The Geometry of Musical Logarithms*

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La géometrie permet de penser le discontinu et le rend susceptible d'une analyse à la limite de la continuité, tandis que le nombre intervient pour compter des éléments apparament continus, et les rend alors discrets.[1]

It is generally acknowledged that Archimedes was close to the invention of logarithms.[2] The related ideas and calculation techniques were redeveloped and advanced in the sixteenth century by Christoff Rudolf and Michael Stifel. The Dutchman Simon Stevin propagated the decimal number system and decimal fractions toward the end of the sixteenth century. This laid the foundation for the calculation techniques developed by John Napier and Jost Bürgi toward the turn of the century, which were of great use in astronomy.

At the end of the year 1618, René Descartes offered his manuscript *Musicæ Compendium* to Isaac Beeckman as a New Year's gift. This early treatise by Descartes was published only in 1650, shortly after Descartes's death. According to H. Floris Cohen, its content is retrospective rather than innovative.[3] As a compendium, however, it is certainly not supposed to develop or propagate a new theory of music. Still, it contains some intriguing diagrams that use the circle as a metaphor for the octave similarity in combination with a logarithmic representation of musical ratios.

In 1618, the renowned publishing house of de Bry in Frankfurt published the second tractate of the first volume of *Utriusque cosmi historia*, the encyclopedic opus magnum by the English physician and philosopher Robert Fludd with some illustra-

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1 Frédéric De Buzon, "Science de la nature et théorie musicale chez Isaac Beeckman," *Revue d'histoire des sciences* 38, no. 2 (1985): 119.

2 Erwin Voellmy, *Jost Bürgi und die Logarithmen* (Basel: Birkhäuser, 1948; repr., 1974), 2–5, and Jörg Waldvogel, "Jost Bürgi and the Discovery of the Logarithms," *Elemente der Mathematik* 69, no. 3 (2014): 91–92.

3 He calls it "Zarlino *more geometrico*" (H. Floris Cohen, *Quantifying Music: The Science of Music at the First Stage of the Scientific Revolution, 1580–1650* [Dordrecht: D. Reidel Publishing, 1984], 163).

tions by Matthäus Merian. This part of *Utriusque cosmi historia* contains the *Templum Musicæ*.[4] Fludd's music theory is based on a Pythagorean mode of thinking, and at its most modern refers to the *ars nova/ars subtilior*-period of the fourteenth century. Fludd was an adherent of the Ptolemaic cosmology, according to which the sun and the other planets circulated around the earth. His interest in and defense of the Rosicrucian movement might have been detrimental for a positive reception of his work on the continent by the philosophers of the new age of mechanization, Kepler, Mersenne, and Gassendi.[5]

In comparison with Descartes's "Zarlinoism," Fludd's tone system appears outdated. While Fludd defended the ratio 81 : 64, the Pythagorean ditonus, Descartes constructed the diatonic scale with consonant thirds in the ratio 5 : 4, as suggested by Gioseffo Zarlino in the second half of the sixteenth century.

In 1619, Johannes Kepler published *Harmonices mundi*.[6] Book 3 (the section on music theory) uses geometry in order to determine the consonant intervals. He claims a correspondence between the regular polygons that are constructible by ruler and compasses and Zarlino's system of consonances. Thus, constructibility acts as a natural selection criterion. Kepler assumed that 5 is the highest prime number for which a regular polygon can be constructed with ruler and compasses. However, in the early nineteenth century Carl Friedrich Gauss proved that the regular polygons with 17 and 257 vertices were also constructible with ruler and compasses. In the appendix of *Harmonices mundi*, Kepler criticized Fludd's *Utriusque cosmi historia* marking the beginning of a long and bitter controversy.[7]

In 1620, *Arithmetische und geometrische Progreß-Tabulen* by Jost Bürgi were printed in Prague. Its title page shows a circular diagram, which is very similar to Descartes's circular diagrams. At the time Bürgi had been using his tables for more than ten years.[8] Because only a few printed copies of Bürgi's excellent *Arithmetische und geometrische Progreß-Tabulen* have survived, it is conceivable that the "publication" in 1620 was merely a test print. And the long period of the Thirty Years' War in Germany, 1618–48, may have prevented Bürgi's work from becoming more generally known.[9]

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4 Fludd's second tractate was shown at the Frankfurt book fairs in spring 1618 where Kepler had seen it; cf. Peter Hauge, *"The Temple of Music" by Robert Fludd* (Burlington, VT: Ashgate, 2011), 22, fn. 80.

5 Cf. Max Caspar in Johannes Kepler, *Gesammelte Werke*, vol. 6, *Harmonice Mundi*, ed. Max Caspar (Munich: Beck'sche Verlagsbuchhandlung, 1940), 513–21.

6 Ibid., 7–377.

7 Ibid., 373–77. Kepler's approach will not be discussed any further here; cf. Cohen, *Quantifying Music*, 13–34.

8 Fritz Staudacher, *Jost Bürgi, Kepler und der Kaiser* (Zurich: NZZ Libro, 2013), 197.

9 Ibid., 203–4.

René Descartes's Aesthetic Principles and Musical Diagrams

In the *Musicæ Compendium*, the young Descartes recapitulated the state of music theoretical thinking as propagated by Zarlino[10] in the sixteenth century. Completed by the end of 1618, he offered the manuscript to his new friend Beeckman. The *Prænotanda* presented a system of aesthetic principles underlying the organization of the horizontal time and the vertical pitch/frequency domains of music theory. Descartes accepted only small integer ratios as fundamental and "understandable by the senses"—seeing and hearing—and he pointedly argued in favor of arithmetic against geometric division of ratios.[11]

He illustrates the two ways of dividing ratios with line segments $2 : 3 : 4$ against $2 : \sqrt{8} : 4$ (see figure 1). The *arithmetic* mean of the outer terms 2 and 4 is 3 because of $3 = {}^1\!/_2 \cdot (2 + 4)$, the *geometric* mean of the outer terms 2 and 4 is $\sqrt{8}$ because of $\sqrt{8} = \sqrt{2 \cdot 4}$.

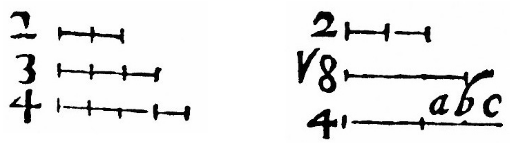

Fig. 1. Arithmetic and geometric division of the octave according to René Descartes, *Musicæ Compendium* (Utrecht: Zijll, 1650), repr. in Descartes, *Œuvres de Descartes*, vol. 10, ed. Charles Adam and Paul Tannery (Paris: Léopold Cerf, 1908), 91 and 92.

The word *geometric* in *geometric mean* originates in the standard geometry problem of transforming a given rectangle into a square of equal area; in Descartes's example, turning the rectangle having sides 2 and 4 into a square of area 8. The sides of this square measure $\sqrt{8}$ units. The fact that the value of the square root of the product of two numbers is always between the two numbers justifies the word *mean* in *geometric mean.*

By combining Euclid's altitude theorem with Thales's theorem, the geometric mean can be found by using a geometrical construction, known also by music theorists. The construction by Lodovico Fogliano in figure 2a shows the construction of the geometric mean of the numbers 80 and 81. In other words, it serves to determine geometrically the square root of $80 \cdot 81 = 6480$, resulting algebraically in 80.498, a value very close to the arithmetic mean ${}^1\!/_2 \cdot (80 + 81) = 80.5$. This example would not be given in a modern geometry text book in order to explain the two kinds of means, because a rectangle with sides of 80 and 81 is almost a square. Descartes's

10 Gioseffo Zarlino, *Le istitutioni harmoniche* (Venice, 1558; 3rd ed., 1573); Zarlino, *Dimostrationi harmoniche* (Venice: Francesco dei Franceschi Senese, 1571).

11 Cohen, *Quantifying Music*, 161–79; Daniel Muzzulini, *Genealogie der Klangfarbe* (Bern: Peter Lang, 2006), 35–37; Muzzulini, "Descartes' Töne—Newtons Farben," in *Musik—Raum—Akkord—Bild: Festschrift zum 65. Geburtstag von Dorothea Baumann*, ed. Antonio Baldassarre (Bern: Peter Lang, 2012), 691–706.

example would be more convincing in this respect, because $\sqrt{8} \approx 2.828$ can be visually distinguished from $1/2 \cdot (2+4) = 3$ (see figure 2b). Note that in the construction of the geometric mean the arithmetic mean equals the radius of the circle.

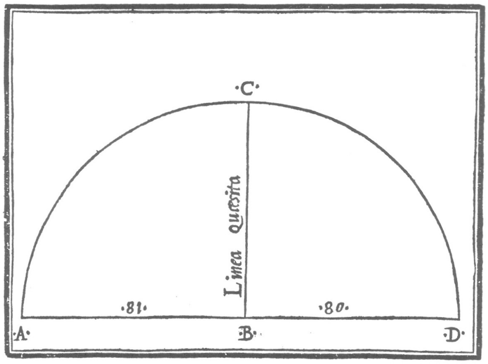

Fig. 2a. Construction of the geometric mean by Fogliano (Lodovico Fogliano, *Musica theorica* [Venezia, 1529], fol. xxxvi).

