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Generalization of Konhauser Polynomials

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Abstract: In this paper, we are showing study of biorthogonal polynomials associated with generalization of Laguere polynomials of Srivastava and Singhal [14]. It happens to generalized Konhauser. here we are trying to obtain the generating functions, recurrence relations, biorthogonality relations, integral representations and also bilinear and bilateral generating relations for the new class of biorthogonal system.

Keywords: Biorthogonal polynomials, Laguere polynomials, generalized Konhauser, generating functions, recurrence relations,

I. INTRODUCTION

The concept of two polynomials explain by Didon [1] and Deruyts [2] considered this concept in some detail. For example, given the set {Pn(x)} the set {Qn(x)} is uniquely determined and conversely over the different interval. considered this concept in some detail, and claimed that for these two polynomials. the set of polynomials in x, $\{P_n(x)\}$ and $\{Q_n(x)\}$, deg. $Q_n(x)=n$, (n=0,1,2,.....) are said to be biorthogonal with respect to distribution $d\alpha$ (x) on interval [a,b] if:

$$\int_a^b P_n(x)Q_n(x)\,da(x) = 0,\ m \neq n\ \neq 0, m = n$$

where $\alpha(x)$ is a distribution function on interval (finite or infinite) with infinitely many points of increase and such

that : $\int_a^b x^n d\alpha(x) < \infty,$ for all n=0, 1,2,....

Not much attention was paid to the study of biorthogonal system of polynomials, till Spencer and Fano [3] encountered a pair of biorthogonal polynomials, while dealing with a problem related to the study of penetration and diffusion of X-Rays, and subsequently studies were made by Preiser [4] in ordinary differential equation of the third order and it also recommend for higer order form .Konhauser [5] , Carlitz [6] , Prabhakar and Kashyap [7] and Prabhakar & Tomar [9] describe some results on biorthognal function suggested by the Laguerre polynomials. Rahman [8] also expressed some explicit function of unearization coefficient of the product of Jacobi polynomials ,

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Madhekar and Thakare [10] has work on Biorthogonal polynomials suggested by the Jacobi polynomials and Al-Salam and Verma [11] also described Analogues of some biorthogonal function. Both Didon and Deruyts paid special attention to the situation in which $P_n(x)$ is a polynomial of degree n in x^k (k fixed)

In this paper, we shall study the generalization of biorthogonal Polynomials suggested by Konhauser polynomials over the interval (0, ∞) with respect to the distribution function $w(x) = x^\alpha \exp(-px^r)dx$ and also Obtain associated generating relations for $Y_n^{(\alpha)}(x,r,p,k)$ and $Z_n^{(\alpha)}(x,r,p,k)$. We recall the polynomials $G_n^{(\alpha)}(x,r,p,k)$, which are introduced by Srivastava and Singhal [14] and in attempt to provide an elegant unification of various known generalized of classical Hermite and Laguerre polynomials. These polynomials are defined by the generalized Rodrigues's formula

$$G_n^{(\alpha)}(x,r,p,k) = x^{-kn-\alpha}\exp(px^r)\left(\frac{1}{n!}\right)(x^{k+1}D_x)^n\{x^\alpha\exp(-px^r)\},$$

where $D_x = d/dx$ and parameters α, k, p and r are unrestricted in general. The explicit expansion is given as

$$G_n^{(\alpha)}(x,r,p,k) = \left(\frac{K^n}{n!}\right)\sum_{i=0}^{n}(px^r)^i\left(\frac{1}{l!}\right)\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{rj+\alpha}{k}\right)_n \quad (1)$$

It is worth mentioning here that Srivastava and Singhal [14], Chandel [1] and Srivastava, P.N. [16] also consider the polynomials defined

$$Y_n^{(\alpha)}(x,k) = k^{-n}G_n^{(\alpha+1)}(x,1,1,k) \quad (2)$$

Thus, we observe that (2) provides a generalization of one member of the pair of Konhauser biorthogonal polynomials. This leads us to consider pair biorthogonal polynomials, one of which is connected with (1)

II. PRELIMINARIES

A. Generalized konhauser polynomials

In this section, we included the different kind of relation which are pair of biorthogonal sets of polynomials see [12,13,15].

$Y_n^{(\alpha)}(x,r,p,k)$ and $Z_n^{(\alpha)}(x,r,p,k)$., where $Z_n^{(\alpha)}(x,r,p,k)$ is a polynomial of degree n in x^k, (k is fixed integer) while $Y_n^{(\alpha)}(x,r,p,k)$ is a polynomial of degree n in, x^r,(r is fixed integer

$$Z_n^{(\alpha)}(x,r,p,k) = \frac{\Gamma\{(\alpha+1+kn)/r\}}{p^{kn/r}n!}\sum_{m=0}^{n}(-1)^m\binom{n}{m}\frac{p^{km/r}x^{km}}{\{(\alpha+1+km)/r\}} \quad (3)$$

$$Y_n^{(\alpha)}(x,r,p,k) = (1/n!)\sum_{i=0}^{n}\frac{p^i x^{ri}}{i!}\sum_{j=0}^{n}(-1)^j\binom{i}{j}\left(\frac{\alpha+1+rj}{k}\right)_n \quad (4)$$

where $(\alpha+1)/r > 0$ and k/r is a positive integer.

B. First biorthogonal relation

The polynomials sets

$Y_n^{(\alpha)}(x,r,p,k)$ and $Z_n^{(\alpha)}(x,r,p,k)$ are biorthogonal with respect to the distribution function w (x) = $x^{(\alpha)}$ exp ($-px^r$) over the interval (0, ∞)

The biorthogonal relation is given by (4)

$$\int_0^\infty x^\alpha \exp\left(-px^r\right) Y_m^{(\alpha)}(x,r,p;k)\, z_n^{(\alpha)}(z,r,p;k)\,dx = \frac{\Gamma\{(\alpha+1+kn)/r\}}{rm!p^{(\alpha+1+kn)/r}}\,\delta m,n$$

where $\delta m, n$ is Kronecker delta and k/r is a positive integer. We shall prove this relation later on.

