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The Complete Proof of the Riemann Hypothesis

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Abstract Robin criterion states that the Riemann Hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. We prove that the Robin inequality is true for all $n > 5040$ which are not divisible by any prime number between 2 and 953. Using this result, we show there is a contradiction just assuming the possible smallest counterexample $n > 5040$ of the Robin inequality. In this way, we prove that the Robin inequality is true for all $n > 5040$ and thus, the Riemann Hypothesis is true.

Keywords Riemann hypothesis · Robin inequality · sum-of-divisors function · prime numbers

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1 Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [7]. As usual $\sigma(n)$ is the sum-of-divisors function of n [3]:

$$\sum_{d \mid n} d$$

where $d \mid n$ means the integer d divides to n and $d \nmid n$ means the integer d does not divide to n. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

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The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and log is the natural logarithm. The importance of this property is:

Theorem 1.1 $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *if and only if the Riemann Hypothesis is true [7].*

It is known that $\mathsf{Robins}(n)$ holds for many classes of numbers n.

Theorem 1.2 $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *that are not divisible by* 2 *[3].*

On the one hand, we prove that $\mathsf{Robins}(n)$ holds for all $n > 5040$ that are not divisible by any prime number between 3 and 953. Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [3]. A natural number n is called superabundant precisely when, for all $m < n$

$$f(m) < f(n).$$

Theorem 1.3 *If n is superabundant, then n is an Hardy-Ramanujan integer [2].*

Theorem 1.4 *The smallest counterexample of the Robin inequality greater than* 5040 *must be a superabundant number [1].*

On the other hand, we prove the nonexistence of such counterexample and therefore, the Riemann Hypothesis is true.

2 A Central Lemma

These are known results:

Lemma 2.1 *[3]. For* $n > 1$*:*

$$f(n) < \prod_{q \mid n} \frac{q}{q-1}. \tag{2.1}$$

Lemma 2.2 *[4].*

$$\prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \tag{2.2}$$

The following is a key lemma. It gives an upper bound on $f(n)$ that holds for all n. The bound is too weak to prove $\mathsf{Robins}(n)$ directly, but is critical because it holds for all n. Further the bound only uses the primes that divide n and not how many times they divide n.

Lemma 2.3 *Let* $n > 1$ *and let all its prime divisors be* $q_1 < \cdots < q_m$*. Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

Proof We use that lemma 2.1:

$$f(n) < \prod_{i=1}^{m} \frac{q_i}{q_i - 1}.$$

Now for $q > 1$,

$$\frac{1}{1 - \frac{1}{q^2}} = \frac{q^2}{q^2 - 1}.$$

So

$$\frac{1}{1 - \frac{1}{q^2}} \times \frac{q+1}{q} = \frac{q^2}{q^2 - 1} \times \frac{q+1}{q} = \frac{q}{q-1}.$$

Then by lemma 2.2,

$$\prod_{i=1}^{m} \frac{1}{1 - \frac{1}{q_i^2}} < \zeta(2) = \frac{\pi^2}{6}.$$

Putting this together yields the proof:

$$\begin{aligned} f(n) &< \prod_{i=1}^{m} \frac{q_i}{q_i - 1} \\ &\leq \prod_{i=1}^{m} \frac{1}{1 - \frac{1}{q_i^2}} \times \frac{q_i + 1}{q_i} \\ &< \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}. \end{aligned}$$

3 About the p-adic order

In basic number theory, for a given prime number p, the p-adic order of a natural number n is the highest exponent $\nu_p \geq 1$ such that p^{ν_p} divides n. This is a known result:

Lemma 3.1 *In general, we know that* $\mathsf{Robins}(n)$ *holds for a natural number* $n > 5040$ *that satisfies either* $\nu_2(n) \leq 19$, $\nu_3(n) \leq 12$ *or* $\nu_7(n) \leq 6$, *where* $\nu_p(n)$ *is the p-adic order of n [5].*

We know the following lemmas:

Lemma 3.2 *[5]. Let* $\prod_{i=1}^{m} q_i^{a_i}$ *be the representation of n as a product of primes* $q_1 < \cdots < q_m$ *with natural numbers as exponents* $a_1, \ldots, a_m$. *Then,*

$$f(n) = \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \right) \times \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i + 1}} \right).$$

Lemma 3.3 *[5]. Let $n > e^{e^{23.762143}}$ and let all its prime divisors be $q_1 < \cdots < q_m$, then*

$$\left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) < \frac{1771561}{1771560} \times e^{\gamma} \times \log\log n.$$

Lemma 3.4 $\mathsf{Robins}(n)$ *holds for all* $10^{10^{10}} \geq n > 5040$ *[5].*

Putting together all these results, then we obtain that

Lemma 3.5 $\mathsf{Robins}(n)$ *holds for* $n > 5040$ *when* $\nu_{31}(n) \leq 3$.