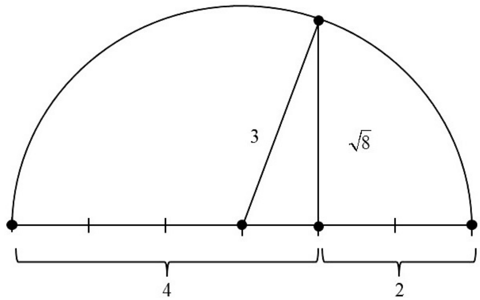

Fig. 2b. Descartes's example in the light of Fogliano's construction.

Both examples are tied to music theoretical questions in the pitch domain. Fogliano's construction bisects the *syntonic comma*, $81 : 80$, into two equal musical intervals.[12] The resulting syntonic *semi-comma* can be used in order to divide the third $(5 : 4)$ into two equal whole tones $(\sqrt{5} : 2 \approx 1.1180 : 1)$ by lowering the major tone $(9 : 8)$ by a semi-comma or by increasing the minor tone $(10 : 9)$ by a semi-comma. These tempered whole steps are used in meantone tuning systems. The syntonic comma is equal to the intervallic *difference* between the major and the minor whole tone. Its ratio is obtained by dividing the ratio of the major tone by the ratio of the minor tone: $9/8 : 10/9 = 9/8 \cdot 9/10 = 81/80 = 81 : 80$.

Descartes's juxtaposition bisects the musical octave $(2 : 1)$ *arithmetically* into a fifth $(3 : 2)$ and a fourth $(4 : 3)$ and *geometrically* into equal semi-octaves, tritones, or diminished fifths of the irrational ratio $\sqrt{2} : 1$.[13]

The comparison of arithmetic and geometric ratios is carried out by Descartes in a continuous "geometrical" context, since the geometric and arithmetic ratios are both visualized by ratios of lengths of line segments. In other words, the discrete integer numbers are understood a priori as a part of a comprehensive continuum. There is no better way to compare arithmetic ratios with geometric ratios in general.

It is essential for Descartes's choice of numbers in figure 1 that the geometric ratio $\sqrt{2}$ is an irrational number. The proportions $8 : 13 : 18$ (arithmetic progres-

12 Equal *musical intervals* are defined by equal *frequency ratios.*

13 Classical music theory distinguishes between harmonic and arithmetic division. For example, $3 : 4 : 5$ is arithmetic division, but $1/3 : 1/4 : 1/5$, the ratio of the reciprocals is harmonic division, a distinction Descartes does not make.

sion with common difference $d = 5$) and $8 : 12 : 18$ (geometric progression with common ratio $r = 3/2$) would not have been sufficient for him to reject geometrical ratios in such a rigorous way: The number 13 from the arithmetic division of the major ninth ($18 : 8 = 9 : 4$) is not used in traditional music theory, whereas the geometric division of the same interval into two fifths would make much more sense.

Because of their irrational ratios, Descartes's aesthetic principles exclude equal temperament as not understandable by the sense of hearing. The *Musicæ Compendium* explains the tone system with three congruent diatonic hexachords separated by fifths sized $3 : 2$. This results in a range of two diatonic major scales centered on f and c with the pitch classes c, d, e, f, g, a, b-flat, b (see figure 3d). Descartes's tone system has two ambiguous tones d ($320/324$) and g ($480/486$ where $c = 360$). The two values of these "mobile tones," d and g, differ by a syntonic comma and are clearly distinguished in the drawings.[14] The numbers in the continued proportions given by Descartes represent string lengths on the monochord (or time periods) and not frequencies. The frequency interpretation of pitch was not yet well established at the time.[15]

The eye-catching feature of Descartes's diagrams (figure 3) is the use of the circle for visualizing the octave similarity. Descartes is the first to express the octave systematically as a full $360°$ angle. However, Robert Fludd, who uses the circle very frequently in his illustration, is very close to such an interpretation in the *Templum Musicæ* published in 1618 (see figure 6a below).

The diagrams in the first printed Latin editions of Descartes's *Musicæ Compendium* (1650 and 1656), are rather accurate in the following sense: equal musical intervals, that is, equal number ratios, are represented by equal circular sectors, so that the full octave corresponds to the full circle of $360°$. Furthermore, the relative size of the different intervals is expressed in the ratios of angles.

A detailed investigation of Descartes's circular diagrams reveals that the angle of the tritone is usually equal to $180°$, and some of the minor thirds are equal to $90°$. Some of the minor tones are even greater than major tones. Essential features of Descartes's diagrams are their inner symmetries, which were deliberately abandoned in the early English edition.[16] The mirror and rotational symmetries follow directly from the logarithmic understanding of pitch.

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14 The musical context decides which of the ambiguous tones is to be used (Descartes, *Musicæ Compendium*, 117–19).

15 Exceptions are Giovanni Battista Benedetti, "De intervalli musicis," in *Diversarum speculationum mathematicarum et physicarum liber* (Turin, 1585), 277–83, and Beeckman (1614), who independently developed a pulse theory of sound: C. de Waard, ed., *Journal tenu par Isaac Beeckman* (The Hague: Martinus Nijhoff, 1939), 1:56–57 (fol. 24v, 1614); cf. Cohen, *Quantifying Music*, 75–78, 94–97, and 127–47.

16 Benjamin Wardhaugh, "Musical Logarithms in the Seventeenth Century: Descartes, Mercator, Newton," *Historia mathematica* 35, no. 1 (2008): section 3; Wardhaugh, ed., *The "Compendium Musicæ" of René Descartes: Early English Responses* (Turnhout: Brepols, 2013), xxxi–xxxii.

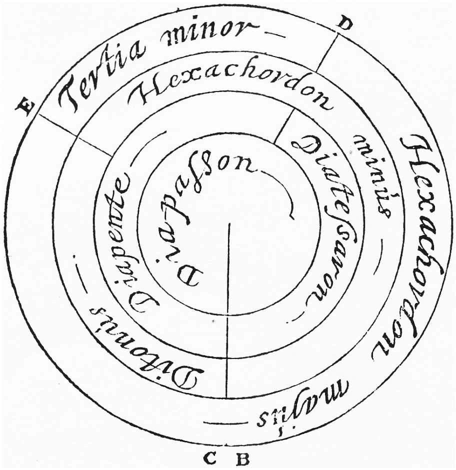

Fig. 3a. The consonant intervals within the octave (Diapason). Descartes, *Musicæ Compendium*, 104.

Fig. 3b. The corresponding diagram in Beeckman's copy of Descartes's manuscript (Ms. Middleburg, fol. 167r): The angle of the minor third is bigger than 90°. Because the radial line at D does not pass through the center of the circles, the related angles cannot be measured accurately.

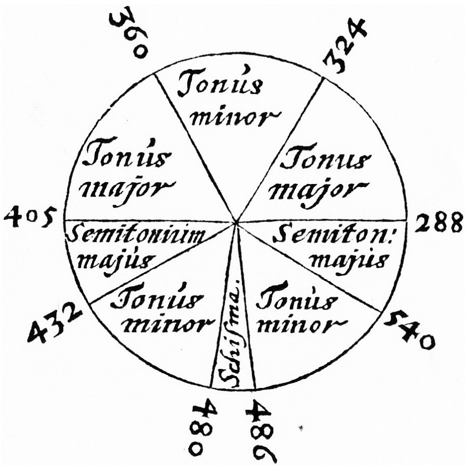

Fig. 3c. The diatonic major scale, starting at ut = 540 in clockwise direction with an ambiguous tone (486/480), separated by a syntonic comma, which Descartes calls "Schisma." The diagram is symmetric about the bisector of the syntonic comma. The radii defining the tritone and the diminished fifth (at 405 and 288) are on the horizontal diameter of the circle. Descartes, *Musicæ Compendium*, 118.

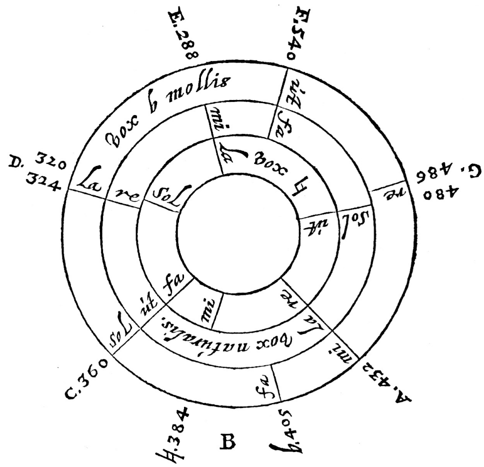

Fig. 3d. The three hexachords from F (540), C (360), and G (480) have congruent angles, each given with relative solmization (ut, re, mi, fa, sol, la). There are two ambiguous pitch classes at G and D. The leading notes B quadratum (384) and E (288) are major thirds (mi) from the tonic (ut) of the adjacent hexachords. Also in this diagram the diminished fifths (540–384 and 405–288) "mi contra fa" (the devil in music) are on diameters of the circle. Descartes, *Musicæ Compendium*, 120.

The manuscript of the *Musicæ Compendium* is lost. The earliest of the extant manuscript copies was made for Isaac Beeckman about 1628.[17] Assuming that Descartes's own drawings were as accurate as in Beeckman's copy, it must be concluded that he had a feeling for logarithms at a time when they just have been made public. Where did he gather the necessary knowledge of mathematics?[18]

In 1614, the Scottish mathematician John Napier published his first tables, *Mirifici logarithmorum descriptio*. However, these tables were of direct use in astronomy, not in musical arithmetic,[19] and the circular diagram by Jost Bürgi, the title page of his *Arithmetische und geometrische Progreß-Tabulen* (1620, see figure 4), which is much easier to understand, had not yet been printed when Descartes composed the *Musicæ Compendium*. Although Bürgi's tables were completed in 1609 or even earlier,[20] it can be excluded that Descartes knew of them, because Bürgi kept them secret.