C. Generating function for $z_N^{(\alpha)}(x,r,p;k)$ and $Y_n^{(\alpha)}(x,r,p;k)$

use (2), we have

$$Z_n^{(\alpha)}(x,r,p;k) = \frac{\Gamma\{(\alpha+1+kn)/r\}}{p^{kn/r}n!}\sum_{m=0}^{n}(-1)^m\binom{n}{m}\frac{p^{km/r}x^{km}}{\Gamma\{(\alpha+1+km)/r\}}$$

$$Z_n^{(\alpha)}(x,r,p;k) = \frac{(\{a+1\}/r)\,q_n}{p^{kn/r}n!}\sum_{m=0}^{n}(-n)_m\frac{p^{km/r}x^{km}}{m!\Gamma\{(\alpha+1+km)/r\}}$$

$$Z_n^{(\alpha)}(x,r,p;k) = \frac{(\{a+1\}/r)\,q_n}{p^{kn/r}n!}\sum_{m=0}^{n}(-n)_m\frac{\left(p^{km/r}x^{km}\right)^m}{m!q^{qm}\prod_{s=1}^{q}[\{(\alpha+1)/r+s-1\}/q]} \quad (5)$$

$$= \frac{(\{\alpha+1\}/r)qn}{p^{km/r}n!}F_q\left[-n,(\alpha+1)/rq\ldots\ldots\{\alpha+1+r(q-1)\}/rq;\left(p^q x^{rq}\right)/q^q\right]$$

thus, $Z_n^{(\alpha)}(x,r,p;k)$ is in the hypergeometric form.

where k/r=q, a positive integer.

Now,

$$\sum_{n=0}^{\infty}Z_n^\alpha(x,r,p;q)\frac{t^n}{((\alpha+1)/r)_{qn}} = \sum_{n=0}^{\infty}\frac{t^n}{n!p^{qn}}\sum_{m=0}^{\infty}(-1)^m\binom{n}{m}\frac{p^{mq}x^{km}}{((\alpha+1)/r)_{qn}}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{m+n}\binom{n+m}{n}\frac{t^{n+m}x^{km}}{(n+m)!p^{q(n+m)}\prod_{s=1}^{q}[\{((\alpha+1)/r)+s-1\}/q]}$$

$$= \sum_{n=0}^{\infty}\frac{(-t)^n}{p^{kn/r}}\sum_{n=0}^{\infty}\frac{(-1)^m x^{km}t^m}{m!\prod_{s=1}^{q}[\{((\alpha+1)/r)+s-1\}/q]} \quad (6)$$

$$= \exp\left(-t/p\right)^{k/r}\,{}_0F_q\left[\ldots\ldots;(\alpha+1)/rq\ldots\ldots\{\alpha+1+r(q-1)\}/rq,(-x/q)^{rq}t\right]$$

which is the generating function for $Z_n^\alpha(x,r,p;q)$.

use (1) and (2), we observe that

$$Y_n^\alpha(x,r,p;k) = k^{-n}G_n^{(\alpha+1)}(x,r,p;k) \quad (7)$$

Hence $Y_n^\alpha(x,r,p;k) = \dfrac{x^{-\alpha-kn-1}\exp\left(px^r\right)}{k^n\, n!}\left(x^{k+1}D\right)^n\left[x^{\alpha+1}\exp\left(-px^r\right)\right]$

Now using the analogues result, we get a generating function for $Y_n^\alpha(x,r,p;k)$ as:

$$\sum_{n=0}^{\infty}Y_n^\alpha(x,r,p;k)\,t^n = (1-t)^{-(\alpha+1)/k}\exp\left[px^r\left\{1-(1-t)^{-r/k}\right\}\right] \quad (8)$$

D. Second biorthogonal relation

We have to prove relation (4)

Consider,,,

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_0^\infty x^\alpha \exp\left(-px^r\right)Y_n^\alpha(x,r,p;k)Z_m^\alpha(x,r,p;k)\left[\frac{1}{(\{1+\alpha\}/r)_{qm}}\right]u^m t^n dx$$

$$= \int_0^\infty x^\alpha \exp\left(-px^r\right)\left[\sum_{n=0}^{\infty}Y_n^\alpha(x,r,p;k)\,t^n\right]\left[\sum_{m=0}^{\infty}Z_m^\alpha(x,r,p;k)\left[\frac{1}{(\{\alpha+1\}/r)_{qn}}\right]u^m\right]dx$$

$$= \int_0^\infty x^\alpha \exp\left(-px^r\right)\left[(1-t)^{\{-(1+\alpha)/k\}}\exp\left[-px^r\left\{-1-t^{-r/k}\right\}\right]\right]\exp\left(-t/p\right)^{k/r}$$

$$= {}_0F_q\left[\ldots..,(\alpha+1)/rq,\ldots..\{\alpha+1+r(q-1)\}/rq;(x/q)^m u\right]$$

[Using (4) and (5)]

$$= (1-t)^{\{-(1+\alpha)/k\}}\exp\left(-u/p\right)^{k/r}\sum_{m=0}^{\infty}\frac{(-u)^m}{((1+\alpha)/r)_{qm}}\int_0^\infty x^{\alpha+qm}\exp\left[-px^r\left\{(1-t)^{-r/k}\right\}\right]dx$$

after some simplification, we get; (9)

$$\int_0^\infty x^\alpha \exp\left(-px^r\right)\left[\sum_{n=0}^{\infty}Y_n^\alpha(x,r,p;k)t^n\right]\left[\sum_{m=0}^{\infty}Z_m^\alpha(x,r,p;k)\left[\frac{1}{(\{\alpha+1\}/r)_{qm}}\right]u^m\right]dx =$$

$$= \frac{\Gamma\{(\alpha+1)/r\}}{r.p^{\{(\alpha+1)/r\}}}\exp\left(ut/p^q\right)\frac{\Gamma\{(\alpha+1)/r\}}{\Gamma p^{\{(\alpha+1)/r\}}}\sum_{m=0}^{\infty}(1/m!)\left[(ut/p^q)\right]^m \quad (9)$$

Comparing the coefficient of $u^m t^n$ on both sides of (9) we see that the coefficient of $u^m t^n$ when n ≠ m, then the right hand member of (9) is zero and when n=m then the right hand member is non zero.

