# Chromatic Coloring of Distance Graphs I

**V. Yegnanarayanan**

*Abstract*: *The primary aim of this paper is to publicize various problems regarding chromatic coloring of finite, simple and undirected graphs. A simple motivation for this work is that the coloring of graphs gives models for a variety of real world problems such as scheduling. We prove some interesting results related to the computation of chromatic number of certain distance graphs and also discuss some open problems.*

*Keywords*: *Graph coloring, chromatic number, distance graph.*

## I. INTRODUCTION

A simple graph G with (V, E) as (vertex set, edge set) pair is considered as a unit distance graph if $f : V \rightarrow \mathbb{R}^2$ exists as an embedding with $|f(x) - f(y)| = 1$ for each pair $(x, y) \in E$. By $| \cdot |$ we deem an Euclidean norm. Erdos propagated the problem of determination of $\chi(G(\mathbb{R}^2, \{1\}))$ due to Hadwiger-Nelson. Here {1} stands for the distance set D with 1 as only element. So the edge set E of such a graph G evolves when introducing an edge between two elements of $\mathbb{R}^2$ separated by a unit distance and $\chi$ stands for the chromatic number. By the chromatic number of a graph we comprehend the least number of colors needed to properly color the vertices of G in such a way that any two vertices that are linked by an edge receive distinct colors. Lately, Grey in [1] found that $\chi(G(\mathbb{R}^2, \{1\}))$ is at least 5 by exhibiting a 5-chromatic graph G on 1581 vertices of unit distance. It was further improved by Heule in [2] and his unit distance graph $G(\mathbb{R}^2, \{1\})$ had 553 vertices.

## II. MOTIVATION

What is it that is common between conducting of examination and organizing sports events? It is clear that both the tasks demand huge planning and scheduling. It is stimulating to note that both the tasks derive a lot of benefits from a portion of graph theory named chromatic coloring.

### Scheduling Task

Suppose that we have a collection of tasks endowed with the counsel regarding which of them cannot be handled simultaneously. The aim is to determine the least time in which all tasks can be completed. We can model this scheduling task with a graph whose vertices denote tasks and the edges among them for conflicts associated with it. That

is, an edge is introduced between two vertices iff the respective tasks could not be performed simultaneously. The least time required to complete the task corresponds to the problem of computing the chromatic number of the respective graph. The same idea can be extended to the task of conducting examinations.

In general, it is known that the task of computation of chromatic number unless it is one or two belongs to the class of NP-complete problems. It means no polynomial time step-by-step procedure could find the chromatic number of any given graph.

### Mobile / Cellular Networks

Chromatic coloring could be involved for mobile based communication. GSM is a mobile network where a land area is split into cells (often in the shape of a hexagon) and each cell can be deemed as a transceiver linking the cellular devices within it. To overcome interference, we should avoid using the same channel for two adjacent cells. By the famous 4-color Theorem of graph colorings, we require only 4 distinct channels to assure this constraint independent of the cell shape.

## III. FORBIDDEN GRAPHS

It has become a practice to call a graph G as forbidden if it is not a unit distance graph. The problem of deciding whether a given graph is of unit distance belongs to the class of NP-hard problems as it shares the identical complexity with the problem of assessing the validity of sentences within the realm of $\mathbb{R}$'s existential theory [3].

In [4] the authors have demonstrated that one of the following two graphs in Fig. 1, namely $G_1$, is of unit distance whereas the other, namely $G_2$, is forbidden by introducing the concept of minimality. They called a graph that is forbidden is minimal if every proper subgraph of it is a unit distance graph.
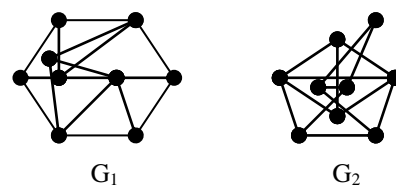


$G_1$          $G_2$

**Fig. 1. $G_1$ is a Unit Distance Graph and $G_2$ is a Forbidden Graph**

Note that $K_4$, the complete graph on 4 vertices and $K_{2,3}$, the complete bipartite graph with two vertices in one partite set and three vertices in the other partite set are minimal forbidden graphs. It is widely known that all graphs on and up to less than or equal to five vertices are unit distance graphs provided they do not admit $K_4$ or $K_{2,3}$ as subgraph.