Jost Bürgi's Mathematical Diagram in Its Relation to Descartes's Circular Pitch Diagrams

Descartes's diagrams (figures 3c and 3d) and the diagram on the title page of Bürgi's *Arithmetische und geometrische Progreß-Tabulen* (figure 4) are closely related. The (red) numbers of the outer circle in Bürgi's diagram are in arithmetic progression, where equal steps correspond with equal angles. They form a linear scale for angles. The (black) numbers of the inner circle are in geometric progression, so that equal angles correspond to equal ratios of black numbers. A full rotation results in a multiplication by 10. Bürgi's black numbers play the same role as the numbers in Descartes's diagrams, where equal ratios also have equal angles. A full rotation in Descartes's diagrams results in a multiplication by 2. The use of the same interval names for equal sectors is the application of a linear scale in the pitch domain comparable with Bürgi's scale on the outer red circle.

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17 René Descartes, *Abrégé de musique: Compendium Musicæ*, ed. Frédéric de Buzon, 2nd ed. (Paris: Presses universitaires de France, 2012); Wardhaugh, "Musical Logarithms in the Seventeenth Century," section 3.

18 The mathematical and musical formation that Descartes obtained at the Jesuits' College La Flèche is discussed in Stephen Gaukroger, *Descartes: An Intellectual Biography* (Oxford: Clarendon Press, 1995), 55–59; Chikara Sasaki, *Descartes's Mathematical Thought* (Dordrecht: Kluwer Academic Publishers, 2003), 13–44; Ivo Schneider, "Trends in German Mathematics at the Time of Descartes' Stay in Southern Germany," in *Mathématiciens français du XVII^e^ siècle: Descartes, Fermat, Pascal*, ed. Michel Serfati and Dominique Descotes (Clermont-Ferrand: Presses universitaires Blaise-Pascal, 2008), 45–68.

19 Thomas Sonar, *3000 Jahre Analysis: Geschichte, Kulturen, Menschen* (Heidelberg: Springer, 2011), 296–301.

20 Staudacher, *Jost Bürgi, Kepler und der Kaiser*, 197; Waldvogel, "Jost Bürgi and the Discovery of the Logarithms," 89.

Fig. 4. Jost Bürgi, *Arithmetische und geometrische Progreß-Tabulen* (Prague: Paul Sessen, 1620), title page. The outer ring is colored red in the original print.

Bürgi's circle closes at 10, because the decimal number system repeats with a multiplication by 10. Descartes's circles close with a multiplication by 2, because it generates notes of the same pitch class. In other words, 135, 270, and 540 denote equivalent notes in different octaves. In the surrounding text Descartes insists that the natural way of studying the consonant intervals and musical scales is by bringing them into an octave,[21] however, he does not comment on the angles in his diagrams, since he must have thought their meaning self-evident.[22] The mathematical transformation from frequencies to pitch classes is summarized in figure 5.

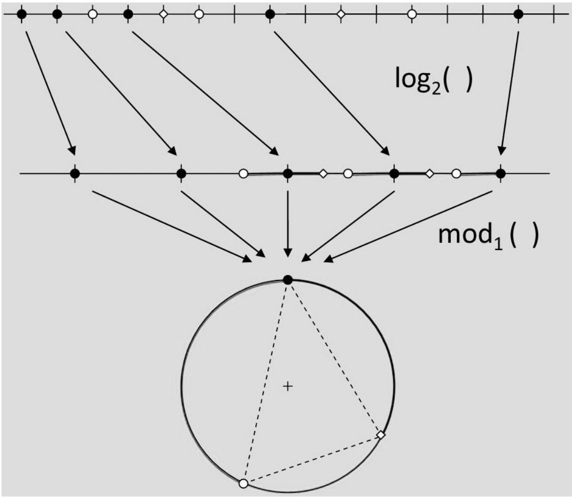

Fig. 5. Pitch classes are formed by a composite mathematical transformation. The first, base 2 logarithm, maps frequency onto pitch, the second, modulo 1 (times 360°), maps pitch onto pitch classes on the circle line. The clockwise oriented triangle corresponds to a major triad (4 : 5 : 6) with the tonic on top of the circle.

Robert Fludd's Circular Diagrams

The circle in its perfect symmetry is present in every topic Robert Fludd studied—from divine numbers to the colors of urine. The following two examples are taken from the *Templum Musicæ* and from *De Numero et Numeratione.*[23]

The first drawing (figure 6a) resembles Descartes's diagrams in many ways: It is a circular arrangement, it consists of concentric circles, and it is about music. Furthermore, it uses a "logarithmic" presentation by equating the size of musical intervals with distances. It shows three octaves of the chromatic scale on concentric rings. The letters indicate the position of the frets on the lute ("barbitum") in alphabetical order, and the numbers indicate the strings with its lowest 1 and highest 6. Both dimensions, the radial and the angular, display chromatic scales. The radial direction covers one octave and the circular scale uniformly covers three octaves.

21 Descartes, *Musicæ Compendium*, 98–105.

22 He only states that a full circle comprises an octave (Descartes, *Musicæ Compendium*, 103–4).

23 Robert Fludd, *Utriusque cosmi maioris scilicet et minoris metaphysica, physica atque technica historia in duo volumina … divisa*, vol. 1, tract. 2, part 2, lib. 6, *De Instrumentis Musicis vulgariter notis* (Oppenheim: Aere Johan Theodori de Bry, 1617), 232; ibid., vol. 1, tract. 2, part 1, lib. 1, *De Numero et Numeratione*, 9.

In the surrounding text Fludd tells us that the vertical scale (North) is meant to be a spinning pointer, so that the drawing represents a mechanical device for lute players, allowing them to transpose their part quickly if necessary.[24] The tuning of the lute can be guessed by analyzing the cells with two designations. It turns out to be G-C-F-A-D-G, which is consistent with Fludd's drawing of the fretboard of the lute with the notes indicated by their position in the stave (see figure 7).[25]

The three octaves express a metaphysical idea manifest in many of Fludd's drawings. The number Three is a holy number for Fludd because it is the first that has a beginning, a middle part, and an end. In the same mode of thinking, the perfect division in the Middle Ages is ternary and not binary.[26]

The second diagram (figure 6b) is taken from *De Numero et Numeratione*.[27] It explains the decimal number system within a logarithmic presentation giving the powers of 10 in counterclockwise direction. The nine digits 1 to 9 of the outermost circle get equal sectors. The second circle groups the nine sectors as three times 1-2-3, which indicates the three positions Ones, Tens, and Hundreds within each of the three groups, Units, Thousands, and Millions.[28]

Notice that 0 does not occur as a number of its own right in Fludd's *Speculum*; the digits in the inner circles run from 1 to 9. Zero is not seen as a proper number—it is merely an articulation sign indicating an empty position. Forming numbers from the *Speculum* is combinatorics: selecting from each sector a digit or an articulator. Thereby, every positive integer number less than one billion can be formed.

Actually, zero (the devil in numbers) would disturb the "perfect order" of three times three. In the diagram neither the closing of the system with nine digit numbers nor the choice of the circular arrangement is intrinsically motivated. As a mechanical device, however, it would be more easily made with rotating disks and a spinning pointer than with straight sliders.[29] The diagram resembles Ramon Llull's (ca. 1232–1316) concentric circles, which are also mechanical tools.[30] In order to

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24 Fludd uses the word "rota" ("wheel"): "Tunc convertendo rotam L invenio a.6. in loco ejusdem spharae. 25. & sub ipso in orbe. A.re.c.6." (ibid., vol. 1, tract. 2, part 2, lib. 6, 232 [= Hauge, *"The Temple of Music" by Robert Fludd*, 188]).

25 Ibid., vol. 1, tract. 2, part 2, lib. 6, 230.

26 Ibid., vol. 2, tract. 1, sect. 1, lib. 1, *De Numeris Divinis*, 26 and 35–36.

27 Ibid., vol. 1, tract. 2, part 1, lib. 1, 9. At the end of this book, Fludd recommends Michael Stifel for further reading (ibid., vol. 1, tract 2, part 1, lib. 3, *De Arithmetica Cossica, Epilogus*, 79). Stifel treats musical intervals and scales in a way very similar to Zarlino (Michael Stifel, *Arithmetica Integra* [Nuremberg: Johannes Petreius, 1544]).

28 Since the numbering starts at 1 and not at 0, the formula $10^n 10^m = 10^{n+m}$ remains hidden.

29 For the history of slide rules, cf. Florian Cajori, *History of the Logarithmic Slide Rule* (Colorado Springs: School of Engineering, Colorado College, 1909), and Cajori, "On the History of Gunter's Scale and the Slide Rule during the Seventeenth Century," *University of California Publications in Mathematics* 1, no. 9 (1920): 187–209.