III. INTEGRAL REPRESENTATIONS

A. integral representation for $Y_n^{(\alpha)}(x,r,p,k)$

Osler [5] has given a fractional derivative formula as:

$$D_{g(x)}^{(\alpha)}\{f(z)\} = D_{h(x)}^{(\alpha)}\left[\frac{f(z)g(z)}{h(z)}\left(\frac{h(z)-h(w)}{g(z)-g(w)}\right)^{\alpha+1}\right]_{w=z}$$

where $D_{g(x)}^{(\alpha)}\{f(z)\}$ denotes the fractional derivation of order α with respect to g (z).

For $\alpha = m$ and $h(z) = z$, we have

$$D_{g(x)}^{(\alpha)}\{f(z)\} = D^m\left[f(z)g(z)\left(\frac{z-w}{g(z)-y(w)}\right)^{m+1}\right]_{w=z} \quad \forall m\in\{0,12\ldots\ldots\ldots\} \quad (10)$$

For the relatively more familiar derivative of order m now from (10), we have

$$Y_n^{(\alpha)}(x,r,p,k) = \frac{x^{-1+nk-\alpha}}{k^n n!}\exp\left(px^r\right)\left(x^{k+1}D_x\right)^n\left\{x^{\alpha+1}\exp\left(-px^r\right)\right\},$$

$$= \frac{kx^{k-\alpha-1}}{n!}\exp\left(-px^r\right)(Dx)^n\left[x^{\alpha+kn}\exp\left(-px^r\right)\left[\left(\frac{x-y}{x^k-y^k}\right)^{n+1}\right]\right]_{y=x} \quad (11)$$

from (11), we get integral representation as :

$$Y_n^\alpha(x,r,p;k) = \frac{kr^{-\alpha-1+kn}}{2\pi i}\int_c\left[\frac{\exp\{p(u^r-x^r)\}u^{\alpha+kn}}{\left[(u^k-x^k)\right]^{n+1}}\right]du \quad (12)$$

taking u=y (1+t) and after simple manipulation, we get

$$Y_n^\alpha(x,r,p;k) = \frac{k}{2\pi i}D_t^n\left[\frac{(1+t)^{a+kn}\exp\left\{-px^r(1+t)^r-1\right\}t^{n+1}}{\left((1+t)^k-1\right)^{n+1}}\right]_{t=0}$$

(13)

from (15), we easily write the integral representation for $Y_n^\alpha(x,r,p;k)$ as:

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$$Y_n^{\alpha}(x,r,p;k)=\frac{k}{2\pi i}\int_c \frac{(1+t)^{\alpha+kn}\exp\{-px^r(1+t)^r-1\}}{((1+t)^k-1)^{n+1}}\,dt,$$

(14)

which C is a closed contour enclosing t=0, but excluding t=1 and the roots of the equation $(t+1)^k-1=0$

B. Integral representation for $Z_n^{\alpha}(x,r,p;k)$

We consider,

$$\int_0^{\infty}\exp(-ps^rx^r)t^{\beta}Z_n^{\alpha}(x,r,p;k)\,dt$$

$$=\frac{\Gamma\{(\alpha+1+kn)/r\}}{p^{kn/r}n!}\sum_{m=0}^{n}(-n)_m\frac{p^{km/r}x^{km}}{m!\Gamma\{(\alpha+1+km)/r\}}\int_0^{\infty}t^{\beta+km}\exp(-ps^rx^r)\,dt$$

$$=\frac{\Gamma\{(\alpha+1+kn)/r\}}{p^{kn+\beta+1/r}r.n!s^{1+\beta}}\sum_{m=0}^{n}(-n)_m\frac{\Gamma\{(km+\beta+1)/r\}}{m!\Gamma\{(\alpha+1+km)/r\}}(x/s)^k$$

(15)

$$=\frac{\Gamma\{(kn+\alpha+1)/r\}}{p^{(kn+\beta+1)/r}\,r.n!s^{1+\beta}}*{}_{q+1}F_q\left[\begin{matrix}-n,(\beta+1)/rq\ldots\ldots\{\beta+1+r(q-1)\}/rq\\(\alpha+1)/rq,\{(\alpha+1+r)/rq\ldots\ldots\{\alpha+1+r(q-1)/rq\}\}\end{matrix};(x/s)^k\right]$$

In particular for $\alpha=\beta$, (15) reduces to the following form :

$$\int_0^{\infty}\exp(-ps^rx^r)t^{\alpha}Z_n^{(\alpha)}(x,r,p;k)\,dt=\frac{\Gamma\{(\alpha+1+kn)/r\}(s^k-x^k)^n}{p^{(kn+\alpha+1)/r}r.n!s^{1+\alpha+k_n}}r.n!s^{1+\alpha+kn}$$