Proof From lemma 3.2, we note that

$$f(n) = \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) \times \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \leq \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) \times \left(1 - \frac{1}{31^{\nu_{31}(n)+1}}\right)$$

when $\nu_{31}(n) \leq 3$. We only need to look at the case where $\nu_{31}(n) = 3$ since the weaker cases follow because

$$\left(1 - \frac{1}{31^{1+1}}\right) < \left(1 - \frac{1}{31^{2+1}}\right) < \left(1 - \frac{1}{31^{3+1}}\right).$$

In this way, we obtain that

$$f(n) \leq \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) \times \left(1 - \frac{1}{31^{3+1}}\right) = \frac{923520}{923521} \times \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right)$$

when $\nu_{31}(n) \leq 3$. With lemma 3.3, we have for $n > e^{e^{23.762143}}$

$$\frac{923520}{923521} \times \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) < \frac{923520}{923521} \times \frac{1771561}{1771560} \times e^{\gamma} \times \log\log n < e^{\gamma} \times \log\log n$$

since $\frac{923520}{923521} \times \frac{1771561}{1771560} < 1$. In light of lemma 3.4 and the fact that $e^{e^{23.762143}} < 10^{10^{10}}$, we then conclude that $\mathsf{Robins}(n)$ holds for $n > 5040$ when $\nu_{31}(n) \leq 3$.

4 A Particular Case

We can easily prove that $\mathsf{Robins}(n)$ is true for certain kind of numbers:

Lemma 4.1 $\mathsf{Robins}(n)$ *holds for* $n > 5040$ *when $q \leq 7$, where q is the largest prime divisor of n.*

Proof Let $n > 5040$ and let all its prime divisors be $q_1 < \cdots < q_m \leq 5$, then we need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

that is true when

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq e^{\gamma} \times \log\log n$$

according to the lemma 2.1. For $q_1 < \cdots < q_m \leq 5$,

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^{\gamma} \times \log\log(5040) \approx 3.81.$$

However, we know for $n > 5040$

$$e^{\gamma} \times \log\log(5040) < e^{\gamma} \times \log\log n$$

and therefore, the proof is complete when $q_1 < \cdots < q_m \leq 5$. The remaining case is for $n > 5040$ when all its prime divisors are $q_1 < \cdots < q_m \leq 7$. $\mathsf{Robins}(n)$ holds for $n > 5040$ when $\nu_7(n) \leq 6$ according to the lemma 3.1 [5]. Hence, it is enough to prove this for those natural numbers $n > 5040$ when $7^7 \mid n$. For $q_1 < \cdots < q_m \leq 7$,

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5 \times 7}{1 \times 2 \times 4 \times 6} = 4.375 < e^{\gamma} \times \log\log(7^7) \approx 4.65.$$

However, for $n > 5040$ and $7^7 \mid n$, we know that

$$e^{\gamma} \times \log\log(7^7) \leq e^{\gamma} \times \log\log n$$

and as a consequence, the proof is complete when $q_1 < \cdots < q_m \leq 7$.

5 A Better Bound

This is a known result:

Lemma 5.1 *[8]. For $x > 1$:*

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + B + \frac{1}{\log^2 x} \tag{5.1}$$

where

$$B = 0.2614972128\cdots$$

denotes the (Meissel-)Mertens constant [8].

We show a better result:

Lemma 5.2 *For $x \geq 11$, we have*

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - 0.12.$$

Proof Let's define $H = \gamma - B$. The lemma 5.1 is the same as

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - (H - \frac{1}{\log^2 x}).$$

For $x \geq 11$,

$$(H - \frac{1}{\log^2 x}) > (0.31 - \frac{1}{\log^2 11}) > 0.12$$

and thus,

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - (H - \frac{1}{\log^2 x}) < \log\log x + \gamma - 0.12.$$

6 On a Square Free Number

We know the following results:

Lemma 6.1 *[3]. For* $0 < a < b$*:*

$$\frac{\log b - \log a}{b-a} = \frac{1}{(b-a)} \int_a^b \frac{dt}{t} > \frac{1}{b}. \tag{6.1}$$

Lemma 6.2 *[3]. For* $q > 0$*:*

$$\log(q+1) - \log q = \int_q^{q+1} \frac{dt}{t} < \frac{1}{q}. \tag{6.2}$$

We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$ [3]. $\mathsf{Robins}(n)$ holds for all $n > 5040$ that are square free [3].