30 Ramon Llull, *Ars brevis Illuminati Doctoris Magistri Raymundi Lull* (Lugduni: Stephanus Baland, 1514), fig. 4. The author owes the conjecture of a possible link between Llull's and Descartes's diagrams to a conversation with Angela Lohri (Vienna). One of Descartes's combinatorial matrix diagrams not shown here occurs also in Ramon Llull's *Ars brevis.*

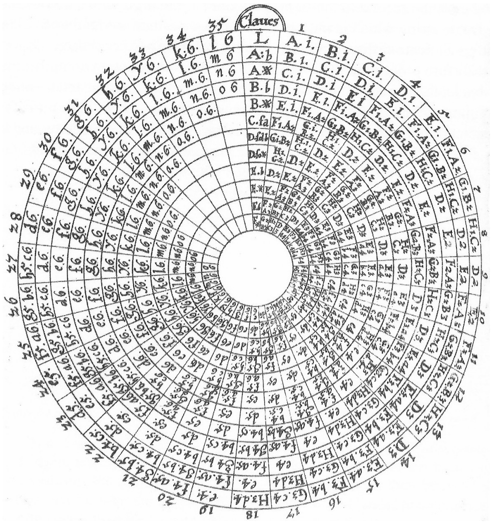

Fig. 6a. Transposition circle for the lute (barbitum).

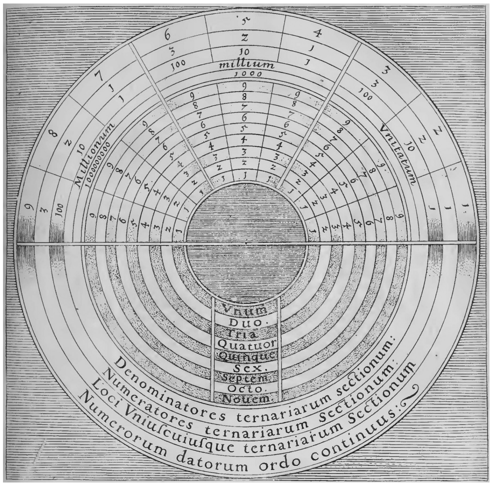

Fig. 6b. Numerationis Speculum.

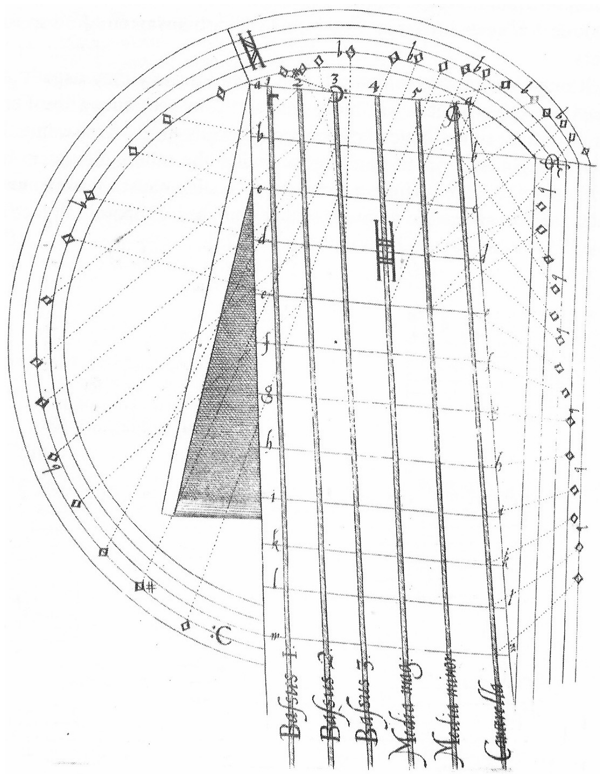

Fig. 7. Fretboard of the barbitum (Fludd, *Utriusque cosmi*, vol. 1, tract. 2, part 2, lib. 6, 230). This picture confirms the tuning derived from figure 6a.

turn Fludd's *Speculum* into a working mechanical tool, it would be necessary to interchange the roles of the radial and the angular dimension, so that the individual numbers could be read off in radial direction.

To sum up, the transposition circle (figure 6a), as a two-dimensional arrangement of chromatic scales with equal semitones, is a double logarithmic representation in polar coordinates, whereas the *Numerationis Speculum* (figure 6b) combines a radial linear dimension with a logarithmic angular dimension.

Constructing Geometric Progressions

Descartes's *Musicæ Compendium* shows that in 1618 he was familiar with geometric sequences and possibly also with fractional powers. He certainly knew that multiplying ratios corresponded to adding musical intervals. This was already evident from the numbers added to the circular diagrams. However, this was standard in music theory since Boethius and well known through the sixteenth century.[31] In *La Géometrie* (1637) Descartes referred to the problem—posed by Pappus of Alexandria (ca. 290–350 CE)—of determining several intermediate proportional numbers between two given numbers.[32] In order to solve this problem, Descartes depicted a mechanical instrument that permitted drawing the graphs of power functions, if a unit length was defined in the geometric plane (see figure 8).[33]

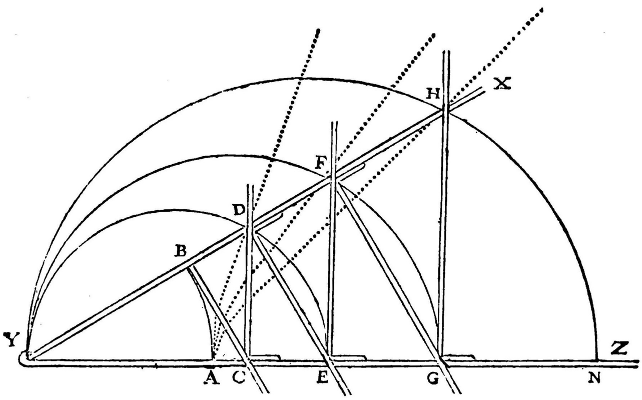

Fig. 8. Descartes's instrument (mesolabe compasses) for constructing geometric progressions and power functions (dotted curves).

31 Fogliano, *Musica theorica*; Stifel, *Arithmetica Integra*; Zarlino, *Le istitutioni harmoniche*; Zarlino, *Dimostrationi harmoniche.*

32 René Descartes, "La Géometrie," in Descartes, *Discours de la méthode pour bien conduire sa saison et chercher la vérité dans les sciences* (Leiden: Jan Maire, 1637), repr. in Descartes, *Œuvres de Descartes*, 6:306; cf. Gaukroger, *Descartes: An Intellectual Biography*, 93–99.

33 Descartes, "La Géometrie," 318 and 370–71.

For any opening angle (less than 90°) of the legs YX and YZ, the line segments between the two legs form a geometric progression, because all triangles are similar. This instrument is called the "mesolabe compasses." With many and sufficiently long rulers, it admits—in principle—the construction of arbitrary powers of any real base greater than 1, which is impossible with ruler and traditional compasses. With twelve rulers the values of the twelve-tempered equal tuning could be found.

The main ideas in Descartes's analytical geometry were developed in the 1620s, soon after the completion of the *Musicæ Compendium*.[34] Descartes in 1637 did not give a hint that the mesolabe compasses could be used mechanically to determine the frets of lutes, and he never again published on music theory.[35]

Comparing the Octave Indirectly with the Syntonic Comma

Syntonic versus Pythagorean Comma

Boethius knew the estimation $75/74 < 531{,}441/524{,}288 < 74/73$ of the Pythagorean comma. Faber Stapulensis (1496; 1551) and Michael Stifel (1544) carried out the more demanding measuring of the Pythagorean semitone ($256 : 243$) in terms of Pythagorean commas.[36]

The syntonic comma ($81 : 80$) is a little bit smaller than the Pythagorean comma. It is defined by a super-particular ratio ($n+1/n$ where $n = 80$) and has a short decimal representation 1.0125, whereas the ratio of the Pythagorean comma is neither simple nor super-particular. Therefore, the syntonic comma can serve as a *unit interval* in order to measure the size of the other intervals. Without decimal fractions, the use of super-particular ratios of small intervals (big values of *n*) as *multiplicative units* is the simplest way of comparing the size of musical intervals. In the Pythagorean tone system, however, the syntonic comma simply does not exist.

Vincenzo Galilei's Semitones

In 1581, Vincenzo Galilei remarked that $18 : 17$ provided an excellent approximation of the semitone in equal temperament. It was indeed the best "semitone" of the form $n+1/n$ to approximate twelve-tempered equal tuning: $(18/17)^2 = 1.9856 \approx 2$. Already

34 Gaukroger, *Descartes: An Intellectual Biography*, 99–103.

35 In his correspondence with Mersenne music theoretical questions are addressed frequently, for example overtones in 1633, Descartes, *Œuvres de Descartes*, 1:267–68. Cf. Muzzulini, *Genealogie der Klangfarbe*, 126–29.

36 Boethius, *De institutione arithmetica libri duo, De institutione musica libri quinque*, vol. 3, *De institutione musica*, ed. Godofredus Friedlein (Leipzig: Teubner, 1867), 286: "In qua numerorum proportione sit comma et quoniam in ea, quae maior sit quam .LXXV. ad .LXXIIII. minor quam .LXXIIII. ad .LXXIII." Jacobi Faber Stapulensis (= Lefèvre d'Etaples), *Musica libris quatuor demonstrata* (Paris: Gulielmum Cauellat, 1552), 2:35; Stifel, *Arithmetica Integra*, no pagination between 72 and 76.

Ptolemy knew the relationship $^{18}/_{17} < \sqrt{^{9}/_{8}} < {}^{17}/_{16}$,[37] which was true because the middle term is the geometric mean of the outer terms (see figure 9). The inequality states that two semitones $18 : 17$ are smaller than a major tone $9 : 8$ and that two semitones $17 : 16$ are greater than a major tone.

