# Disproof of the Riemann Hypothesis

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**Abstract**

We define the function $\upsilon(x) = \frac{3\times\log x+5}{8\times\pi\times\sqrt{x}+1.2\times\log x+2} + \frac{\log x}{\log(x+C\times\sqrt{x}\times\log\log\log x)} - 1$ for some positive constant $C$ independent of $x$. We prove that the Riemann hypothesis is false when there exists some number $y \geq 13.1$ such that for all $x \geq y$ the inequality $\upsilon(x) \leq 0$ is always satisfied. We know that the function $\upsilon(x)$ is monotonically decreasing for all sufficiently large numbers $x \geq 13.1$. Hence, it is enough to find a value of $y \geq 13.1$ such that $\upsilon(y) \leq 0$ since for all $x \geq y$ we would have that $\upsilon(x) \leq \upsilon(y) \leq 0$. Using the tool *gp* from the project PARI/GP, we found the first zero $y$ of the function $\upsilon(y)$ in $y \approx 8.2639316883312400623766461031726662911\ E5565708$ for $C \geq 1$. In this way, we claim that the Riemann hypothesis could be false.

*Keywords:* Riemann hypothesis, Nicolas inequality, Chebyshev function, prime numbers
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## 1. Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. Let $N_n = 2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times p_n$ denotes a primorial number of order $n$ such that $p_n$ is the $n^{th}$ prime number. Say $\mathsf{Nicolas}(p_n)$ holds provided

$$\prod_{q \mid N_n} \frac{q}{q-1} > e^{\gamma} \times \log\log N_n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, log is the natural logarithm, and $q \mid N_n$ means the prime number $q$ divides to $N_n$. The importance of this property is:

**Theorem 1.1.** *[2], [3].* $\mathsf{Nicolas}(p_n)$ *holds for all prime numbers* $p_n > 2$ *if and only if the Riemann hypothesis is true.*

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where $p \leq x$ means all the prime numbers $p$ that are less than or equal to $x$. We know this property for this function:

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**Theorem 1.2.** *[4]. There are infinitely many values of x such that*

$$\theta(x) > x + C \times \sqrt{x} \times \log \log \log x$$

*for some positive constant C independent of x.*

We also know that

**Theorem 1.3.** *[5]. If the Riemann hypothesis holds, then*

$$\left( \frac{e^{-\gamma}}{\log x} \times \prod_{q \leq x} \frac{q}{q-1} - 1 \right) < \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}$$

*for all numbers* $x \geq 13.1$.

Let's define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant [6]. We know from the constant $H$, the following formula:

**Theorem 1.4.** *[7].*

$$\sum_{q} \left( \log(\frac{q}{q-1}) - \frac{1}{q} \right) = \gamma - B = H.$$

For $x \geq 2$, the function $u(x)$ is defined as follows

$$u(x) = \sum_{q > x} \left( \log(\frac{q}{q-1}) - \frac{1}{q} \right).$$

We use the following theorems:

**Theorem 1.5.** *[8]. For* $x > -1$*:*

$$\frac{x}{x+1} \leq \log(1 + x).$$

**Theorem 1.6.** *[9]. For* $x \geq 1$*:*

$$\log(1 + \frac{1}{x}) < \frac{1}{x + 0.4}.$$

Let's define:

$$\delta(x) = \left( \sum_{q \leq x} \frac{1}{q} - \log \log x - B \right).$$

**Definition 1.7.** *We define another function:*

$$\varpi(x) = \left( \sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B \right).$$

Putting all together yields the proof that the inequality $\varpi(x) > u(x)$ is satisfied for a number $x \geq 3$ if and only if $\mathsf{Nicolas}(p)$ holds, where $p$ is the greatest prime number such that $p \leq x$. In this way, we introduce another criterion for the Riemann hypothesis based on the Nicolas criterion. Using this new criterion, we claim that the Riemann hypothesis could be false.

## 2. Results

**Theorem 2.1.** *The inequality $\varpi(x) > u(x)$ is satisfied for a number $x \geq 3$ if and only if* $\mathsf{Nicolas}(p)$ *holds, where $p$ is the greatest prime number such that $p \leq x$.*

*Proof.* We start from the inequality:

$$\varpi(x) > u(x)$$

which is equivalent to

$$\left(\sum_{q \leq x} \frac{1}{q} - \log\log\theta(x) - B\right) > \sum_{q > x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right).$$