(16)

now applying inverse Laplace transform techniques to (16),we get the integral representation for $Z_n^{(\alpha)}(x,r,p;k)$

$$\frac{m!u^{1+\alpha-r}}{\Gamma\{(1+\alpha+kn)/r\}}Z_n^{(\alpha)}(u^{1/r},r,p;k)=\frac{1}{2\pi i}\int_C\frac{\exp(ut)[(t/p)^{k/r}-1]^n}{t^{\{(1+\alpha+kn)/r\}}}\,dt$$

(17)

putting $u=x^r$ and $t=ps^r$ in (17), we get:

$$\frac{n!\,p^{(1+\alpha-r+kn)/r}x^{\alpha-r+1}}{\Gamma\{(1+\alpha+kn)/r\}}Z_n^{(\alpha)}(x,r,p;k)=\frac{r}{2\pi i}\int_C\frac{\exp(ps^rx^r)[s^k-1]^n}{s^{(2+\alpha-r+kn)}}\,ds$$

(18)

where c is contour enclosing s=0 when α,r,k and n are integers.

We also have differential formula for $Z_n^{(\alpha)}(x,r,p;k)$ as

$$\frac{p^{(1+\alpha-r+kn)/r}x^{\alpha-r+1}}{\Gamma\{(1+\alpha+kn)/r\}}Z_n^{(\alpha)}(x,r,p;k)=\frac{r}{(1+\alpha-r+kn)!}D_s^{(1+\alpha-r+kn)!}\left[\frac{\exp(ps^rx^r)[s^k-1]^n}{s^{(2+\alpha-r+kn)}}\right]_{s=0}$$

(19)

In particular, the above result reduces to the corresponding result Spencer and Feno (3) and Konhauser (5).

IV. RECURRENCE RELATIONS

The polynomials $Y_n^{(\alpha)}(x,r,p;k)$ and $Z_n^{(\alpha)}(x,r,p;k)$ satisfy the recurrence relation

A. The recurrence relation for $Y_n^{(\alpha)}(x,r,p;k)$

$$(D_x-prx^{r-1})Y_n^{(\alpha)}(x,r,p;k)=(-prx^{r-1})Y_n^{(\alpha+r)}(x,r,p;k)$$

(20)

$$(p^{-1}r^{-1}x^{1-r}D_{x-1})Y_n^{(\alpha)}(x,r,p;k)=-Y_n^{(\alpha+r)}(x,r,p;k)$$

(21) $$(p^{-1}r^{-1}x^{1-r}D_x-1)^m\,Y_n^{(\alpha)}(x,r,p;k)=(-1)^m\,Y_n^{(\alpha+mr)}(x,r,p;k)$$

(22) $$(-p^{-1}r^{-1}x^{1-r}D_x+1)^q\,Y_n^{(\alpha)}(x,r,p;k)=Y_n^{(\alpha+k)}(x,r,p;k)$$

(23)

where k/r=q is a positive integer.

$$[(-p^{-1}r^{-1}x^{1-r}D_x+1)^q-1]\,Y_n^{(\alpha)}(x,r,p;k)=Y_{n-1}^{(\alpha+k)}(x,r,p;k)\qquad(24)$$

$$Y_n^{(\alpha+k)}(x,r,p;k)-Y_n^{(\alpha)}(x,r,p;k)=Y_{n-1}^{(\alpha+k)}(x,r,p;k)\qquad(25)$$

$$(xD_x+\alpha+kn-prx^r)Y_n^{(\alpha)}(x,r,p;k)=k(n+1)\,Y_n^{(\alpha)}(x,r,p;k)\qquad(26)$$

$$(xD_x+\alpha+1-k-prx^r)Y_n^{(\alpha)}(x,r,p;k)=k(n+1)Y_{n+1}^{(\alpha-k)}(x,r,p;k)\qquad(27)$$

$$(\alpha+1-k)\,Y_n^{(\alpha)}(x,r,p;k)=prx^r\,Y_n^{(\alpha+r)}(x,r,p;k)+k(n+1)Y_{n+1}^{(\alpha-k)}(x,r,p;k)\qquad(28)$$

$$k(n+1)\,Y_{n+1}^{(\alpha)}(x,r,p;k)=(\alpha+kn+1)\,Y_n^{(\alpha)}(x,r,p;k)-prx^r\,Y_{n+1}^{(\alpha+r)}(x,r,p;k)\qquad(29)$$

B. The recurrence relation for $Z_N^{(\alpha)}(x,r,p;k)$

$$D_xZ_n^{(\alpha)}(x,r,p;k)=-kx^{k-1}Z_{n-1}^{(\alpha+k)}(x,r,p;k)\qquad(29)$$

$$(x^{1-k}D_x)^m\,Z_n^{(\alpha)}(x,r,p;k)=(-k)^m Z_{n-m}^{(\alpha+km)}(x,r,p;k)\qquad(30)$$

$$(xD_x-k_n)\,Z_n^{(\alpha)}(x,r,p;k)=\frac{kp^{-k}\Gamma\{(1+\alpha+kn)/r\}}{r\Gamma\{(1+\alpha+k(n-1))/r\}}Z_{n-1}^{(\alpha)}(x,r,p;k)\qquad(31)$$

$$((xD_x+\alpha-r)+1)\,Z_n^{(\alpha)}(x,r,p;k)=(1+\alpha-r+kn)\,Z_n^{(\alpha-r)}(x,r,p;k)\qquad(32)$$

$$(1+\alpha-r+kn)Z_n^{(\alpha)}(x,r,p;k)-(1+\alpha-r+kn)\,Z_n^{(\alpha-r)}(x,r,p;k)=\frac{kp^{-1}\Gamma\{(1+\alpha+kn)/r\}}{r\Gamma\{(1+\alpha+k(n-1))/r\}}Z_{n-1}^{(\alpha)}(x,r,p;k)\qquad(33)$$

$$[(p^{-1}r^{-1}x^{1-r}D_x)^q-1]\,x^{1+\alpha-r}Z_n^{(\alpha)}(x,r,p;k)=(n+1)x^{1+\alpha-r-k(\alpha-k)}Z_{n+1}^{(\alpha)}(x,r,p;k)\qquad(34)$$

where q=k/r.