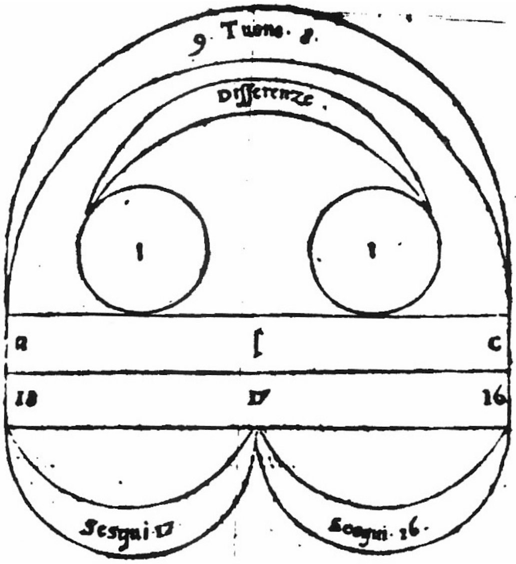

Fig. 9. Division of the whole tone $9 : 8$ into two semi-tones $17 : 16$ and $18 : 17$ (Zarlino, *Dimostrationi harmoniche*, 166).

The ratio $18 : 17$ was used to determine the positions of the frets in lutes, for which, for practical reasons, equal temperament was early accepted as a compromise. In 1619, Kepler calculated the related string lengths and compared them with his own chromatic scale.[38]

Comparing the syntonic comma $81 : 80$ with the semitones $^{18}/_{17} \approx 1.05882$ leads to approximately nine syntonic commas per two semitones $18 : 17$, and to $9 \cdot 6 = 54$ syntonic commas per octave. The exact value is 55.8 syntonic commas per octave, which gives a syntonic comma of $^{360°}/_{55.8} = 6.45°$.[39] Determining the whole tone $9 : 8$ as approximately nine syntonic commas and the octave as approximately six whole tones also gives approximately 54 syntonic commas per octave.

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37 Cris Forster, *Musical Mathematics: On the Art and Science of Acoustic Instruments* (San Francisco: Chronicle Books, 2010), 354.

38 Cohen, *Quantifying Music*, 68; Kepler, *Gesammelte Werke*, 5:143. However, he could have picked these values directly from Stevin's tables of interest: *Tafel van Interest van den penninck 17* (1582). Dirk J. Struik, ed., *The Principal Works of Simon Stevin*, vol. 2, *Mathematics* (Amsterdam: Swets & Zeitlinger, 1958), 75.

39 The angles of the syntonic comma in the diagrams of the early French and Latin editions of Descartes's compendium vary between 5° and 14°, Descartes, *Abrégé de musique*, 100–1 and 104–5. See figure 3 above for reproductions of some of the diagrams.

It can be seen directly that the whole tone $9:8$ must be smaller than ten syntonic commas: $9/8 = 90/89 \cdot 89/88 \cdot 88/87 \cdot 87/86 \cdot 86/85 \cdot 85/84 \cdot 84/83 \cdot 83/82 \cdot 82/81 \cdot 81/80 < (81/80)^{10}$. The last fraction in the product representation of the whole tone is the biggest factor, which implies the inequality. Since the octave is less than six whole tones $9:8$, the octave must be smaller than sixty syntonic commas, resulting in a syntonic comma greater than 6°. Even a syntonic comma of 6° leads to diagrams that are at least as accurate as those in Beeckman's copy and the early Latin printed editions of Descartes's *Musicæ Compendium.*

Comparing the Tritone with the Diminished Fifth

In order to create a circular diagram of the diatonic scale for didactical purposes it would be desirable not only to distinguish visually the major whole tone $(9:8)$ from the minor whole tone $(10:9)$ but also to make the tritone (two major whole tones plus one minor whole tone) different from $180°$: The difference between the diminished fifth $(64:45)$ and the tritone $(45:32)$ is equal to $64/45 : 45/32 = 2048 : 2025 = 1.01136$, which gives $177.1°$ for the augmented fourth $45:32$ and $182.9°$ for the diminished fifth $64:45$. In other words, the difference between the two angles is comparable with the angle of a syntonic comma. Descartes's diagrams do not make this distinction at all.

Stevin and Beeckman

Today it seems to be clear that the twelve-tempered intervals, multiples of $30°$ (a semitone), could also have served as points of reference in Descartes's diagrams. As already mentioned, some of the minor thirds in Descartes's diagrams are indeed equal to $90°$ (three semitones) and the tritone is usually equal to $180°$ (six semitones).

The first and very accurate numerical values of the ratios of the twelve-tempered equal tuning can be found in Simon Stevin's *Vande Spiegheling der Singconst.* A negative feedback by the organist Abraham Verheyen in ca. 1608 might have prevented Stevin from publishing this text.[40] In 1624, Beeckman borrowed the manuscript from Stevin's widow,[41] and in his diary he mentioned Stevin's description of the fifth as the twelfth part of seven octaves by $\sqrt[12]{128}$.[42] Beeckman, however, knew Stevin's Mathematical Memoirs (1605/8) much earlier. He referred to Stevin's writings from 1612 onwards, with respect to music theory and geometric division of musical ratios in 1613/14 and in 1618.[43]

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40 Cohen, *Quantifying Music*, 61–63.

41 Later on, Stevin's manuscript was in the hands of Constantin Huygens and eventually published in 1884; cf. Waard, *Journal tenu par Isaac Beeckman*, 2:292, 2:Appendix (fol. 228r–v), and 403–5.

42 Ibid., 2:291–92 (fol. 194r, 16–24 June 1624, Flemish); cf. Cohen, *Quantifying Music*, 185.

43 Waard, *Journal tenu par Isaac Beeckman*, 1:29 (fol. 14r, July 1613–April 1614); ibid., 1:180–81 (fol. 74v, April–25 June 1618).

In 1616, Beeckman wrote that multiplying ratios corresponds to adding musical intervals and he distinguished between the Pythagorean and the syntonic thirds explicitly: "Verus enim ditonus est 80/64, id est 5/4, eorum verò 81/64 a duplicatâ ratione 9/8."[44] Notice the collocation "duplicatâ ratione": doubling a ratio is squaring its fraction. The traditional Latin term "ditonus" for the major third makes a clear statement about interval size. Possibly, Beeckman had influenced Descartes's interval calculations in 1618, which resulted in the circular diagrams, but Descartes could have also learned these basic facts about musical intervals from studying Zarlino at La Flèche in the years before his friendship with Beeckman.

This implies that no table of logarithms is needed to find the angles in the circular diagrams as they are given in the printed versions of the *Musicæ Compendium* and its extant manuscripts. However, in order to create a "circular musical slide rule" with astronomical precision, Bürgi's *Progreß-Tabulen* (see below) would be helpful.

Interlude: A Rosicrucian Link?

In 1620, Descartes visited the famous mathematician Johann Faulhaber (1580–1635) in order to study with him. Faulhaber, the founder of a mathematical school in Ulm (1600), was also interested in alchemy and in the Rosicrucian Society: "On 21 January 1618 he wrote to Rudolph von Bünau: '… I am not sparing any efforts in inquiring about the commendable Rosicrucian Society.'"[45]

Apparently, in 1619, Descartes planned to write a book provisionally titled *The Thesaurus of Polybius Cosmopolitanus* and to dedicate it to the Rosicrucians. According to an extant copy of its summary, which is similar in content to Rule 4 of the *Regulæ ad directionem ingenii*, its intention was to create a new science that would merge algebra with geometry.[46] At that time Descartes was already fascinated by the compasses, later described in *La Géometrie* (1637, see figure 8 above).[47]

Seemingly, Descartes had also tried to find out about the Rosicrucian Society without success[48] and in this he was in good company with Robert Fludd. Gary L. Stewart claims that not only Faulhaber but also Descartes and Beeckman were members of the secret order of the Rosicrucians, however he conceals that no hard facts such as membership cards or lists have survived.[49]

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44 Ibid., 1:88–89 (fol. 40r, 6 February–23 December 1616); at the same time he also quotes Faber Stapulensis (ibid., 1:84 [fol. 38v, March 1615–6 February 1616]).

45 Paul A. Kirchvogel, "Faulhaber, Johann," in *Complete Dictionary of Scientific Biography*, accessed 18 September 2015, http://www.encyclopedia.com/doc/1G2-2830901390.html.

46 Descartes, *Musicæ Compendium*, 371–78.

47 Gaukroger, *Descartes: An Intellectual Biography*, 99–103.

48 In 1624 Nicolaes Wassenar claimed in Historich Verhal, that Descartes was a Rosicrucian; cf. Gary L. Stewart, "Determining Rosicrucian Affiliation: René Des-Cartes (1596–1650)," in *Rosicrucian Library*, accessed 18 September 2015, http://www.crcsite.org/affiliation.htm.

49 Ibid.

Kepler as well as Mersenne and Gassendi fought against Robert Fludd's Neoplatonism. Descartes, however, remained silent about this issue. We wonder whether he had seen the first parts of Robert Fludd's *Utriusque cosmi historia* (published in 1617 and in spring 1618) during his stay at Breda so that he could have got his inspiration from the circular diagrams in the *Templum Musicæ*. It is certain, however, that he knew Ramon Llull's diagrams.[50]

Jost Bürgi's Calculations

The calculations given above to estimate the angles of the musical intervals in Descartes's circular diagrams were carried out with super-particular ratios: The idea was to express greater intervals, for example the octave $(2 : 1)$, as powers of super-particular ratios from smaller intervals. There is no exact representation in the form $({}^{n+1}\!/_{n})^k = 2$ with integer values *k* and $n > 1$, since *n* and $n+1$ are relatively prime numbers. From the equality $2 = ({}^{81}\!/_{80})^{55.798\cdots}$ the angle $360°/55.798$ of the syntonic comma in Descartes's circular diagrams is obtained. How can the power index be determined with elementary methods?