Let's add the following formula to the both sides of the inequality,

$$\sum_{q \leq x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right)$$

and due to the theorem 1.4, we obtain that

$$\sum_{q \leq x} \log(\frac{q}{q-1}) - \log\log\theta(x) - B > H$$

because of

$$H = \sum_{q \leq x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right) + \sum_{q > x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right)$$

and

$$\sum_{q \leq x} \log(\frac{q}{q-1}) = \sum_{q \leq x} \frac{1}{q} + \sum_{q \leq x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right).$$

Let's distribute it and remove $B$ from the both sides:

$$\sum_{q \leq x} \log(\frac{q}{q-1}) > \gamma + \log\log\theta(x)$$

since $H = \gamma - B$. If we apply the exponentiation to the both sides of the inequality, then we have that

$$\prod_{q \leq x} \frac{q}{q-1} > e^{\gamma} \times \log\theta(x)$$

which means that $\mathsf{Nicolas}(p)$ holds, where $p$ is the greatest prime number such that $p \leq x$. The same happens in the reverse implication. □

**Theorem 2.2.** *The Riemann hypothesis is true if and only if the inequality $\varpi(x) > u(x)$ is satisfied for all numbers $x \geq 3$.*

*Proof.* This is a direct consequence of theorems 1.1 and 2.1. □

**Theorem 2.3.** *If the Riemann hypothesis holds, then*

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log\theta(x)} > 1$$

*for all numbers $x \geq 13.1$.*

*Proof.* Under the assumption that the Riemann hypothesis is true, then we would have

$$\prod_{q \leq x} \frac{q}{q-1} < e^{\gamma} \times \log x \times \left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right)$$

after of distributing the terms based on the theorem 1.3 for all numbers $x \geq 13.1$. If we apply the logarithm to the both sides of the previous inequality, then we obtain that

$$\sum_{q \leq x} \log(\frac{q}{q-1}) < \gamma + \log \log x + \log\left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right).$$

That would be equivalent to

$$\sum_{q \leq x} \frac{1}{q} + \sum_{q \leq x} \left(\log(\frac{q}{q-1}) - \frac{1}{q}\right) < \gamma + \log \log x + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

where we know that

$$\begin{aligned}\log\left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right) &< \frac{1}{\frac{8 \times \pi \times \sqrt{x}}{3 \times \log x + 5} + 0.4} \\ &= \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 0.4 \times (3 \times \log x + 5)} \\ &= \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}\end{aligned}$$

according to theorem 1.6 since $\frac{8 \times \pi \times \sqrt{x}}{3 \times \log x + 5} \geq 1$ for all numbers $x \geq 13.1$. We use the theorem 1.4 to show that

$$\sum_{q \leq x} \left(\log(\frac{q}{q-1}) - \frac{1}{q}\right) = H - u(x)$$

and $\gamma = H + B$. So,

$$H - u(x) < H + B + \log \log x - \sum_{q \leq x} \frac{1}{q} + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

which is the same as

$$H - u(x) < H - \delta(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}.$$

We eliminate the value of $H$ and thus,

$$-u(x) < -\delta(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

which is equal to

$$u(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).$$

We know from the theorem 2.1 that $\varpi(x) > u(x)$ for all numbers $x \geq 13.1$ and therefore,

$$\varpi(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).$$

Hence,

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \log \log \theta(x) - \log \log x.$$

Suppose that $\theta(x) = \epsilon \times x$ for some constant $\epsilon > 1$. Then,

$$\begin{aligned}
\log \log \theta(x) - \log \log x &= \log \log(\epsilon \times x) - \log \log x \\
&= \log (\log x + \log \epsilon) - \log \log x \\
&= \log \left( \log x \times (1 + \frac{\log \epsilon}{\log x}) \right) - \log \log x \\
&= \log \log x + \log(1 + \frac{\log \epsilon}{\log x}) - \log \log x \\
&= \log(1 + \frac{\log \epsilon}{\log x}).
\end{aligned}$$

In addition, we know that

$$\log(1 + \frac{\log \epsilon}{\log x}) \geq \frac{\log \epsilon}{\log \theta(x)}$$

using the theorem 1.5 since $\frac{\log \epsilon}{\log x} > -1$ when $\epsilon > 1$. Certainly, we will have that

$$\log(1 + \frac{\log \epsilon}{\log x}) \geq \frac{\frac{\log \epsilon}{\log x}}{\frac{\log \epsilon}{\log x} + 1} = \frac{\log \epsilon}{\log \epsilon + \log x} = \frac{\log \epsilon}{\log \theta(x)}.$$

Thus,

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \frac{\log \epsilon}{\log \theta(x)}.$$

If we add the following value of $\frac{\log x}{\log \theta(x)}$ to the both sides of the inequality, then