V. MAIN RESULTS

A. proofs from (20) to (31)

Rewrite equation (15) in the following form:

$$\exp(-px^r)Y_n^{\alpha}\left(x,r,p;k=\frac{k}{2\pi i}\int_C\left[\frac{(1+t)^{\alpha+kn}\exp\{-px^r(1+t)^r\}}{((1+t)^k-1)^{n+1}}\right]\right)dt$$

differentiating both sides with respect to x, we get :

$$(D_x-prx^{r-1})Y_n^{\alpha}\left(x,r,p;k=\frac{k(-prx^{r-1})}{2\pi i}\int_C\left[\frac{(1+t)^{\alpha+r+kn}\exp\{-px^r(1+t)^r\}}{((1+t)^k-1)^{n+1}}\right]\right)dt$$

Using (26), we get :

$$(D_x-prx^{r-1})Y_n^{(\alpha)}(x,r,p;k)=(-prx^{r-1})Y_n^{(\alpha+r)}(x,r,p;k)$$

which proves (19)

Now multiplying (19) both sides by $p^{-1}r^{-1}x^{1-r}$ and after simplification we get result (20)

result (21) and (22) are obvious iterations of (20).

- **proof of (20)**

  Subtracting (23) both sides by $Y_n^{(\alpha)}(x,r,p;k)$

and after some simplification we get:

$$[(-p^{-1}r^{-1}x^{1-r}D_x+1)^q-1]\quad Y_n^{(\alpha)}(x,r,p;k)=Y_{n-1}^{(\alpha+k)}(x,r,p;k)$$

- **proof of (21)**

  From equation (22) and (23), we get :

$$Y_n^{(\alpha+k)}(x,r,p;k)-Y_n^{(\alpha)}(x,r,p;k)=Y_{n-1}^{(\alpha+k)}(x,r,p;k)$$

- **proof of (22)**

Rewrite equation (15), in the following form :

$$\exp(-px^r)Y_n^{\alpha}\left(x,r,p;k=\frac{k}{2\pi i}\int_C\left[\frac{(x)^{-\alpha-1-kn}\exp(-pu^r)(u)^{\alpha+kn}}{((u+x)^k-1)^{n+1}}\right]\right)du$$

differentiating both sides with respect to 'x' and after simplification we get :

$$\left(xD_x+\alpha+kn-prx^r\right)Y_n^{(\alpha)}(x,r,p;k)=k(n+1)Y_n^{(\alpha)}(x,r,p;k)$$

- **proof of (23)**

Rewrite equation (15), in the following form :

$$\exp\left(-px^r\right)Y_n^{\alpha}\left(x,r,p;k=\frac{k}{2\pi i}\int_C\left[\frac{\exp\left(-pu^r\right)(u/x)^{\alpha+kn}}{\left((u/x)^k-1\right)^{n+1}}\right]\right)du$$

differentiating both sides with respect to x and rearranging terms, we get

$$\left(xD_x+\alpha+1-k-prx^r\right)Y_n^{(\alpha)}(x,r,p;k)=k(n+1)Y_{n+1}^{(\alpha-k)}(x,r,p;k)$$

which proves (26).

- **proof of (24)**

Eliminating the term $xD_x\ Y_n^{(\alpha)}(x,r,p;k)$ between (21) and (23) we get:

$$(\alpha+1-k)Y_n^{(\alpha)}(x,r,p;k)-prx^rY_n^{(\alpha+r)}(x,r,p;k)=k(n+1)Y_{n+1}^{(\alpha-k)}(x,r,p;k)$$

on transposition we get :

$$(\alpha+1-k)Y_n^{(\alpha)}(x,r,p;k)-prx^rY_n^{(\alpha+r)}(x,r,p;k)=k(n+1)Y_{n+1}^{(\alpha-k)}(x,r,p;k)$$

- **proof of (25)**

Eliminating the term $xD_x\ Y_n^{(\alpha)}(x,r,p;k)$ between (21) and (26), we get:

$$(\alpha+kn+1)Y_n^{(\alpha)}(x,r,p;k)-prx^rY_{n+1}^{(\alpha+r)}(x,r,p;k)=k(n+1)Y_{n+1}^{(\alpha+k)}(x,r,p;k)$$

- **proof of (26)**

Now putting u=sx in relation (19), we get:

$$\frac{n!p^{(1+\alpha-r+kn)/r}}{\Gamma\{(1+\alpha+kn)/r\}}Z_n^{(\alpha)}(x,r,p;k)=\frac{r}{2\pi i}\int_C\frac{\exp\left(pu^r\right)\left[u^k-x^k\right]^n}{u^{(2+\alpha-r+kn)/r}}du$$

differentiating both the sides with respect to x and after simplification we get:

$$\frac{n!\,p^{(1+\alpha-r+kn)/r}}{\Gamma\{(1+\alpha+kn)/r\}}D_x\left[Z_n^{(\alpha)}(x,r,p;k)\right]=\frac{rnkx^{-k-1}}{2\pi i}\int_C\frac{\exp\left(pu^r\right)\left[u^k-x^k\right]^{n-1}}{u^{(2+\alpha-r+k(n-1))/r}}du$$

after some simplification we get

$$D_x\,Z_n^{(\alpha)}(x,r,p;k)=-kx^{k-1}Z_{n-1}^{(\alpha+k)}(x,r,p;k)$$

- **proof of (27)**

From equation (31), we have

$$x^{1-k}D_xZ_n^{(\alpha)}(x,r,p;k)=-kZ_{n-1}^{(\alpha+k)}(x,r,p;k)$$

which on integration 'n' times further gives the recurrence relation:

$$\left(x^{1-k}D_k\right)^m\ Z_n^{(\alpha)}(x,r,p;k)=(-k)^m\,Z_{n-m}^{(\alpha+km)}(x,r,p;k)$$