Jost Bürgi's *Progreß-Tabulen* originate in a similar problem. Bürgi used the number 1.0001 as a base and made a fine-grained table of powers, covering the range of values from 1 to 10. Thereby, he found that $10 = 1.0001^{23027.0022}$. Because the values are given in finite precision, the practical question for Bürgi was, how to calculate them efficiently by hand, so that the results are correct to the number of digits required.[51] In the following section we give a hint of how Bürgi might have created his excellent tables. The basic idea is then used to show how Descartes's angles can be determined quickly in an elementary way.

The "black values" in Bürgi's table are 9-digit values of the integer powers 1.0001^n. They form a geometric sequence with 23,027 values between 1 and 10. The "red values" are the corresponding power indices *n* running from 1 to $23,027$. Although the graph of this geometrical sequence is concave-up, the first 100 values of 1.0001^n rounded to 4 decimal places form an arithmetic sequence with the common difference $d = 0.0001$ (see figure 10a). As can be seen in the first five lines of figure 10a the decimal representations contain the binomial coefficients of Pascal's triangle[52] (see figure 10b) filled up with zeroes. It is probable that Bürgi used binomial coefficients to calculate some well distributed values of the 23,027 entries of his table very accurately and then used interpolation between them.

50 Sasaki, *Descartes's Mathematical Thought*, 105–8.

51 Jörg Waldvogel has shown that Bürgi's table contains no systematic mistakes; see Waldvogel, "Jost Bürgi and the Discovery of the Logarithms," 104–15.

52 Blaise Pascal (1623–1662). The triangular table that incorrectly has Pascal's name was known by Nicolo Tartaglia (1523), Girolamo Cardano (1539), and Michael Stifel (1554) and can be traced back to Greek antiquity; cf. A. W. F. Edwards, "The Arithmetical Triangle," in *Combinatorics: Ancient & Modern*, ed. John J. Watkins and Robin Wilson (New York: Oxford University Press, 2013), 166–80.

1.0001^0	$= 1$	$= 1.0000$
1.0001^1	$= 1.0001$	$= 1.0001$
1.0001^2	$= 1.00020001$	$= 1.0002$
1.0001^3	$= 1.000300030001$	$= 1.0003$
1.0001^4	$= 1.0004000600040001$	$= 1.0004$
...		
1.0001^{100}	$= 1.01004966...$	$= 1.0100$
1.0001^{101}	$= 1.01015067...$	$= 1.0\mathbf{102}$

Fig. 10a. Some values from Bürgi's geometric progression. The second equality sign in each row means "is equal to … when rounded to 4 decimal places."

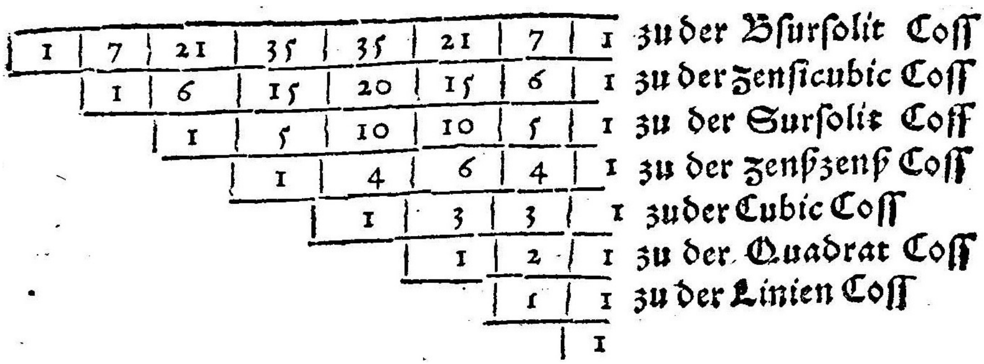


Fig. 10b. Pascal's Triangle according to Michael Stifel (Christoff Rudolff and Michael Stifel, *Die Coss Christoffs Rudolffs: Mit schönen Exempeln der Coss* . . . [Königsperg i. Pr.: Alexander Berm, 1553], 45). The German text on the right gives the then usual designations for powers of the unknown: "Coss" from cosa (it.) = *x*, e.g., *Sursolit* indicates the fifth power of *x*.

How Bürgi Would Have Calculated Descartes's Angles

If Bürgi had needed to calculate the angles for Descartes's circular diagrams, he could have created a new table with the base $b = 1.0125 = {}^{81}/_{80}$, black values running from 1 to 2 and corresponding red values from 0 to 55.798. However, more likely he would have used his own *Progreß-Tabulen* in order to determine Descartes's angles.

To illustrate this, an estimation of the angle for the major third (5 : 4) is determined by using Bürgi's diagram (see figure 11). The digits of the ratio ${}^5/_4 = 1.250$ are found between the black numbers **122**139055 and **128**400937 corresponding to the red numbers 20,000 and 25,000. So the arithmetic mean red 22,500 corresponds well with the major third. The best black number for the octave 2 = 2.000 is black 201368223 corresponding to red 70,000. The ratio between the two red values gives the ratio between the angle of the major third and the angle of the octave resulting in the angle ${}^{22500}/_{70000} \cdot 360° = 115.7°$ for the major third (correct 115.9°).

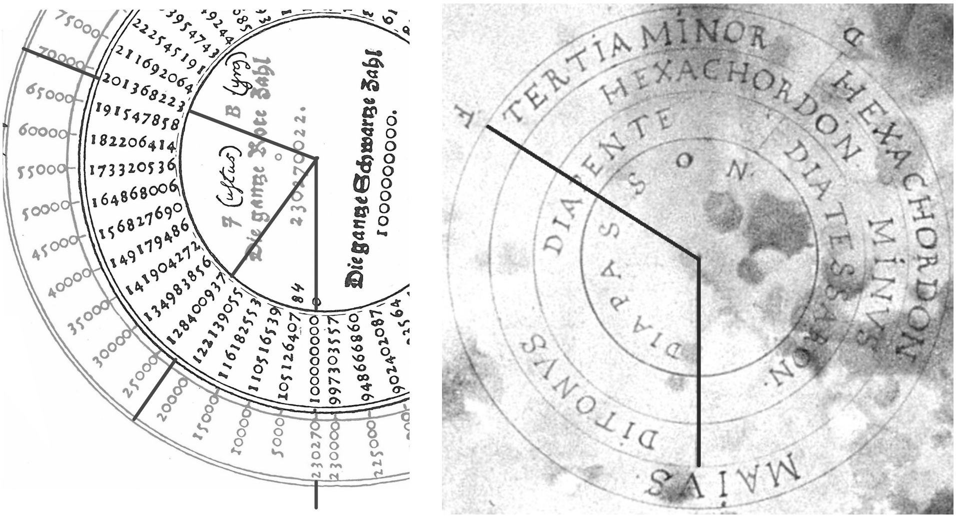


Fig. 11. From the ratio 22.5 : 70 of the (red) numbers the angle of the major third ratio 5 : 4 results in 115.7°. The angle in Beeckman's copy of Descartes's diagram is equal to 120° corresponding to a twelve-tempered major third.

Octagesimal and Heptadecimal Number Systems

The more digits a number system has, the fewer calculations are needed to find accurate values of powers. Therefore, in the Babylonian number system with base 60 iterative tasks such as extracting square roots can be done with fewer steps than in the decimal number system. In the following, some of the above calculations for Descartes's circular diagrams are repeated by using base 80 and base 17 number systems and Pascal's triangle.

We apply an ad hoc notational convention to describe "octagesimal fractions." The "octagesimal digits," the numbers between 0 and 79 are given in decimal notation. Octagesimal digits are separated by colons, a double colon separates the integral from the fractional part of an octagesimal fraction. For example, $1 :: 25 : 3$ stands for $1 + {}^{25}/_{80} + {}^{3}/_{6400}$ and the syntonic comma is represented by $1 :: 1$ because of $1 + {}^{1}/_{80} = {}^{81}/_{80}$.

In order to calculate the size of nine syntonic commas by using the octagesimal numbers we use the ninth row $1, 9, 36, 84, \ldots$ of Pascal's triangle to obtain $({}^{81}/_{80})^9 \approx 1 :: 9 : 36 : 84 = 1 :: 9 : 37 : 4 \approx 1 :: 9 : 37 = 1 + {}^{9}/_{80} + {}^{37}/_{6400} = \frac{6400+9\cdot 80+37}{6400} = \frac{7157}{6400} = 1.11828$. This value is between the whole tones ${}^{10}/_{9} = 1.11111$ and ${}^{9}/_{8} = 1.125$, and very close to two Galilean semitones $({}^{18}/_{17})^2 = \frac{324}{289} = 1.12111$.

Likewise, in order to estimate the size of ten syntonic commas the tenth row in Pascal's triangle $1, 10, 45, 121, \ldots$ gives $({}^{81}/_{80})^{10} = 1 :: 10 : 45 : 121 : \ldots > 1 :: 10 : 46 = 1 + {}^{10}/_{80} + {}^{46}/_{6400} = \frac{6400+10\cdot 80+46}{6400} = \frac{7246}{6400} \approx 1.132 > 1.125 = {}^{9}/_{8}$.

Therefore, the ratio of ten syntonic commas is between $9 : 8 = 1.125$ and $8 : 7 = 1.143$.

