Disproof of the Riemann Hypothesis

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Abstract

We define the function $\upsilon(x) = \frac{3\times\log x+5}{8\times\pi\times\sqrt{x}+1.2\times\log x+2} + \frac{\log x}{\log(x+C\times\sqrt{x}\times\log\log\log x)} - 1$ for some positive constant C independent of x. We prove that the Riemann hypothesis is false when there exists some number $y \geq 13.1$ such that for all $x \geq y$ the inequality $\upsilon(x) \leq 0$ is always satisfied. We know that the function $\upsilon(x)$ is monotonically decreasing for all sufficiently large numbers $x \geq 13.1$. Hence, it is enough to find a value of $y \geq 13.1$ such that $\upsilon(y) \leq 0$ since for all $x \geq y$ we would have that $\upsilon(x) \leq \upsilon(y) \leq 0$. That value of $y \geq 13.1$ exists since we know that $\lim_{x\to\infty} \upsilon(x) = -1$ for all positive value of C and $\upsilon(x)$ is monotonically decreasing. In this way, the Riemann hypothesis is false.

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1. Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. Let $N_n = 2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times p_n$ denotes a primorial number of order n such that p_n is the n^{th} prime number. Say $\mathsf{Nicolas}(p_n)$ holds provided

$$\prod_{q \mid N_n} \frac{q}{q-1} > e^{\gamma} \times \log\log N_n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, log is the natural logarithm, and $q \mid N_n$ means the prime number q divides to N_n. The importance of this property is:

Theorem 1.1. *[2], [3].* $\mathsf{Nicolas}(p_n)$ *holds for all prime numbers* $p_n > 2$ *if and only if the Riemann hypothesis is true.*

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where $p \leq x$ means all the prime numbers p that are less than or equal to x. We know this property for this function:

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Theorem 1.2. *[4]. There are infinitely many values of x such that*

$$\theta(x) > x + C \times \sqrt{x} \times \log\log\log x$$

for some positive constant C independent of x.

We also know that

Theorem 1.3. *[5]. If the Riemann hypothesis holds, then*

$$\left(\frac{e^{-\gamma}}{\log x} \times \prod_{q \leq x} \frac{q}{q-1} - 1 \right) < \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}$$

for all numbers $x \geq 13.1$.

Let's define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant [6]. We know from the constant H, the following formula:

Theorem 1.4. *[7].*

$$\sum_{q} \left(\log(\frac{q}{q-1}) - \frac{1}{q} \right) = \gamma - B = H.$$

For $x \geq 2$, the function $u(x)$ is defined as follows

$$u(x) = \sum_{q > x} \left(\log(\frac{q}{q-1}) - \frac{1}{q} \right).$$

We use the following theorems:

Theorem 1.5. *[8]. For* $x > -1$*:*

$$\frac{x}{x+1} \leq \log(1 + x).$$

Theorem 1.6. *[9]. For* $x \geq 1$*:*

$$\log(1 + \frac{1}{x}) < \frac{1}{x + 0.4}.$$

Let's define:

$$\delta(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log\log x - B \right).$$

Definition 1.7. *We define another function:*

$$\varpi(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log\log \theta(x) - B \right).$$

Putting all together yields the proof that the inequality $\varpi(x) > u(x)$ is satisfied for a number $x \geq 3$ if and only if Nicolas(p) holds, where p is the greatest prime number such that $p \leq x$. In this way, we introduce another criterion for the Riemann hypothesis based on the Nicolas criterion. Using this new criterion, we claim that the Riemann hypothesis is false.

2. Results

Theorem 2.1. *The inequality $\varpi(x) > u(x)$ is satisfied for a number $x \geq 3$ if and only if* $\mathsf{Nicolas}(p)$ *holds, where p is the greatest prime number such that $p \leq x$.*

Proof. We start from the inequality:

$$\varpi(x) > u(x)$$

which is equivalent to

$$\left(\sum_{q \leq x} \frac{1}{q} - \log\log\theta(x) - B\right) > \sum_{q > x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right).$$

Let's add the following formula to the both sides of the inequality,

$$\sum_{q \leq x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right)$$

and due to the theorem 1.4, we obtain that

$$\sum_{q \leq x} \log(\frac{q}{q-1}) - \log\log\theta(x) - B > H$$

because of

$$H = \sum_{q \leq x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right) + \sum_{q > x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right)$$

and

$$\sum_{q \leq x} \log(\frac{q}{q-1}) = \sum_{q \leq x} \frac{1}{q} + \sum_{q \leq x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right).$$