Lemma 6.3 *For a square free number*

$$n = q_1 \times \cdots \times q_m$$

such that $q_1 < q_2 < \cdots < q_m$ *are odd prime numbers,* $q_m \geq 11$ *and* $3 \nmid n$*, then:*

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^{\gamma} \times n \times \log\log(2^{19} \times n).$$

Proof By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n [3]. Put $\omega(n) = m$ [3]. We need to prove the assertion for those integers with $m = 1$. From a square free number n, we obtain

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \cdots \times (q_m + 1) \tag{6.3}$$

when $n = q_1 \times q_2 \times \cdots \times q_m$ [3]. In this way, for every prime number $q_i \geq 11$, then we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{q_i}) \leq e^{\gamma} \times \log\log(2^{19} \times q_i). \tag{6.4}$$

For $q_i = 11$, we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{11}) \leq e^{\gamma} \times \log\log(2^{19} \times 11)$$

is actually true. For another prime number $q_i > 11$, we have

$$(1 + \frac{1}{q_i}) < (1 + \frac{1}{11})$$

and

$$\log\log(2^{19} \times 11) < \log\log(2^{19} \times q_i)$$

which clearly implies that the inequality (6.4) is true for every prime number $q_i \geq 11$. Now, suppose it is true for $m-1$, with $m \geq 2$ and let us consider the assertion for those square free n with $\omega(n) = m$ [3]. So let $n = q_1 \times \cdots \times q_m$ be a square free number and assume that $q_1 < \cdots < q_m$ for $q_m \geq 11$.

Case 1: $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1+1) \times \cdots \times (q_{m-1}+1) \leq e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

and hence

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1+1) \times \cdots \times (q_{m-1}+1) \times (q_m+1) \leq$$

$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times (q_m+1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

when we multiply the both sides of the inequality by (q_m+1). We want to show

$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times (q_m+1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}) \leq$$

$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times q_m \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = e^{\gamma} \times n \times \log\log(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) \geq (q_m+1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} \geq$$

$$\frac{\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}.$$

We can apply the inequality in lemma 6.1 just using $b = \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ and $a = \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$. Certainly, we have

$$\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) =$$

$$\log \frac{2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \cdots \times q_{m-1}} = \log q_m.$$

In this way, we obtain

$$\frac{q_m \times (\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} >$$

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)}.$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)} \geq \frac{\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}$$

which is trivially true for $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ [3].

Case 2: $q_m < \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

We need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^{\gamma} \times \log\log(2^{19} \times n).$$

We know $\frac{3}{2} < 1.503 < \frac{4}{2.66}$. Nevertheless, we could have

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times \sigma(n)}{3 \times n} \times \frac{\pi^2}{2 \times 2.66}$$

and therefore, we only need to prove

$$\frac{\sigma(3 \times n)}{3 \times n} \times \frac{\pi^2}{5.32} \leq e^{\gamma} \times \log\log(2^{19} \times n)$$

where this is possible because of $3 \nmid n$. If we apply the logarithm to the both sides of the inequality, then we obtain

$$\log(\frac{\pi^2}{5.32}) + (\log(3+1) - \log 3) + \sum_{i=1}^{m} (\log(q_i + 1) - \log q_i) \leq \gamma + \log\log\log(2^{19} \times n).$$

In addition, note that $\log(\frac{\pi^2}{5.32}) < \frac{1}{2} + 0.12$. However, we know

$$\gamma + \log\log q_m < \gamma + \log\log\log(2^{19} \times n)$$

since $q_m < \log(2^{19} \times n)$. We use that lemma 6.2 for each term $\log(q+1) - \log q$ and thus,

$$0.12 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq 0.12 + \sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log\log q_m$$

where $q_m \geq 11$. Hence, it is enough to prove

$$\sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log\log q_m - 0.12$$

but this is true according to the lemma 5.2 for $q_m \geq 11$. In this way, we finally show the lemma is indeed satisfied.

7 Robin on Divisibility

Robins(n) holds for every $n > 5040$ that is not divisible by 2 [3]. We extend this property to other prime numbers:

Lemma 7.1 Robins(n) *holds for all* $n > 5040$ *when* $3 \nmid n$. *More precisely: every possible counterexample* $n > 5040$ *of the Robin inequality must comply with* $(2^{20} \times 3^{13}) \mid n$.

Proof We will check the Robin inequality is true for every natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that $q_1, q_2, \cdots, q_m$ are distinct prime numbers, $a_1, a_2, \cdots, a_m$ are natural numbers and $3 \nmid n$. We know this is true when the greatest prime divisor of $n > 5040$ is lesser than or equal to 7 according to the lemma 4.1. Therefore, the remaining case is when the greatest prime divisor of $n > 5040$ is greater than or equal to 11. We need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^{\gamma} \times \log\log n$$

according to the lemma 2.3. Using the formula (6.3) for the square free numbers, then we obtain that is equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^{\gamma} \times \log\log n$$

where $n' = q_1 \times \cdots \times q_m$ is the square free kernel of the natural number n [3]. The Robin inequality has been proved for all integers n not divisible by 2 (which are bigger than 10) [3]. Hence, we only need to prove the Robin inequality is true when $2 \mid n'$. In addition, we know that $\mathsf{Robins}(n)$ holds for every $n > 5040$ when $\nu_2(n) \leq 19$ according to the lemma 3.1 [5]. Consequently, we only need to prove that $\mathsf{Robins}(n)$ holds for $n > 5040$ when $2^{20} \mid n$ and thus,

$$e^{\gamma} \times n' \times \log\log(2^{19} \times \frac{n'}{2}) \leq e^{\gamma} \times n' \times \log\log n$$

because of $2^{19} \times \frac{n'}{2} \leq n$ where $2^{20} \mid n$ and $2 \mid n'$. So,