- **proof of (28)**

Now putting u=sx in relation (18), we get :

$$\frac{n!\,p^{(1+\alpha-r+kn)/r}}{\Gamma\{(1+\alpha+kn)/r\}}x^{-kn}Z_n^{(\alpha)}(x,r,p;k)=\frac{r}{2\pi i}\int_C\frac{\exp\left(pu^r\right)\left[(u/x)^k-1\right]^n}{u^{(2+\alpha-r+kn)/r}}du$$

differentiating both the sides with respect to x and after simplification we get

$$\frac{n!p^{(1+\alpha-r+kn)/r}}{\Gamma\{(1+\alpha+kn)/r\}}\left[D_x-knx^{-1}\right]\left[Z_n^{(\alpha)}(x,r,p;k)\right]=\frac{-rnkx^{-k-1}}{2\pi i}\int_C\frac{\exp\left(pu^r\right)\left[(u/x)^k-1\right]^{n-1}}{u^{(2+\alpha-r+k(n-1))/r}}du$$

or

$$\left(xD_x-kn\right)Z_n^{(\alpha)}(x,r,p;k)=\frac{-kp^{-k}\Gamma\{(1+\alpha+kn)/r\}}{r\Gamma\{(1+\alpha+k(n-1))/r\}}Z_{n-1}^{(\alpha)}(x,r,p;k)$$

- **proof of (29)**

Differentiating (19), both the sides with respect to x we get :

$$\left(xD_x+\alpha-r+1\right)Z_n^{(\alpha)}(x,r,p;k)=\frac{r\Gamma[(1+\alpha+kn)/r]}{\Gamma[(1+\alpha-r+kn)/r]}Z_n^{(\alpha-r)}(x,r,p;k)\text{ or}$$

$$\left(\left(xD_x+\alpha-r\right)+1\right)Z_n^{(\alpha)}(x,r,p;k)=(1+\alpha-r+kn)\,Z_n^{(\alpha-r)}(x,r,p;k)$$

- **proof of (30)**

Eliminating the term $xD_xZ_n^{(\alpha)}(x,r,p;k)$ from (30) and (32), we obtain the following recurrence relation:

$$(1+\alpha-r+kn)Z_n^{(\alpha)}(x,r,p;k)-(1+\alpha-r+kn)Z_n^{(\alpha-r)}(x,r,p;k)=\frac{kp^{-1}\Gamma\{(1+\alpha+kn)/r\}}{r\Gamma\{(1+\alpha+k(n-1))/r\}}Z_{n-1}^{(\alpha)}(x,r,p;k)$$

- **proofs of (31)**

Rewrite the equation (10) in the following form :

$$\frac{n!p^{(1+\alpha-r+kn)/r}x^{\alpha-r+1}}{\Gamma\{(1+\alpha+kn)/r\}}Z_n^{(\alpha)}(x,r,p;k)=\frac{r}{2\pi i}\int_C\frac{\exp\left(ps^rx^r\right)p^qs^k\left[s^k-1\right]^n}{s^{(2+\alpha-r+kn)}}ds$$

where k/r=q,

$$\text{or}\ =\frac{rp^q}{2\pi i}\int_C\frac{\exp\left(ps^rx^r\right)\left[s^k-1\right]^n}{s^{(2+\alpha-r+kn)}}ds+\frac{rp^q}{2\pi i}\int_C\frac{\exp\left(ps^rx^r\right)\left[s^k-1\right]^n}{s^{(2+(\alpha-k)-r+k(n+1))}}ds$$

differentiating both sides with respect to x, q times, we get :

$$\left[\left(p^{-1}r^{-1}x^{1-r}D_x\right)^q-1\right]x^{1+\alpha-r}Z_n^{(\alpha)}(x,r,p;k)=(n+1)x^{1+\alpha-r-k(\alpha-k)}Z_{n+1}^{(\alpha)}(x,r,p;k)$$

VI. SPECIAL CASES

The following known special cases of (11) and (12) are

- **spencer and fano polynomials**

Taking, k=2, r=1, p=1; we get:

$$Z_1^{(\Omega)}(x)=\gamma_1^{(\Omega)}(x,1,1,2)\text{ and }Z_1^{(\Omega)}(x,1,1,2)=\gamma_1^{(\Omega)}(x)$$

- **konhauser polynomials**

Taking, r=1, p=1; we get:

$$\gamma_n^{(\Omega)}(x,k)=\gamma_n^{(\Omega)}(x,1,1,k)\text{ and }Z_n^{(\alpha)}(x,k)=Z_n^{(\alpha)}(x,1,1k)$$

- **laguerre polynomials**

Taking, k=1, r=1, p=1, we get:

$$L_n^{(\alpha)}(x)=\gamma_n^{(\alpha)}(x,1,1,1)=Z_n^{(\alpha)}(x,1,1,1)$$

- **bessel polynomials**

Other than above (11) and (12) also give rise to biorthogonal polynomials sets associated with Bessel polynomials given below for k=-1 and r = -1, we get :

$$\gamma_n^{(\alpha)}(x,-1,\beta-1)=[(-1)/n!](\beta/n)^n\,\gamma_n^{(\alpha)}(x,\alpha+\beta-2n,\beta)$$

where $\gamma_n^{(\alpha)}(x,\alpha,\beta)$ is generalized Bessel polynomials defined by

$$\gamma_n(x,-1,\beta,-1)=\beta^{-n}x^{-\alpha+2}\exp(\beta/n)\,D^n\left[x^{\alpha-2+2n}\exp(-\beta/n)\right]$$

clearly the above polynomials satisfy the biorthogonal property,

$$\int_0^\infty x^\alpha e^{-\frac{\beta}{x}}\left(\frac{\beta}{x}\right)^n\gamma_n(x,\alpha+\beta-2n,\beta)\,q_m(x,\alpha+\beta,\beta)\,dx=\frac{(-1)^{n-1}n!\Gamma(m-\alpha+1)}{m!\,\beta^{m-\alpha+1}}\delta_{m,n}$$

VII. BILINEAR AND BILATERAL GENERATING RELATIONS

In the section, we have derived some generating function for

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$Y_n^{(\alpha)}(x,r,p,k)$ and $Z_n^{(\alpha)}(x,r,p,k)$.