Let's distribute it and remove B from the both sides:

$$\sum_{q \leq x} \log(\frac{q}{q-1}) > \gamma + \log\log\theta(x)$$

since $H = \gamma - B$. If we apply the exponentiation to the both sides of the inequality, then we have that

$$\prod_{q \leq x} \frac{q}{q-1} > e^{\gamma} \times \log\theta(x)$$

which means that $\mathsf{Nicolas}(p)$ holds, where p is the greatest prime number such that $p \leq x$. The same happens in the reverse implication. □

Theorem 2.2. *The Riemann hypothesis is true if and only if the inequality $\varpi(x) > u(x)$ is satisfied for all numbers $x \geq 3$.*

Proof. This is a direct consequence of theorems 1.1 and 2.1. □

Theorem 2.3. *If the Riemann hypothesis holds, then*

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log\theta(x)} > 1$$

for all numbers $x \geq 13.1$.

Proof. Under the assumption that the Riemann hypothesis is true, then we would have

$$\prod_{q\leq x}\frac{q}{q-1} < e^{\gamma}\times\log x\times\left(1+\frac{3\times\log x+5}{8\times\pi\times\sqrt{x}}\right)$$

after of distributing the terms based on the theorem 1.3 for all numbers $x \geq 13.1$. If we apply the logarithm to the both sides of the previous inequality, then we obtain that

$$\sum_{q\leq x}\log(\frac{q}{q-1}) < \gamma+\log\log x+\log\left(1+\frac{3\times\log x+5}{8\times\pi\times\sqrt{x}}\right).$$

That would be equivalent to

$$\sum_{q\leq x}\frac{1}{q}+\sum_{q\leq x}\left(\log(\frac{q}{q-1})-\frac{1}{q}\right) < \gamma+\log\log x+\frac{3\times\log x+5}{8\times\pi\times\sqrt{x}+1.2\times\log x+2}$$

where we know that

$$\begin{aligned}\log\left(1+\frac{3\times\log x+5}{8\times\pi\times\sqrt{x}}\right) &< \frac{1}{\frac{8\times\pi\times\sqrt{x}}{3\times\log x+5}+0.4}\\ &= \frac{3\times\log x+5}{8\times\pi\times\sqrt{x}+0.4\times(3\times\log x+5)}\\ &= \frac{3\times\log x+5}{8\times\pi\times\sqrt{x}+1.2\times\log x+2}\end{aligned}$$

according to theorem 1.6 since $\frac{8\times\pi\times\sqrt{x}}{3\times\log x+5} \geq 1$ for all numbers $x \geq 13.1$. We use the theorem 1.4 to show that

$$\sum_{q\leq x}\left(\log(\frac{q}{q-1})-\frac{1}{q}\right) = H-u(x)$$

and $\gamma = H + B$. So,

$$H-u(x) < H+B+\log\log x-\sum_{q\leq x}\frac{1}{q}+\frac{3\times\log x+5}{8\times\pi\times\sqrt{x}+1.2\times\log x+2}$$

which is the same as

$$H-u(x) < H-\delta(x)+\frac{3\times\log x+5}{8\times\pi\times\sqrt{x}+1.2\times\log x+2}.$$

We eliminate the value of H and thus,

$$-u(x) < -\delta(x)+\frac{3\times\log x+5}{8\times\pi\times\sqrt{x}+1.2\times\log x+2}$$

which is equal to

$$u(x)+\frac{3\times\log x+5}{8\times\pi\times\sqrt{x}+1.2\times\log x+2} > \delta(x).$$

We know from the theorem 2.1 that $\varpi(x) > u(x)$ for all numbers $x \geq 13.1$ and therefore,

$$\varpi(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).$$

Hence,

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \log \log \theta(x) - \log \log x.$$

Suppose that $\theta(x) = \epsilon \times x$ for some constant $\epsilon > 1$. Then,

$$\begin{aligned}
\log \log \theta(x) - \log \log x &= \log \log(\epsilon \times x) - \log \log x \\
&= \log(\log x + \log \epsilon) - \log \log x \\
&= \log\left(\log x \times (1 + \frac{\log \epsilon}{\log x})\right) - \log \log x \\
&= \log \log x + \log(1 + \frac{\log \epsilon}{\log x}) - \log \log x \\
&= \log(1 + \frac{\log \epsilon}{\log x}).
\end{aligned}$$