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In [5] the authors showed that the graphs in Figure 2 which constitute a set of graphs that are minimal forbidden on less than equal to 7 vertices. So it reveals that the necessary and sufficient that any graph on and upto 7 vertices is considered as forbidden if it includes a subgraph that is $\cong$ to one of the graphs given in Fig. 2. One can also refer to [6] for more on such forbidden graphs on $\leq$ 9 vertices.
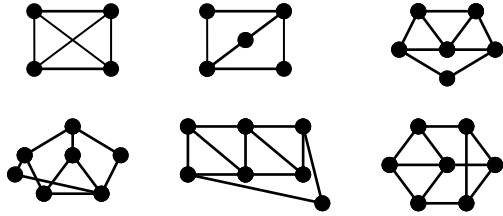


**Fig. 2. Set of Graphs that are minimal forbidden on less than or equal to seven vertices**

## IV. CHROMATIC COLORING OF DISTANCE GRAPHS

As the problem of computing the exact value of $\chi(G(\mathbb{R}^2, \{1\}))$ is extremely difficult, researchers considered problems of the next lower levels namely that of finding the exact value of $\chi(G(Z, D))$ where Z has replaced $\mathbb{R}^2$ and $\{1\}$ is replaced by a well-defined subset of either Z or $\mathbb{N}$.

Voigt [7-11] carried out this investigation of these type of integer distance graphs and have achieved several interesting results. Note that any graph G with finite vertex set can be deemed as a distance graph with integers as its distance set elements. That is, for every such G, one can always find a distance set D, and a graph generated with D, namely $G_1(D_1)$ includes in it an induced subgraph $G_2 \cong G$.

By a k-coloring of a distance graph G(Z, D) we mean a mapping $g : Z \to A$ where $|A| = k$. It is further called a periodic coloring with period s, if $g(j + s) = g(j)$ for each $j \in Z$. Next, we refer g as T-congruous if $g(j + t) \neq g(j)$ for each $j \in Z$ and each $t \in T$. Suppose $T = \{t\}$, we then name such g as t-congruous. Hence, in the case of g being a proper k-coloring of G(Z, D) it is evident that it must be D-congruous. Further one can observe that if g is t-congruous periodic with period s > t then it is also $(ns \pm t)$-congruous for each $n \in \mathbb{N}$.

**Proposition 1.** If $m \equiv 0 \pmod 3$ then the chromatic number of the distance graph G(Z, D) with D = {1, 2, m} is four.

**Proof.** Assume that G(Z, D) admits a proper three coloring $g : Z \to \{1, 2, 3\}$. In such a case, g must properly three color a subgraph of G(Z, D). Call such a subgraph as $G(Z, D_1)$ with $D_1 = \{1, 2\}$. Now observe that g is not t-congruous if $t \equiv 0 \pmod 3$. So g is not m-congruous and hence is not a proper three coloring of G(Z, D). But then this is a contradiction and so the number of colors required to properly color G(Z, D) is four.

**Proposition 2.** Let D = $\{u_1, u_2, u_3\} \subseteq \mathbb{N}$. If G(Z, D) permits a proper three coloring $g : Z \to \{1, 2, 3\}$ and g also properly three colors $\sum_{i=1}^{3} u_i$ vertices of G(Z, D) that are arranged in increasing order with a distance of separation as unity among any two consecutive vertices then the chromatic number of G(Z, D) is at most three.

**Proof.** Suppose that an arrangement of such a set of $\sum_{i=1}^{3} u_i$ vertices is a, a+1, a+2, ..., $u_1+u_2+u_3$ and it is properly three colored by g. Let us now attempt to allot a color not used previously to color one after the other the integers left out of

$$\left\{ a, a+1, a+2, ..., \sum_{i=1}^{3} u_i \right\}, \qquad \text{namely}$$

$$\left\{ \sum_{i=1}^{3} u_i + 1, \sum_{i=1}^{3} u_i + 2, ..., 0, -1, -2, ... \right\}.$$ Presume that g cannot

have such an extension. Suppose that w is the least integer such that the colors 1, 2, 3 are assigned to the neighbors of w previously. Let it by $g(w-u_1) = 1$, $g(w-u_2) = 2$, $g(w-u_3) = 3$. Then $g(w-(u_1+u_2)) = 3$, $g(w-(u_1+u_3)) = 2$ and $g(w-(u_2+u_3)) = 1$. But then this means the existence of a vertex allotted with

a color and it is $w - \left( \sum_{i=1}^{3} u_i \right) > 1$ and it cannot be assigned

with the color 1, 2 or 3. This is against our supposition that w is least such vertex. Hence the chromatic number of G(Z, D) is at most three.

### On r-distance k-coloring

An r-distance k-coloring of a simple graph G is a function $f : V(G) \to \{1, 2, ..., k\}$ with the attribute that any two vertices separated by a distance at most r receive different colors. $\chi_j(G)$ is the r-distance chromatic number if k is the least integer with respect to which G has a r-distance k-coloring. If r = 2, then it is easy to see that $\chi_2(G)$ equal to $\chi(G^2)$ where $G^2$ is obtained by having $V(G^2)$ as V(G) and $u_1v_1 \in E(G^2)$ iff $d(u_1, v_1)$ is at most 2. For more on the case when r = 2 one can see [12, 13]. A lot of work can be done for other values of r.