Similarly, by using the heptadecimal number system (base 17) and row 12 of Pascal's triangle we can quickly check Vincenzo Galilei's approximation of the octave by twelve semitones sized $18 : 17$: $(^{18}/_{17})^{12} \approx 1 :: 12 : 66 : 220 \approx 1 :: 12 : 66 + 13 = 1 :: 12 : 79 = 1 :: 16 : 11 = 1 + ^{16}/_{17} + ^{11}/_{289} = \frac{572}{289} \approx 1,980$.

Therefore, the octave measures approximately $9 \cdot 6 = 54$ syntonic commas and certainly less than sixty syntonic commas. Alternatively, row 56 of Pascal's triangle gives $(^{81}/_{80})^{56} > 1 :: 56 : 1540 : 27720 : 36729 \approx 2.003$ showing that the octave is a little bit smaller than fifty-six syntonic commas.

Musical Power Tables Derived from the Tetraktys

In music theory, logarithmic thinking is standard since Pythagoras's time.[53] The *tetraktys* is usually depicted in triangular form as shown in figure 12a. The pattern can be interpreted as the number 10 given as a triangular number $10 = 1 + 2 + 3 + 4$. The number pairs from the constituents $1, 2, 3$, and 4 are the Pythagorean consonances. The smallest of these intervals, the ratio $4 : 3$, the *sesquitertia*, the fourth, is used to subdivide the octave into two fourths separated by a major tone resulting in the continuous proportion $6 : 8 : 9 : 12$, a proportion sometimes called *tetraktys of the second kind* (see figure 12b). The ratio $9 : 8$ of the middle terms, a major tone, is not among the Pythagorean consonances, but one of the primary melodic intervals.

In the next step of the derivation, the fourths ($8 : 6$ and $12 : 9$) are divided into two whole steps and a half step $256 : 243$. Eventually, the Pythagorean diatonic scale consists of five whole steps $9 : 8$ and two half steps $256 : 243$. The Pythagorean chromatic genus and an early logarithmic representation of a double octave (Bisdiapason) by Faber Stapulensis is shown in figure 12d.[54]

The intervals of the Pythagorean tone system can be expressed as products of integer powers of the form $2^k \cdot 3^m$. It is therefore straightforward to visualize the Pythagorean ratios as a part of a triangle as in the figures 12c and 13a. The diagram 12c is also given by Kepler,[55] who could have picked it from Fludd's *Templum Musicæ* (1618).

53 Kepler gives a concise summary of Pythagorean music theory: Kepler, *Gesammelte Werke*, 6:95–101.

54 Faber Stapulensis is quoted by Beeckman as early as 1616 (Waard, *Journal tenu par Isaac Beeckman*, 1:84 [fol. 38v, March 1615–6 February 1616]). In 1630, in a letter to Marin Mersenne, Descartes judged Beeckman's knowledge of music theory as very poor by claiming that Beeckman never surpassed what he had learnt from Faber Stapulensis; cf. Buzon, "Science de la nature et théorie musicale chez Isaac Beeckman," 99.

55 Kepler, *Gesammelte Werke*, 3:94–95.

1
1 1
1 1 1
1 1 1 1

Fig. 12a. The Pythagorean tetraktys as a triangular number: $1 + 2 + 3 + 4 = 10$.

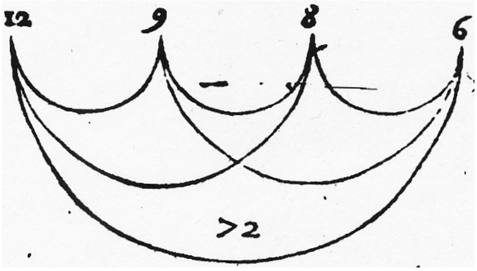


Fig. 12b. Division of the octave $6 : 8 : 9 : 12$ according to Zarlino, *Dimostrationi harmoniche*, 112. A detail of the *School of Athens* (1509–11) by Raphael displays the diagrams of figures 12a and 12b.

1
2 3
4 6 9
8 12 18 27

Fig. 12c. Power table $2^k \cdot 3^m$ for positive integers *k* and *m*.

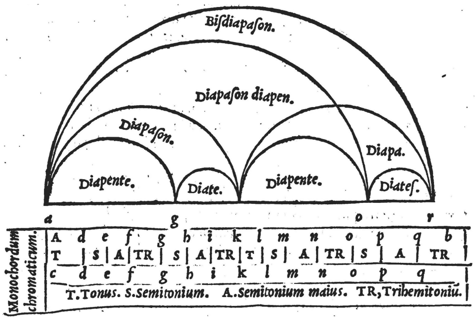


Fig. 12d. Division of the double octave (2 : 3 : 4 : 6 : 8) and the Pythagorean chromatic genus by Stapulensis, *Musica libris quatuor demonstrata*, fol. 32v.

These triangular diagrams were used to illustrate the combinations of binary and ternary durations in the *ars nova*/*ars subtilior* period, and they were rather popular from the late fourteenth into the sixteenth century in British sources.[56] Robert Fludd, in *Templum Musicæ*, has not only copied the diagram from Johannes Torkesey *Declaratio et Scuti* but also copied from its text.[57]

The same kind of diagram is already used in a copy of Boethius's *Arithmetic* of the tenth century, where the underlying number pairs $2/3$, $3/4$, and $4/5$ are in a super-particular ratio (see figure 13b). The numbers in the columns form geometrical sequences with the common ratios $3/2$, $4/3$, and $5/4$.

56 Gilbert Reaney and André Gilles, eds., *Breviarum Regulare Musicæ: MS. Oxford, Bodley 842 (Willelmus)* (Rome: American Institute of Musicology, 1966), 9; Laurie Koehler, *Pythagoreisch-platonische Proportionen in Werken der ars nova und ars subtilior* (Kassel: Bärenreiter, 1990), 1:46–51, 2:1–3.

57 Reaney and Gilles, *Breviarum Regulare Musicæ*, 57.

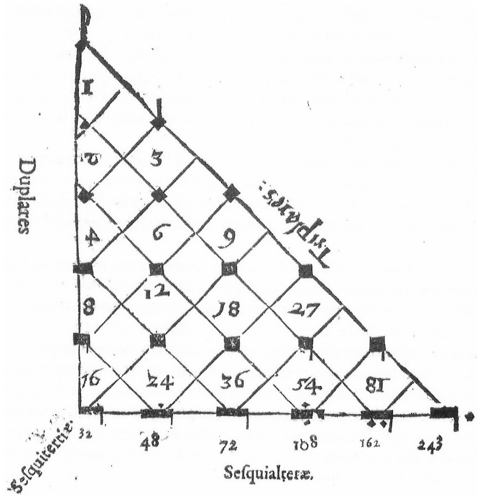


Fig. 13a. Robert Fludd's triangle (Fludd, *Utriusque cosmi*, vol. 1, tract. 2, part 2, lib. 4, *De Temporibus Musicis*, 204).

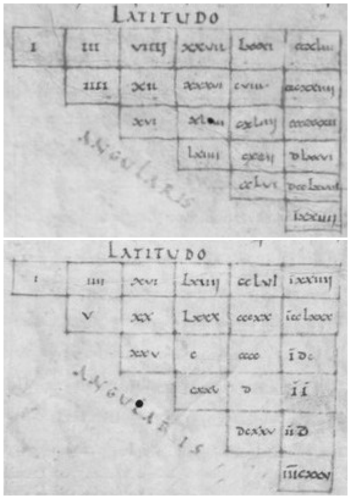


Fig. 13b. Arithmetic triangles by Boethius, *Institutio Arithmetica* (Medeltidhandskrift 1 [Mh 1], Lund University Library, fol. 4v) for $3/4$ and for $4/5$.

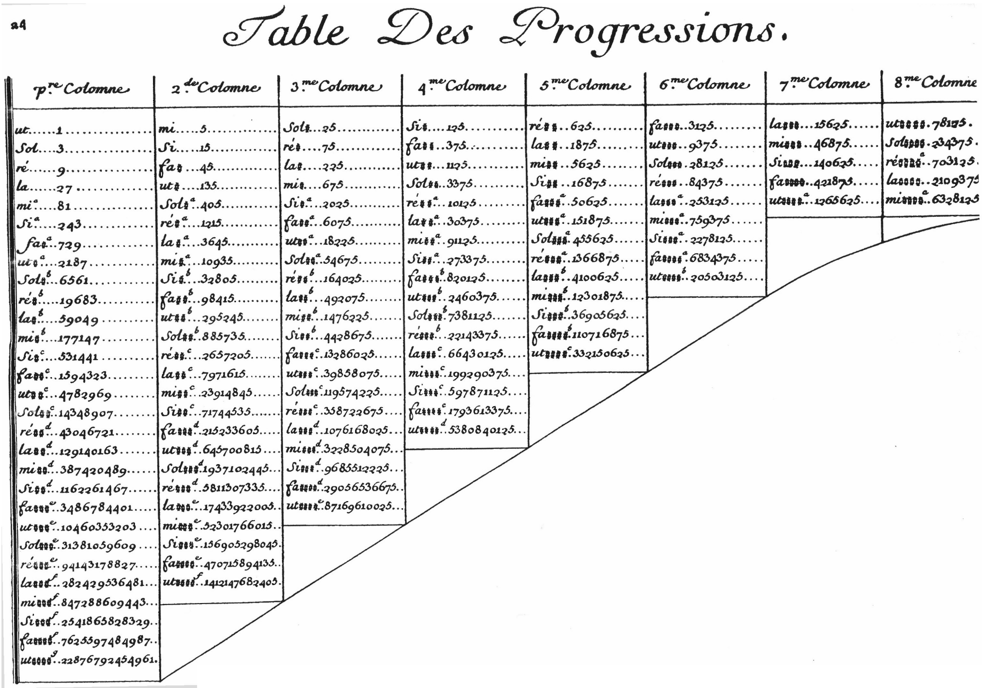
In Fludd's diagram (figure 13a), the numbers on parallel lines through grid points form geometric progressions with the common ratios $2, 3, 3/2$, and $4/3$. The labels added to the triangle make clear that Fludd is aware of this fact. He indicates clearly the directions of the Duplares ($2 : 1$), Triplares ($3 : 1$), Sesquialteræ ($3 : 2$) and Sesquitertiæ ($4 : 3$).