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^{\gamma} \times n' \times \log\log(2^{19} \times \frac{n'}{2}).$$

According to the formula (6.3) for the square free numbers and $2 \mid n'$, then,

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times 2 \times \frac{n'}{2} \times \log\log(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times \frac{n'}{2} \times \log\log(2^{19} \times \frac{n'}{2})$$

where this is true according to the lemma 6.3 when $3 \nmid \frac{n'}{2}$. In addition, we know that $\mathsf{Robins}(n)$ holds for every $n > 5040$ when $\nu_3(n) \leq 12$ according to the lemma 3.1 [5]. Hence, we only need to prove that $\mathsf{Robins}(n)$ holds for every $n > 5040$ when $2^{20} \mid n$ and $3^{13} \mid n$. To sum up, the proof is complete.

Let's state the following known properties:

Lemma 7.2 *$\sigma(n)$ and $f(n)$ are multiplicatives [3]. Besides, for a prime number q and a positive integer $a \geq 0$, we have that $\sigma(q^a) = \frac{q^{a+1}-1}{q-1}$ [3]. We know that $f(q^a) < \frac{q}{q-1}$ and $f(q^{a+1}) > f(q^a)$ for all primes q and all $a \geq 0$.*

Lemma 7.3 Robins(n) *holds for all $n > 5040$ when $5 \nmid n$ or $7 \nmid n$.*

Proof We need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

when $(2^{20} \times 3^{13}) \mid n$. Suppose that $n = 2^a \times 3^b \times m$, where $a \geq 20$, $b \geq 13$, $2 \nmid m$, $3 \nmid m$ and $5 \nmid m$ or $7 \nmid m$. Therefore, we need to prove

$$f(2^a \times 3^b \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times m).$$

We know

$$f(2^a \times 3^b \times m) = f(3^b) \times f(2^a \times m)$$

since f is multiplicative [3]. In addition, we know $f(3^b) < \frac{3}{2}$ for every natural number b [3]. In this way, we have

$$f(3^b) \times f(2^a \times m) < \frac{3}{2} \times f(2^a \times m).$$

However, that would be equivalent to

$$\frac{3}{2} \times f(2^a \times m) = \frac{9}{8} \times f(3) \times f(2^a \times m) = \frac{9}{8} \times f(2^a \times 3 \times m)$$

where $f(3) = \frac{4}{3}$ since f is multiplicative [3]. Nevertheless, we have

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(5) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 5 \times m)$$

and

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(7) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 7 \times m)$$

where $5 \nmid m$ or $7 \nmid m$, $f(5) = \frac{6}{5}$ and $f(7) = \frac{8}{7}$. We know the Robin inequality is true for $2^a \times 3 \times 5 \times m$ and $2^a \times 3 \times 7 \times m$ when $a \geq 20$, since this is true for every natural number $n > 5040$ when $\nu_3(n) \leq 12$ according to the lemma 3.1 [5]. Hence, we would have

$$f(2^a \times 3 \times 5 \times m) < e^{\gamma} \times \log\log(2^a \times 3 \times 5 \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times m)$$

and

$$f(2^a \times 3 \times 7 \times m) < e^{\gamma} \times \log\log(2^a \times 3 \times 7 \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times m)$$

when $b \geq 13$.

Lemma 7.4 Robins(n) *holds for all $n > 5040$ when a prime number $11 \leq q \leq 47$ complies with $q \nmid n$.*

Proof We know that $\mathsf{Robins}(n)$ holds for every $n > 5040$ when $\nu_7(n) \leq 6$ according to the lemma 3.1 [5]. We need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

when $(2^{20} \times 3^{13} \times 7^{7}) \mid n$. Suppose that $n = 2^a \times 3^b \times 7^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 7$, $2 \nmid m$, $3 \nmid m$, $7 \nmid m$, $q \nmid m$ and $11 \leq q \leq 47$. Therefore, we need to prove

$$f(2^a \times 3^b \times 7^c \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times 7^c \times m).$$