In addition, we know that

$$\log(1 + \frac{\log \epsilon}{\log x}) \geq \frac{\log \epsilon}{\log \theta(x)}$$

using the theorem 1.5 since $\frac{\log \epsilon}{\log x} > -1$ when $\epsilon > 1$. Certainly, we will have that

$$\log(1 + \frac{\log \epsilon}{\log x}) \geq \frac{\frac{\log \epsilon}{\log x}}{\frac{\log \epsilon}{\log x} + 1} = \frac{\log \epsilon}{\log \epsilon + \log x} = \frac{\log \epsilon}{\log \theta(x)}.$$

Thus,

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \frac{\log \epsilon}{\log \theta(x)}.$$

If we add the following value of $\frac{\log x}{\log \theta(x)}$ to the both sides of the inequality, then

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} > \frac{\log \epsilon}{\log \theta(x)} + \frac{\log x}{\log \theta(x)} = \frac{\log \epsilon + \log x}{\log \theta(x)} = \frac{\log \theta(x)}{\log \theta(x)} = 1.$$

We know this inequality is satisfied when $0 < \epsilon \leq 1$ since we would obtain that $\frac{\log x}{\log \theta(x)} \geq 1$. Therefore, the proof is done. □

Theorem 2.4. *If there exists some number $y \geq 13.1$ such that for all $x \geq y$ the inequality $\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log \log x)} \leq 1$ is satisfied for some positive constant C independent of x, then the Riemann hypothesis should be false.*

Proof. From the theorem 1.2, we know that there are infinitely many values of x such that

$$\theta(x) > x + C \times \sqrt{x} \times \log \log \log x$$

for some positive constant C independent of x. That would be equivalent to

$$\log \theta(x) > \log(x + C \times \sqrt{x} \times \log \log \log x)$$

and so,

$$\frac{1}{\log \theta(x)} < \frac{1}{\log(x + C \times \sqrt{x} \times \log \log \log x)}$$

for all numbers $x \geq 13.1$. Hence,

$$\frac{\log x}{\log \theta(x)} < \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log \log x)}.$$

If the Riemann hypothesis holds, then

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log \log x)} > 1$$

for those values of x that complies with

$$\theta(x) > x + C \times \sqrt{x} \times \log \log \log x$$

due to the theorem 2.3. By contraposition, if there exists some number $y \geq 13.1$ such that for all $x \geq y$ the inequality

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log \log x)} \leq 1$$

is satisfied for some positive constant C independent of x, then the Riemann hypothesis should be false, because of there are infinitely many values of x which satisfy the inequality in the theorem 1.2 and comply with $x \geq y$ no matter how big could be y. □

Definition 2.5. *Let's define the function* $\upsilon(x) = \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log \log x)} - 1$ *for some positive constant C independent of x.*

Theorem 2.6. *The Riemann hypothesis is false.*

Proof. From the theorem 2.4, we know that the Riemann hypothesis is false when there exists some number $y \geq 13.1$ such that for all $x \geq y$ the inequality $\upsilon(x) \leq 0$ is always satisfied. We know that the function $\upsilon(x)$ is monotonically decreasing for all sufficiently large numbers $x \geq 13.1$. Let $\upsilon'(x)$ be the derivative of $\upsilon(x)$. We can check the value of $\upsilon'(x)$ is lesser than zero for all sufficiently large numbers $x \geq 13.1$ and all positive value of C. Indeed, a function $\upsilon(x)$ of a real variable x is monotonically decreasing in some interval if the derivative of $\upsilon(x)$ is lesser than zero and the function $\upsilon(x)$ is continuous over that interval [10]. In this way, it is enough to find a value of $y \geq 13.1$ such that $\upsilon(y) \leq 0$ since for all $x \geq y$ we would have that $\upsilon(x) \leq \upsilon(y) \leq 0$. That value of $y \geq 13.1$ exists since we know that $\lim_{x \to \infty} \upsilon(x) = -1$ for all positive value of C and $\upsilon(x)$ is monotonically decreasing. Certainly, we have that

$$\lim_{x \to \infty} \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} = 0.$$

Moreover, if we define $f(x) = \log x$ and $g(x) = \log(x + C \times \sqrt{x} \times \log \log \log x)$, then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = 0$$

for all positive value of C by the L'Hospital's rule since $f(x)$ and $g(x)$ are differentiable and continuous for $x \geq 13.1$. Therefore, the proof is done. □

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