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} > \frac{\log \epsilon}{\log \theta(x)} + \frac{\log x}{\log \theta(x)} = \frac{\log \epsilon + \log x}{\log \theta(x)} = \frac{\log \theta(x)}{\log \theta(x)} = 1.$$

We know this inequality is satisfied when $0 < \epsilon \leq 1$ since we would obtain that $\frac{\log x}{\log \theta(x)} \geq 1$. Therefore, the proof is done. □

**Theorem 2.4.** *If there exists some number $y \geq 13.1$ such that for all $x \geq y$ the inequality $\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log \log x)} \leq 1$ is satisfied for some positive constant C independent of x, then the Riemann hypothesis should be false.*

*Proof.* From the theorem 1.2, we know that there are infinitely many values of $x$ such that

$$\theta(x) > x + C \times \sqrt{x} \times \log \log \log x$$

for some positive constant $C$ independent of $x$. That would be equivalent to

$$\log \theta(x) > \log(x + C \times \sqrt{x} \times \log \log \log x)$$

and so,

$$\frac{1}{\log \theta(x)} < \frac{1}{\log(x + C \times \sqrt{x} \times \log \log \log x)}$$

for all numbers $x \geq 13.1$. Hence,

$$\frac{\log x}{\log \theta(x)} < \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log \log x)}.$$

If the Riemann hypothesis holds, then

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log \log x)} > 1$$

for those values of $x$ that complies with

$$\theta(x) > x + C \times \sqrt{x} \times \log \log \log x$$

due to the theorem 2.3. By contraposition, if there exists some number $y \geq 13.1$ such that for all $x \geq y$ the inequality

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log \log x)} \leq 1$$

is satisfied for some positive constant $C$ independent of $x$, then the Riemann hypothesis should be false, because of there are infinitely many values of $x$ which satisfy the inequality in the theorem 1.2 and comply with $x \geq y$ no matter how big could be $y$. □

**Definition 2.5.** *Let's define the function* $\upsilon(x) = \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log \log x)} - 1$ *for some positive constant C independent of x.*

**Theorem 2.6.** *The Riemann hypothesis could be false.*

*Proof.* From the theorem 2.4, we know that the Riemann hypothesis is false when there exists some number $y \geq 13.1$ such that for all $x \geq y$ the inequality $\upsilon(x) \leq 0$ is always satisfied. We know that the function $\upsilon(x)$ is monotonically decreasing for all sufficiently large numbers $x \geq 13.1$. Let $\upsilon'(x)$ be the derivative of $\upsilon(x)$. We can check the value of $\upsilon'(x)$ from this web site https://www.wolframalpha.com/input and see that $\upsilon'(x)$ is lesser than zero for all sufficiently large numbers $x \geq 13.1$. Indeed, a function $\upsilon(x)$ of a real variable $x$ is monotonically decreasing in some interval if the derivative of $\upsilon(x)$ is lesser than zero and the function $\upsilon(x)$ is continuous over that interval [10]. In this way, it is enough to find a value of $y \geq 13.1$ such that $\upsilon(y) \leq 0$ since for all $x \geq y$ we would have that $\upsilon(x) \leq \upsilon(y) \leq 0$. We found the first zero $y$ of the function $\upsilon(y)$ in $y \approx 8.2639316883312400623766461031726662911\ E5565708$ for $C \geq 1$ using the tool *gp* from the web site https://pari.math.u-bordeaux.fr. Consequently, we claim that the Riemann hypothesis could be false. □

## Appendix

We use the following input:

*(3*log(x)+5)/(8*pi*sqrt(x)+1.2*log(x)+2)+log(x)/log(x+C*sqrt(x)*log(log(log(x))))-1*

from the web site https://www.wolframalpha.com/input. Besides, we use the following input into a single command line:

$$solvestep(X = 1000000!, 5000000!, 1$$

$$, (3*log(X)+5)/(8*3.14*sqrt(X)+1.2*log(X)+2)+log(X)/log(X+sqrt(X)*log(log(log(X))))-1, 1)$$

using the tool *gp* from the web site https://pari.math.u-bordeaux.fr. In the project PARI/GP, the method $solvestep(X = a, b, 1, F(X), 1)$ finds the first zero of the function $F(X)$ in the interval $[a, b]$. We found the first zero $X \approx 8.2639316883312400623766461031726662911\ E5565708$ of our $F(X)$ in the interval $[1000000!, 5000000!]$ where $(\ldots)!$ means the factorial function. We use $\pi$ as 3.14 and $C = 1$ in $F(X)$, but we know there must exist a zero of the function $F(X)$ also for $C > 1$ and a more accurate value of $\pi$.

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