We know

$$f(2^a \times 3^b \times 7^c \times m) = f(7^c) \times f(2^a \times 3^b \times m)$$

since f is multiplicative [3]. In addition, we know $f(7^c) < \frac{7}{6}$ for every natural number c [3]. In this way, we have

$$f(7^c) \times f(2^a \times 3^b \times m) < \frac{7}{6} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{7}{6} \times f(2^a \times 3^b \times m) = \frac{49}{48} \times f(7) \times f(2^a \times 3^b \times m) = \frac{49}{48} \times f(2^a \times 3^b \times 7 \times m)$$

where $f(7) = \frac{8}{7}$ since f is multiplicative [3]. In addition, we know

$$\frac{49}{48} \times f(2^a \times 3^b \times 7 \times m) < f(q) \times f(2^a \times 3^b \times 7 \times m) = f(2^a \times 3^b \times 7 \times q \times m)$$

where $q \nmid m$, $f(q) = \frac{q+1}{q}$ and $11 \leq q \leq 47$. Nevertheless, we know the Robin inequality is true for $2^a \times 3^b \times 7 \times q \times m$ when $a \geq 20$ and $b \geq 13$, since this is true for every natural number $n > 5040$ when $\nu_7(n) \leq 6$ according to the lemma 3.1 [5]. Hence, we would have

$$\begin{aligned} f(2^a \times 3^b \times 7 \times q \times m) &< e^{\gamma} \times \log\log(2^a \times 3^b \times 7 \times q \times m) \\ &< e^{\gamma} \times \log\log(2^a \times 3^b \times 7^c \times m) \end{aligned}$$

when $c \geq 7$ and $11 \leq q \leq 47$.

Lemma 7.5 $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when a prime number* $53 \leq q \leq 953$ *complies with* $q \nmid n$.

Proof We know that $\mathsf{Robins}(n)$ holds for every $n > 5040$ when $\nu_{31}(n) \leq 3$ according to the lemma 3.5. We need to prove that

$$f(n) < e^{\gamma} \times \log\log n$$

when $(2^{20} \times 3^{13} \times 31^{4}) \mid n$. Suppose that $n = 2^a \times 3^b \times 31^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 4$, $2 \nmid m$, $3 \nmid m$, $31 \nmid m$, $q \nmid m$ and $53 \leq q \leq 953$. Therefore, we need to prove that

$$f(2^a \times 3^b \times 31^c \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times 31^c \times m).$$

We know that

$$f(2^a \times 3^b \times 31^c \times m) = f(31^c) \times f(2^a \times 3^b \times m)$$

since f is multiplicative [3]. In addition, we know that $f(31^c) < \frac{31}{30}$ for every natural number c [3]. In this way, we have that

$$f(31^c) \times f(2^a \times 3^b \times m) < \frac{31}{30} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{31}{30} \times f(2^a \times 3^b \times m) = \frac{961}{960} \times f(31) \times f(2^a \times 3^b \times m) = \frac{961}{960} \times f(2^a \times 3^b \times 31 \times m)$$

where $f(31) = \frac{32}{31}$ since f is multiplicative [3]. In addition, we know that

$$\frac{961}{960} \times f(2^a \times 3^b \times 31 \times m) < f(q) \times f(2^a \times 3^b \times 31 \times m) = f(2^a \times 3^b \times 31 \times q \times m)$$

where $q \nmid m$, $f(q) = \frac{q+1}{q}$ and $53 \leq q \leq 953$. Nevertheless, we know the Robin inequality is true for $2^a \times 3^b \times 31 \times q \times m$ when $a \geq 20$ and $b \geq 13$, since this is true for every natural number $n > 5040$ when $\nu_{31}(n) \leq 3$ according to the lemma 3.5. Hence, we would have that

$$\begin{aligned} f(2^a \times 3^b \times 31 \times q \times m) &< e^{\gamma} \times \log\log(2^a \times 3^b \times 31 \times q \times m) \\ &< e^{\gamma} \times \log\log(2^a \times 3^b \times 31^c \times m) \end{aligned}$$

when $c \geq 4$ and $53 \leq q \leq 953$.

8 Helpful Lemmas

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

where $q \leq x$ means all the prime numbers q that are less than or equal to x.

Lemma 8.1 *[8]. For $x \geq 41$:*

$$\theta(x) > (1 - \frac{1}{\log(x)}) \times x.$$

Besides, we know that

Lemma 8.2 *[8]. For $x \geq 286$:*

$$\prod_{q \leq x} \frac{q}{q-1} < e^{\gamma} \times (\log x + \frac{1}{2 \times \log(x)}).$$

For the counting prime function $\pi(x)$, we know that

Lemma 8.3 *[8]. For $x \geq 17$:*

$$\frac{x}{\log x} < \pi(x) < 1.25506 \times \frac{x}{\log x}.$$

The following lemma is crucial in our proof

Lemma 8.4 *[6]. For $x > -1$:*

$$\frac{x}{x+1} \leq \log(1+x) \leq x.$$

The smallest counterexample of the Robin inequality greater than 5040 complies with

Lemma 8.5 *If $n > 5040$ is the smallest counterexample of the Robin inequality, then $q < \log n$ where q denotes the largest prime factor of n [3].*

We show some tools that could help us in the final proof.