Syntonic Tone System and Higher Dimensional Grids

Similar grids can be created for any pair of relatively prime numbers. Musically meaningful in Western music theory are the primes 3 and 5 to represent the fifth and the major third of the syntonic tuning system. Such grids were studied by Rameau (1726) and Euler (1739) in order to describe the ratios between the pitch classes of syntonic tuning systems in geometrical terms. In order to create syntonic scales within an octave the powers $3^x \cdot 5^y$ are reduced into values between 1 and 2 by adding or subtracting one or several octaves, that is, by multiplying these numbers by suitable powers of 2. In other words, numbers of the form $2^x \cdot 3^y \cdot 5^z$ are studied, resulting in a three-dimensional grid of numbers.[58]

58 For details see Guerino Mazzola, *Geometrie der Töne: Elemente der mathematischen Musiktheorie* (Basel: Birkhäuser, 1990), 63–84. Descartes was aware of the fundamental role of the first three prime numbers for constructing tone systems and called them "numeros sonoros" (Descartes, *Musicæ Compendium*, 105).

Rameau's grid (see figure 14) is obtained by neglecting the octave information *x*. In this grid of pitch classes the consonant major and minor triads form right-angled triangles with legs of unit length.



24

Table Des Progressions.

1.re Colomne	2.de Colomne	3.me Colomne	4.me Colomne	5.me Colomne	6.me Colomne	7.me Colomne	8.me Colomne
ut … 1	mi … 5	Sol# … 25	Si# … 125	ré## … 625	fa### … 3125	la### … 15625	ut#### … 78125
Sol … 3	Si … 15	ré# … 75	fa## … 375	la## … 1875	ut### … 9375	mi### … 46875	Sol#### … 234375
ré … 9	fa# … 45	la# … 225	ut## … 1125	mi## … 5625	Sol### … 28125	Si### … 140625	ré####^{a} … 703125
la … 27	ut# … 135	mi# … 675	Sol## … 3375	Si## … 16875	ré### … 84375	fa#### … 421875	la#### … 2109375
mi^{a} … 81	Sol#^{a} … 405	Si#^{a} … 2025	ré##^{a} … 10125	fa###^{a} … 50625	la###^{a} … 253125	ut####^{a} … 1265625	mi####^{a} … 6328125
Si^{a} … 243	ré#^{a} … 1215	fa##^{a} … 6075	la##^{a} … 30375	ut###^{a} … 151875	mi###^{a} … 759375		
fa#^{a} … 729	la#^{a} … 3645	ut##^{a} … 18225	mi##^{a} … 91125	Sol###^{a} … 455625	Si###^{a} … 2278125		
ut#^{a} … 2187	mi#^{a} … 10935	Sol##^{a} … 54675	Si##^{a} … 273375	ré###^{a} … 1366875	fa####^{a} … 6834375		
Sol#^{b} … 6561	Si#^{b} … 32805	ré##^{b} … 164025	fa###^{b} … 820125	la###^{b} … 4100625	ut####^{b} … 20503125		
ré#^{b} … 19683	fa##^{b} … 98415	la##^{b} … 492075	ut###^{b} … 2460375	mi###^{b} … 12301875			
la#^{b} … 59049	ut##^{b} … 295245	mi##^{b} … 1476225	Sol###^{b} … 7381125	Si###^{b} … 36905625			
mi#^{b} … 177147	Sol##^{b} … 885735	Si##^{b} … 4428675	ré###^{b} … 22143375	fa####^{b} … 110716875			
Si#^{c} … 531441	ré##^{c} … 2657205	fa###^{c} … 13286025	la###^{c} … 66430125	ut####^{c} … 332150625			
fa##^{c} … 1594323	la##^{c} … 7971615	ut###^{c} … 39858075	mi###^{c} … 199290375				
ut##^{c} … 4782969	mi##^{c} … 23914845	Sol###^{c} … 119574225	Si###^{c} … 597871125				
Sol##^{c} … 14348907	Si##^{c} … 71744535	ré###^{c} … 358722675	fa####^{c} … 1793613375				
ré##^{d} … 43046721	fa###^{d} … 215233605	la###^{d} … 1076168025	ut####^{d} … 5380840125				
la##^{d} … 129140163	ut###^{d} … 645700815	mi###^{d} … 3228504075					
mi##^{d} … 387420489	Sol###^{d} … 1937102445	Si###^{d} … 9685512225					
Si##^{d} … 1162261467	ré###^{d} … 5811307335	fa####^{d} … 29056536675					
fa###^{e} … 3486784401	la###^{e} … 17433922005	ut####^{e} … 87169610025					
ut###^{e} … 10460353203	mi###^{e} … 52301766015						
Sol###^{e} … 31381059609	Si###^{e} … 156905298045						
ré###^{e} … 94143178827	fa####^{e} … 470715894135						
la###^{f} … 282429536481	ut####^{f} … 1412147682405						
mi###^{f} … 847288609443							
Si###^{f} … 2541865828329							
fa####^{f} … 7625597484987							
ut####^{g} … 22876792454961							

Fig. 14. Syntonic grid of pitch classes by Rameau (Jean Philippe Rameau, *Nouveau système de musique* [Paris: Ballard, 1726], 24) combining powers of 3 (fifths) vertically and powers of 5 (major thirds) horizontally.

By admitting higher prime numbers in the same way higher dimensional structures are obtained. Therefore, some authors, such as Christiaan Huygens in the seventeenth and Martin Vogel in the twentieth century,[59] have suggested an additional musical dimension for powers of 7. Kepler, in possession of Gauss's result, however, would have taken 17 instead of 7.

Conclusions

Two ways of representing frequency ratios used in the early seventeenth century have been at the center of this essay. The common property of the related diagrams is the use of spatial distance for measuring musical intervals. Whereas the two dimensional "Cartesian representation" used by Robert Fludd within a discrete straight

59 Martin Vogel, *Die Lehre von den Tonbeziehungen* (Bonn: Orpheus, 1975).

line coordinate system can be traced back to Boethius, Descartes's circular diagrams do not have early forerunners, but there are apparent similarities to one of Robert Fludd's musical diagrams and to a mathematical diagram by Jost Bürgi. The latter was printed two years after Descartes composed his *Musicæ Compendium*, whereas the former had already occurred in print eight months earlier.

It has been highlighted that thinking in musical intervals and scales is genuinely logarithmic and predates the invention of logarithms as calculation techniques by the end of the sixteenth century in Scotland and Switzerland by John Napier and Jost Bürgi. The equivalency of adding musical intervals and multiplying their frequency ratios is a music theoretical truism, which is manifest in the traditional Latin terms *ditonus*, *tritonus*, *bisdiapason*, etc. In traditional music theory and arithmetic, the standard operation on ratios is multiplication and not addition. This is the state of the art already in Boethius's reception of Greek music theory and arithmetic as handed down through the Middle Ages.

The most remarkable element of Descartes diagrams is not the logarithmic representation of musical interval size per se, but its combination with a circular topology capturing the octave similarity as a perceptual phenomenon. These diagrams visualize pitch classes as locations on the circle line and intervals as central angles of circular sectors. In one of the diagrams (figures 3a and 3b) it is shown that the set of consonances described by Zarlino's *senario* is closed under octave addition as well as under octave complements. The diagram with the hexachords (figure 3d) equates transposition of scales with rotation about the center of the circle. It is suited to show the potential infinity[60] of the syntonic diatonic tone system.

It remains unclear whether Descartes knew Fludd's circular diagrams when he composed the *Musicæ Compendium*. Basic knowledge of music theory and mathematics and a glance at Ramon Llull's diagrams might have been all Descartes needed in order to develop his manner of representing pitch and interval classes.

There are no early three dimensional visualizations of the syntonic tone system. In the pitch grids of the eighteenth century by Rameau and Euler built from the same principle as Boethius's triangles, the octaves are reduced to points. These points symbolize classes of pitches with an unspecified octave. A "natural" geometrical representation of the syntonic tone system taking account of the octave in both directions, fifths and thirds, would be a torus, the combination of two Cartesian pitch circles.[61]

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60 In a sketch Isaac Newton generalizes the diagram to five concentric circles of diatonic scales; Penelope Gouk, *Music, Science, and Natural Magic in Seventeenth-Century England* (New Haven, CT: Yale University Press, 1999), 140; Wardhaugh, *The "Compendium Musicæ" of René Descartes*, 85–128.

61 Roger Shepard even proposes a four-dimensional pitch model (double helix on a helical cylinder): Roger N. Shepard, "Pitch Perception and Measurement," in *Music, Cognition, and Computerized Sound: An Introduction to Psychoacoustics*, ed. Perry R. Cook (Cambridge, MA: MIT Press, 2001), 163.