mathsf{inl}\ x];\ \ ) \\ \mathsf{inr}\ y \rightarrow \mathbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y\ \mathbf{of}\ [x] \rightarrow [\mathsf{inr}\ x]\end{array} \quad \{\mathit{case\ distrib.}\} \\
= & \delta_{r,s}([\![A\oplus B]\!]^{\Sigma}_{\mathsf{pull}}\ z) \quad \{\mathit{defn.}\}
\end{array}
$$

- $T = A \otimes B$

$$
\begin{aligned}
& \Box_r \llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{pull}} (\llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{pull}}\ (\llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{fmap}}\ \delta_{r,s} z)) \\
= \;& (\lambda x'.\mathbf{case}\ x'\ \mathbf{of}\ [x] \to [\llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{pull}} x])((\llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{pull}} (\llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{fmap}}\ (\lambda z.\mathbf{case}\ z\ \mathbf{of}\ [x] \to [[x]])\ z))) && \{\mathit{defn.}\} \\
= \;& (\lambda x'.\mathbf{case}\ x'\ \mathbf{of}\ [x] \to [\llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{pull}} x])(\llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{pull}} (\mathbf{case}\ z\ \mathbf{of}\ (x',y') \to (\llbracket A \rrbracket^{\Sigma}_{\mathsf{fmap}}\ \delta x', \llbracket B \rrbracket^{\Sigma}_{\mathsf{fmap}}\ \delta\ y'))) && \{\mathit{defn.} + \beta\} \\
= \;& (\lambda x'.\mathbf{case}\ x'\ \mathbf{of}\ [x] \to [\llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{pull}} x])(\\
& \qquad \mathbf{case}\ z\ \mathbf{of}\ (x,y) \to \mathbf{case}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{fmap}}\ \delta x'), \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ (\llbracket B \rrbracket^{\Sigma}_{\mathsf{fmap}}\ \delta\ y'))\ \mathbf{of}\ ([u],[v]) \to [(u,v)]) && \{\mathit{defn.} + \beta\} \\
= \;& \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ (x,y) \to \\
& \qquad \mathbf{case}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{fmap}}\ \delta x), \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ (\llbracket B \rrbracket^{\Sigma}_{\mathsf{fmap}}\ \delta\ y))\ \mathbf{of}\ ([u],[v]) \to [(u,v)])\ \mathbf{of}\ [x] \to [\llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{pull}} x] && \{\beta\} \\
= \;& \mathbf{case}\ z\ \mathbf{of}\ (x,y) \to \mathbf{case}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{fmap}}\ \delta x), \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ (\llbracket B \rrbracket^{\Sigma}_{\mathsf{fmap}}\ \delta\ y))\ \mathbf{of}\ ([u],[v]) \to [\llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{pull}} (u,v)] && \{\beta + \mathit{case\ assoc.}\} \\
= \;& \mathbf{case}\ z\ \mathbf{of}\ (x,y) \to \mathbf{case}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{fmap}}\ \delta x), \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ (\llbracket B \rrbracket^{\Sigma}_{\mathsf{fmap}}\ \delta\ y))\ \mathbf{of}\ ([u],[v]) \to \\
& \qquad\qquad [\mathbf{case}\ (u,v)\ \mathbf{of}\ (x,y) \to \mathbf{case}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ x, \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ y)\ \mathbf{of}\ ([u],[v]) \to [(u,v)]] && \{\mathit{defn.} + \beta\} \\
= \;& \mathbf{case}\ z\ \mathbf{of}\ (x,y) \to \mathbf{case}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{fmap}}\ \delta x), \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ (\llbracket B \rrbracket^{\Sigma}_{\mathsf{fmap}}\ \delta\ y))\ \mathbf{of}\ ([u],[v]) \to \\
& \qquad\qquad [\mathbf{case}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ u, \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ v)\ \mathbf{of}\ ([u],[v]) \to [(u,v)]] && \{\beta\} \\
= \;& \mathbf{case}\ z\ \mathbf{of}\ (x,y) \to \mathbf{case}\ (\Box_r \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}} (\llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{fmap}}\ \delta x)), \Box_r \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}} (\llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ (\llbracket B \rrbracket^{\Sigma}_{\mathsf{fmap}}\ \delta\ y)))\ \mathbf{of} \\
& \qquad\qquad ([[u]],[[v]]) \to [[(u,v)]] && \{\eta + \mathit{case\ assoc}\} \\
= \;& \mathbf{case}\ z\ \mathbf{of}\ (x,y) \to \mathbf{case}\ (\delta_{r,s}(\llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}} x), \delta_{r,s}(\llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ y))\ \mathbf{of}\ ([[u]],[[v]]) \to [[(u,v)]] && \{\mathit{induction}\} \\
= \;& \delta_{r,s}(\mathbf{case}\ z\ \mathbf{of}\ (x,y) \to \mathbf{case}\ (\llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}} x, \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}\ y)\ \mathbf{of}\ ([u],[v]) \to [(u,v)]) && \{\beta\eta + \mathit{defn.}\} \\
= \;& \delta_{r,s}(\llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{pull}}\ z) && \{\mathit{defn.}\}
\end{aligned}
$$

- $T = \mu X.A$

$$
\begin{aligned}
& \Box_r \llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{pull}} (\llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{pull}}\ (\llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{fmap}} \delta_{r,s} z)) \\
= \;& (\lambda x'.\mathbf{case}\ x'\ \mathbf{of}\ [x] \to [\mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ f\ x]) \\
& \qquad\qquad ((\mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ f\ ((\llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{fmap}}\ \delta\, z)))) && \{\mathit{defn.}\} \\
= \;& \mathbf{case}\ (\mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ f\ ((\llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{fmap}}\ \delta\, z)))\ \mathbf{of}\ [x] \to \\
& \qquad\qquad [\mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ f\ x] && \{\beta\} \\
= \;& \mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ \delta_{r,s}\ (f\ z) && \{\mathit{induction}\ (+\ \beta)\} \\
= \;& \delta_{r,s}(\llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{pull}}\ z) && \{\mathit{defn.}\}
\end{aligned}
$$

where in the binding of X to f we recursively apply the inductive evidence on A itself that it satisfies the property and thus provides that $X \mapsto f$ satisfies the condition of (**) in the induction step.

Thus from the above we have that the desired property by function extensionality and since the initial environemnt $\Sigma = \emptyset$ trivially satisfies condition (**). □