Lemma 8.6 *Let $q \geq 2$ be a prime and let $b \geq 0$ be a positive integer. If $q^a \| n$, then*

$$f(q^b \times n) = f(n) \times \frac{q^{a+b+1} - 1}{q^{a+b+1} - q^b}$$

where $q^a \| n$ signifies that q^a divides n, but q^{a+1} does not divide n.

Proof We assume that $q^a \| n$. Since $\sigma(n)$ and $f(n)$ are multiplicatives according to the lemma 7.2, then we would only need to study $f(q^{a+b})$ where we know from lemma 7.2 that $\sigma(q^a) = \frac{q^{a+1}-1}{q-1}$. Then,

$$\begin{aligned}
f(q^{a+b}) &= \frac{q^{a+b+1} - 1}{q^{a+b} \times (q-1)} \times \frac{q^{a+1} - 1}{q^a \times (q-1)} \times \frac{q^a \times (q-1)}{q^{a+1} - 1} \\
&= f(q^a) \times \frac{q^{a+b+1} - 1}{q^{a+b} \times (q-1)} \times \frac{q^a \times (q-1)}{q^{a+1} - 1} \\
&= f(q^a) \times \frac{q^{a+b+1} - 1}{q^b} \times \frac{1}{q^{a+1} - 1} \\
&= f(q^a) \times \frac{q^{a+b+1} - 1}{q^{a+b+1} - q^b}.
\end{aligned}$$

Let's see another inequalities:

Lemma 8.7 *If $n > 5040$ is the smallest counterexample of the Robin inequality, then*

$$\frac{\log\log n}{\log q} < \left(1 + \frac{1}{2 \times \log^2 q}\right)$$

and

$$\frac{\log\log\log n}{\log q} < \frac{\log\log q}{\log q} + \frac{1}{2 \times \log^3 q}$$

when we assume that $q \geq 953$ is the largest prime factor of n.

Proof Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of the first m consecutive primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. According to the theorems 1.3 and 1.4, the primes $q_1 < \cdots < q_m$ must be the first m consecutive primes since $n > 5040$ should be an Hardy-Ramanujan integer. We assume that $q_m \geq 953$. For $q_m \geq 953$, we have that

$$\prod_{q \leq q_m} \frac{q}{q-1} < e^{\gamma} \times (\log q_m + \frac{1}{2 \times \log(q_m)})$$

because of the lemma 8.2. We use that lemma 2.1 to show that

$$e^{\gamma} \times \log\log n \leq f(n) < \prod_{q \leq q_m} \frac{q}{q-1} < e^{\gamma} \times (\log q_m + \frac{1}{2 \times \log(q_m)})$$

since we assume that n is a counterexample of the Robin inequality. In this way, we obtain that

$$\log\log n < (\log q_m + \frac{1}{2 \times \log(q_m)})$$

which is the same as

$$\frac{\log\log n}{\log q_m} < (1 + \frac{1}{2 \times \log^2(q_m)}).$$

Besides, if we apply the logarithm to the both sides of the inequality, then

$$\log\log\log n < \log\left(\log q_m \times (1 + \frac{1}{2 \times \log^2(q_m)})\right)$$

that is equivalent to

$$\log\log\log n < \log\log q_m + \log(1 + \frac{1}{2 \times \log^2(q_m)}).$$

We use that lemma 8.4 to show that

$$\log(1 + \frac{1}{2 \times \log^2(q_m)}) \leq \frac{1}{2 \times \log^2(q_m)}.$$

Therefore, we finally have that

$$\frac{\log\log\log n}{\log q_m} < \frac{\log\log q_m}{\log q_m} + \frac{1}{2 \times \log^3 q_m}.$$

Let's show another inequality

Lemma 8.8 *For $q_m \geq 953$, we have that*

$$\sum_{q \leq q_m} \frac{\log\log q}{q_m} > \frac{1}{\log q_m}$$

Proof This is the same as

$$\sum_{q \leq q_m} \log\log q > \frac{q_m}{\log q_m}.$$

According to the lemma 8.3, it is enough to show that

$$\sum_{q \leq q_m} \log\log q \geq \pi(q_m) > \frac{q_m}{\log q_m}$$

when $q_m \geq 953$. We know that for all primes $q_i > q_m \geq 953$, then

$$\log\log q_i > 1.$$

Hence, it is enough to prove that

$$\sum_{q \leq q_m} \log\log q \geq \sum_{q \leq 953} \log\log q \geq \pi(953).$$

We compute that

$$\sum_{q \leq 953} \log\log q > 274.$$

However, we know that $q_{274} = 1759 > 953$ and thus,

$$274 \geq \pi(953).$$

Therefore, the proof is done.