## V. SOME OPEN PROBLEMS

Even though a lot of related developments took place while attempting the exact chromatic number computation of $G(\mathbb{R} \times \mathbb{R}, \{1\})$, not many results were obtained for spaces that are non-euclidean. In [14] the authors have considered the computation of $\chi(\mathbb{H}^2, D)$ where $\mathbb{H}^2$ is the hyperbolic plane. Since $\mathbb{H}^2$ includes no homothety, it is worthwhile to probe its $\chi$ value in the case of D turning out as singleton.

### Johnson-Szlam Problem

Find $\chi(G(Y, \{1\}))$ where Y varies over every possible planar metric spaces.

Deem Y = $(\mathbb{R} \times \mathbb{R}, d)$ where d is a metric with the translation invariant attribute namely $d(u, v) = d(u+w, v+w)$ for all u, v, w $\in \mathbb{R} \times \mathbb{R}$. Note that d creates a standard topology on $\mathbb{R} \times \mathbb{R}$. This implies that every Euclidean disk contains a d disk of identical center. We name such a d as a plane metric and $(\mathbb{R} \times \mathbb{R}, d)$ a planar metric space. Benoit Kloeckner in [15] demonstrated that 3 is a member of PLANAR, where the set PLANAR implies every possible value of $\chi(G(Y, \{1\}))$ and in [16] the authors established that 4 is a member of PLANAR. One can see [8] for more open problems relatex to Johnson-Szham Problem.

## VI. ROLE OF SET THEORY AXIOMS IN THE COMPUTATION OF $\chi(G(\mathbb{R} \times \mathbb{R}, \{1\}))$

In 1951, Erdos and de Bruijn in their seminal paper [17] established that the computation of $\chi(G(\mathbb{R} \times \mathbb{R}, \{1\}))$ is related to the computation of $\chi$ on some finite subgraph $H \subseteq G(\mathbb{R} \times \mathbb{R}, \{1\})$. But their proof technique liberally used the axiom of choice. But a related question that tickles every one is what if we do not involve axiom of choice? In [17] the authors have given a distance graph on $\mathbb{R}$ for illustrative purpose, whose $\chi$ computation relies on set theory based system of axioms. They further obtained a conditional result which states: $\chi(G(\mathbb{R} \times \mathbb{R}, \{1\})) = 5, 6, 7$ in the case of a system that presumes Zermelo-Fraenkel, Dependent Choices and Lebegue Measurability.

## VII. AN USEFUL UPPER BOUND FOR CHROMATIC NUMBER

**Proposition 3.** Suppose that G is any given graph that is finite, simple and undirected. Then the chromatic number of G is at most $\left\lceil \left( \dfrac{t}{t+1} \right)(\Delta + 2) \right\rceil$.

**Proof.** Assume that $f_1 \geq f_2 \geq \ldots \geq f_p$ is the degree based ordering of the vertices of G. Set $s = (t/t+1)(\Delta+2)$, where $\Delta$ stands for the maximum degree among the vertices of G, and t is the greatest number of vertices of identical degree with each of them is greater than or equal to $(\Delta+2)/2$ at least. We assert that $f_s$ is strictly not more them $s-1$. As t is at least 1, the assertion follows when $f_s$ is strictly not more than $y-1$ where $y = (\Delta+2)/2$. Else, we will get that $f_s$ is at least $(\Delta+2)/2$. Note that in the case of $i = 1, 2, \ldots, s$ we have $f_i$ is at most strictly not more than $\Delta - \lceil i/t \rceil$ and hence strictly not more than $\Delta-i/2+1$. This implies that $f_s$ is strictly not more than $\Delta-s/t+1$ and hence not more than s. So the given graph G is s-degenerate in the sense that any subgraph of G includes a vertex of degree at most $s-1$. But then it is already a fact that every s-degenerate graph can be chromatically colored with not more than s colors. So we see that the chromatic number of G is at most $\left\lceil \left( \dfrac{t}{t+1} \right)(\Delta + 2) \right\rceil$.

## VIII. CONCLUSION

We have obtained some interesting results concerning the computation of $\chi$ for a certain class of distance graphs and a useful upper bound for $\chi$ of any given graph. We also discussed some open problems and about the practical value of probing $\chi$. The author Yegnanarayanan has also obtained several interesting results on chromatic coloring of graphs.. One can see [18-32].

## ACKNOWLEDGMENT

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*Retrieval Number: 100.1/ijitee.I92910710921*
*DOI: 10.35940/ijitee.I9291.0710921*
*Journal Website: www.ijitee.org*

*Published By:*
*Blue Eyes Intelligence Engineering and Sciences Publication*
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978-1-5386-0569-I$31.00@2017 IEEE, Pp 1564-1568, 2017 **DOI:** 10.1109/SmartTechCon.2017.8358628 Date Added to IEEE Digital Explore 14th May 2018.

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