9 Proof of Main Theorems

Theorem 9.1 $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when a prime number* $q \leq 953$ *complies with* $q \nmid n$.

Proof This is a compendium of the results from the theorem 1.2 and the lemmas 7.1, 7.3, 7.4 and 7.5.

Theorem 9.2 *Let* $\prod_{i=1}^{m} q_i^{a_i}$ *be the representation of n as a product of the first m consecutive primes* $q_1 < \cdots < q_m$ *with natural numbers as exponents* $a_1, \ldots, a_m$. *We obtain a contradiction just assuming that* $n > 5040$ *is the smallest integer such that* $\mathsf{Robins}(n)$ *does not hold.*

Proof According to the theorems 1.3 and 1.4, the primes $q_1 < \cdots < q_m$ must be the first m consecutive primes since $n > 5040$ should be an Hardy-Ramanujan integer. From the theorem 9.1, we know that necessarily $q_m \geq 953$. Under our assumption, we know that

$$f(n) \geq e^{\gamma} \times \log\log n.$$

For $b = 1$ and the lemma 8.6, we know that

$$f(n) = f(q_i \times m) = f(m) \times \frac{q_i^{a_i+2} - 1}{q_i^{a_i+2} - q_i}$$

for every prime q_i that divides n where $m = \frac{n}{q_i}$. If we subtract $f(m)$ to both sides of the inequality, then we obtain that

$$f(n) - f(m) \geq e^{\gamma} \times \log\log n - f(m).$$

Then,

$$\begin{aligned}
f(n) - f(m) &= f(m) \times \frac{q_i^{a_i+2} - 1}{q_i^{a_i+2} - q_i} - f(m) \\
&= f(m) \times \left(\frac{q_i^{a_i+2} - 1}{q_i^{a_i+2} - q_i} - 1 \right) \\
&= f(m) \times (\frac{q_i - 1}{q_i^{a_i+2} - q_i}) \\
&= f(m) \times (\frac{q_i - 1}{q_i \times (q_i^{a_i+1} - 1)}) \\
&= f(m) \times (\frac{1}{q_i \times \sigma(q_i^{a_i})}) \\
&= f(m') \times f(q_i^{a_i-1}) \times (\frac{1}{q_i \times \sigma(q_i^{a_i})}) \\
&= f(m') \times \frac{\sigma(q_i^{a_i-1})}{q_i^{a_i-1}} \times (\frac{1}{q_i \times \sigma(q_i^{a_i})}) \\
&< f(m') \times \frac{\sigma(q_i^{a_i})}{q_i^{a_i}} \times (\frac{1}{q_i \times \sigma(q_i^{a_i})}) \\
&= f(m') \times \frac{1}{q_i^{a_i+1}}
\end{aligned}$$

where $m' = \frac{n}{q_i^{a_i}}$ and we know that $q_i^{a_i} \| n$ and $\frac{\sigma(q_i^{a_i})}{q_i^{a_i}} > \frac{\sigma(q_i^{a_i-1})}{q_i^{a_i-1}}$ because of the lemma 7.2. In this way, we have that

$$f(m') \times \frac{1}{q_i^{a_i+1}} \geq e^{\gamma} \times \log\log n - f(m).$$

We know that $\mathsf{Robins}(m')$ and $\mathsf{Robins}(m)$ hold, since $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold. Consequently, we only need to prove that

$$\begin{aligned}
e^{\gamma} \times \log\log m' \times \frac{1}{q_i^{a_i+1}} &> f(m') \times \frac{1}{q_i^{a_i+1}} \\
&\geq e^{\gamma} \times \log\log n - f(m) \\
&> e^{\gamma} \times \log\log n - e^{\gamma} \times \log\log m.
\end{aligned}$$

As result, we have that

$$\log\log m' \times \frac{1}{q_i^{a_i+1}} > \log\log(q_i \times m) - \log\log m$$

since $m = \frac{n}{q_i}$. We know that

$$\begin{aligned}\log\log(q_i \times m) - \log\log m &= \log(\log q_i + \log m) - \log\log m \\ &= \log\left(\log m \times (1 + \frac{\log q_i}{\log m})\right) - \log\log m \\ &= \log\log m + \log(1 + \frac{\log q_i}{\log m}) - \log\log m \\ &= \log(1 + \frac{\log q_i}{\log m}).\end{aligned}$$

In addition, we know that

$$\log(1 + \frac{\log q_i}{\log m}) \geq \frac{\log q_i}{\log n}$$

using the lemma 8.4. Certainly, we will have that

$$\log(1 + \frac{\log q_i}{\log m}) \geq \frac{\frac{\log q_i}{\log m}}{\frac{\log q_i}{\log m} + 1} = \frac{\log q_i}{\log q_i + \log m} = \frac{\log q_i}{\log n}.$$

As a consequence, we would have

$$\log\log m' \times \frac{1}{q_i^{a_i+1}} > \frac{\log q_i}{\log n}$$

which is equivalent to

$$\log n \times \log\log m' > q_i^{a_i+1} \times \log q_i.$$

However, we know that

$$\log n \times \log\log n > \log n \times \log\log m'$$

and thus

$$\log n \times \log\log n > q_i^{a_i+1} \times \log q_i.$$

For $n > 10^{10^{10}}$, we have that $\log n \times \log\log n > 1$ according to the lemma 3.4. Moreover, for $q_i \geq 3$, then $q_i^{a_i+1} \times \log q_i > 1$. In addition, for $q_1 = 2$, we have that $q_1^{a_1+1} \times \log q_1 > 1$ since $a_1 \geq 20$ due to the lemma 3.1. Since the both sides of the inequality is greater that 1 for all primes q_i which divides n, then we can multiply the inequalities to obtain

$$(\log n \times \log\log n)^{\pi(q_m)} > n \times N_m \times \prod_{i=1}^{m} \log q_i$$

where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m. If we apply the logarithm to the both sides of the inequality, then we would have

$$\pi(q_m) \times (\log\log n + \log\log\log n) > \log n + \log N_m + \sum_{i=1}^{m} \log\log q_i$$

which is equivalent to

$$\pi(q_m) \times (\log\log n + \log\log\log n) > \log n + \theta(q_m) + \sum_{i=1}^{m} \log\log q_i.$$

If we apply the lemma 8.3, then we would have

$$1.25506 \times \frac{q_m}{\log q_m} \times (\log\log n + \log\log\log n) > \log n + \theta(q_m) + \sum_{i=1}^{m} \log\log q_i.$$

Let's introduce the lemma 8.1 in this inequality and thus

$$1.25506 \times \frac{q_m}{\log q_m} \times (\log\log n + \log\log\log n) > \log n + (1 - \frac{1}{\log q_m}) \times q_m + \sum_{i=1}^{m} \log\log q_i.$$

In addition, we can transform this into

$$1.25506 \times \frac{q_m}{\log q_m} \times (\log\log n + \log\log\log n) > q_m + (1 - \frac{1}{\log q_m}) \times q_m + \sum_{i=1}^{m} \log\log q_i$$

because of the lemma 8.5. If we divide the both sides by q_m, then

$$1.25506 \times \frac{1}{\log q_m} \times (\log\log n + \log\log\log n) > 1 + 1 - \frac{1}{\log q_m} + \sum_{i=1}^{m} \frac{\log\log q_i}{q_m}.$$

According to the lemma 8.8, we know that

$$-\frac{1}{\log q_m} + \sum_{i=1}^{m} \frac{\log\log q_i}{q_m} = \alpha > 0.$$

Consequently, we would have that

$$1.25506 \times (\frac{\log\log n}{\log q_m} + \frac{\log\log\log n}{\log q_m}) > 2 + \alpha.$$

If we use the lemma 8.7, then

$$1.25506 \times (1 + \frac{1}{2 \times \log^2 q_m} + \frac{\log\log q_m}{\log q_m} + \frac{1}{2 \times \log^3 q_m}) > 2 + \alpha.$$

We know that

$$\begin{aligned} &1.25506 \times (1 + \frac{1}{2 \times \log^2 q_m} + \frac{\log\log q_m}{\log q_m} + \frac{1}{2 \times \log^3 q_m}) \\ &\leq 1.25506 \times (1 + \frac{1}{2 \times \log^2 953} + \frac{\log\log 953}{\log 953} + \frac{1}{2 \times \log^3 953}) \end{aligned}$$

and we have that

$$1.25506 \times (1 + \frac{1}{2 \times \log^2 953} + \frac{\log\log 953}{\log 953} + \frac{1}{2 \times \log^3 953}) \approx 1.62266460495.$$

Consequently, we have that

$$2 > 1.25506 \times (1 + \frac{1}{2 \times \log^2 q_m} + \frac{\log \log q_m}{\log q_m} + \frac{1}{2 \times \log^3 q_m}) > 2 + \alpha > 2$$

and

$$2 > 2$$

is a contradiction. To sum up, we obtain a contradiction just assuming that $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold.

Theorem 9.3 $\mathsf{Robins}(n)$ *holds for all* $n > 5040$.

Proof Due to the theorem 9.2, we can assure there is not any natural number $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold.

Theorem 9.4 *The Riemann Hypothesis is true.*

Proof This is a direct consequence of theorems 1.1 and 9.3

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