Now in this section we shall adopt group theoretic method to obtain a new class of bilinear and bilateral generating relations associated with $Y_n^{(\alpha)}(x,r,p,k)$ and $Z_n^{(\alpha)}(x,r,p,k)$ all the result derived here appear in the form of some theorem. We prove the following theorems with application.

Theorem-1

If there exists a generating relation of the form

$$G(x,w)=\sum_{n=0}^{\infty} a_n n!\gamma_n^{(\alpha)}(x,r,p,k)\,w^n \quad (35)$$

$$\exp\left[px^r\left\{1-(1-t)^{\frac{-r}{k}}\right\}\right](1-t)^{-(\alpha+1)/k}\, G\left[x(1-t)^{-1/k}, tv(1-t)\right] \quad (36)$$

Proof: Consider the linear partial differential operator Ω as follows

$$\Omega = xy\frac{\partial}{\partial x}+ky^2\frac{\partial}{\partial x}+\left(\alpha+1-rpx^r\right)y$$

Such that

$$\Omega=\left[Y_n^{(\alpha)}(x,r,p,k)n!\,y^n = k(n+1)y^{n+1}Y_{n+1}^{(\alpha)}(x,r,p,k)\right]$$

(37)

Such that Hence, clearly Ω form a raising Lie-operator for the class of function $\gamma_n^{(\alpha)}(x,r,p;k)$. The multiplier representation of this operator is given by

$$\exp\left((w\Omega)f(x,y)=\exp\left[px^r\left\{1-(1-kwy)^{-r/k}\right\}\right]\right)(1-kwy)^{-(\alpha+1)/k}$$

$$*\ f\left[x(1-kwy)^{-1/k},\ y(1-kwy)^{-1}\right]$$

Let us now consider the following generating relation;

$$G(x,w)=\sum_{n=0}^{\infty} a_n\ n!\gamma_n^{(\alpha)}(x,r,p;k)\,w^n \quad (38)$$

replacing w by wyz in (37), we get;

$$G(x,wyz)=\sum_{n=0}^{\infty} a_n n!\gamma_n^{(\alpha)}(x,r,p;k)(wyz)^n \quad (39)$$

operating both side of (38) by exp (w Ω), we get;

$$\exp(w\Omega)\,G(x,wyz)=\exp(w\Omega),\sum_{n=0}^{\infty} a_n n!\,\gamma_n^{(\alpha)}(x,r,p;k)(wyz)^n$$

(40)

now, using (37) the left hand member of (39) becomes.

$$\exp\left[px^r\left\{1-(1-kwy)^{r/k}\right\}\right]\ (1-kwy)^{-(\alpha+1)/k}$$

$$G\left[x(1-kxy)^{-1/k}, y(1-kwy)^{-1}\right] \quad (41)$$

also using (34) the right hand member of (38) becomes

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n w^{n+m}z^n(1/m!)\Omega^m\left[\gamma_n^{(\alpha)}(x,r,p,k)y^n n!\right]$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} k^m a_n(1/m!)(n+m)!\,w^{n+m}z^n y^{n+m}\left[\gamma_n^{(\alpha)}(x,r,p,k)\right]$$

(42)

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{n-m}\ (1/m!)\ n!(kwy)^{n-m}\left[\gamma_n^{(\alpha)}(x,r,p,k)\right]$$

equating (41) and (42) and then putting kwy=t, zt/k=v, we get the following relation;

$$\exp\left[px^r\left\{1-(1-t)^{-r/k}\right\}\right](1-t)^{-(\alpha+1)/k}G\left[x(1-t)^{-1/k}, tv(1-t)\right]$$

$$=\sum_{n=0}^{\infty}\gamma_n^{(\alpha)}(x,r,p;k)\,t^n\,\sigma_n(v)$$

Where $\sigma_n(v)=\sum_{n=0}^{\infty} m!\,a_m\binom{n}{m}v^m$

This completes proof of the theorem.

Theorem-2

If there exists a bilinear generating relation of the form

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n w^n(n!)^2\ \gamma_n^{(\beta)}(u;s) \quad (43)$$

then there exists a generating relation of the form :

$$(1-tw)^{-1/t}(1-sw)^{-(\beta+1)/s}\exp\left[px^r\left\{1-(1-wt)^{r/t}\right\}+u^r\left\{1-sw\right\}^{-r/s}\right]$$

(44)

$$G\left[x(1-tw)^{-1/t},\ u(1-sw)^{-1/s}, wg(1-wt)^{-1}(1-ws)^{-1}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{j_1=0}^{\infty}(wg)^m\ n!\ f_n^{(t,s)}(w,g,x)\,\gamma_n^{(\beta)}(u;s)$$

Where(44)

$$f_n^{(t,s)}(w,g;x)=\sum_{j_2=0}^{\min(n,j_1)}\frac{a_{n-j2}(wt)^{j_1-j_2}s^{j2}g^{j2}}{(j_1-j_2)!(j_2)!}$$

$$*\left(n+j_1-2j_2\right)!\ \gamma_{n+j_1-2/2}^{(\alpha)}(x;t)$$

Proof:

Consider the linear partial differential operator Ω_i follows

$$\Omega_i=y_i\left(x\partial/\partial x+ky,\partial/\partial y_i+\alpha+1-rpx^r\right);\ i=1,2$$

Such that

$$(45)\ \Omega\left[\gamma_n^{(\alpha)}(x,t)n!y_i\right]^n = k\,(n+1)!y_i^{n+1}\ \gamma_{n+1}^{(\alpha)}(x,t)$$

hence, clearly Ω_i form a raising Lie-operator for the class of function $\gamma_n^{(\alpha)}(x,r,p;k)$. The multiplier representation of this operator is given by

$$\exp(w\Omega_i)f(x,y_i)=(1-kwy_i)^{-(\alpha+1)/k} \quad (46)$$

$$\exp\left[px^r\left\{1-(1-kwy_i)^{-r/k}\right\}\right]f\left[x(1-kwy_i)^{-1/k}, y_i(1-kwy_i)^{-1}\right] \quad (47)$$

assuming, that (46) exists, we substitute wy_1y_2 g in the place of w and operating both sides by exp $(w\Omega_1)\exp(w\Omega_2)$,

we get $\exp(w\Omega_1)\exp(w\Omega_2)\,G(x,u.wy_1y_2g)$

$$=\exp(w\Omega_1)\exp(w\Omega_2)\sum_{n=0}^{\infty} a_n(wy_1y_2g)^n(n!)\,\gamma_n^{(\alpha)}(x,t)\,\gamma_n^{(\beta)}(u;s)$$

now, using (36), the left hand member of (37) becomes

$$\exp(w\Omega_1)\left[(1-swy_2)^{-(\beta+1)/s}\exp\left[pu^r\left\{1-(1-swy_2)^{-r/s}\right\}\right]\right]$$

$$G\left[u\ (1-swy_2)^{-1/s}, y_2\ (1-swy_2)^{-1}\right]$$

$$=(1-twy_1)^{-(\alpha+1)/t}\ (1-swy_2)^{-(\beta+1)/s} \quad (48)$$

$$*\exp\left[px^r\left\{1-(1-kwy_i)^{-r/t}\right\}+pu^r\left\{1-(1-swy_2)^{-r/s}\right\}\right]$$

$$G\left[x(1-twy_1)^{-1/k}, u(1-swy_2)^{-1/s}, wy_1y_2g\,(1-twy_1)^{-1}(1-swy_2)^{-1}\right]$$

also using (36), we see that the right member side of (37) becomes,

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$$\sum_{n=0}^{\infty}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\frac{a_n (wy_1y_2g)^n\ (n!)^2\Omega^{j_1}w^{j_2}\gamma_n^{(\beta)}(x;t)\gamma_n^{(\beta)}(u,s)}{j_i!\,j_2!}$$

$$\sum_{n=0}^{\infty}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\frac{a_n w^{n+j_i+j_2}g^n\left\{\Omega_1^{j_1}\left(\gamma_n^{(\alpha)}(x,t)n!wy_1^n\right)\right\}\left\{\Omega_2^{j_2}\left(\gamma_n^{(\beta)}(u,s)n!y_2^n\right)\right\}}{j_i!\,j_2!}$$

$$\sum_{n=0}^{\infty}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\frac{a_n w^{n+j_i+j_2}g^n}{j_i!\,j_2!}\left\{t^{j_1}\gamma_{n+j_1}^{(\alpha)}(x,t)(n+j_1)^{1+j_1}y_1\right\}\left\{s^{j_2}\gamma_{n+j_2}^{(\alpha)}(u,s)(n+j_2)!\,n+j_1y_2\right\}$$

(49)

$$=\sum_{n=0}^{\infty}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\min(n-j_1)}\frac{w^{n+j_1+j_2}g^{n-j_2}}{(j_1-j_2)j_2!}t^{j_1-j_2}\gamma_{n+j_1-j_2}^{(\alpha)}(u,t)(n+j_1-2j_2)!\,s^{j_2}$$

$$*\ \gamma_n^{(\alpha)}(u,s)n!\,y_1^{n+j_1-2j_2}y_2^n \qquad (50)$$

Theorem-3

If there exists a generating relation of the form

$$G(x,w)=\sum_{n=0}^{\infty}a_n Z_n^{(\alpha)}(x,r,p;k)\,w^n \qquad (51)$$

then there exists a generating relation of the form ;

$$\left[rw^{-1}(-vt/wr)^{1/r}\right]^{(1/r)-1}(-vt/wr)G\left[\frac{x(r^2w^2-vt)^{1/r}}{(-vt)^{1/r}},\frac{x(r^2w^2-vt)^{1/r}}{wr}\right]$$

$$=\sum_{m=0}^{n}t^n\sigma_n(x,v)$$

where

$$\sigma_n(x,v)=\sum_{m=0}^{n}(a_m/m!)(1-\{1+\alpha+km\}/r)_m\ Z_m^{(\alpha-nm)}(x,r,p;k)\,v^m$$

Proof:

Consider the linear partial differential operator Δ as follows :

$$\Delta=y^{-r}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-r+1\right)$$ Such that

$$\Delta\left[Z_n^{(\alpha)}(x,r,p,k)y^{\alpha}\right]=(1+\alpha-r+kn)Z_n^{(\alpha)}(x,r,p,k) \quad (52)$$

Hence, clearly Δ forms a raising Lie-operator for the class of function $Z_n^{(\alpha)}(x,r,p,k)$. The multiplier representation of this operator is given by

$$\exp(w\Delta)f(x,y,)=(rw+y^r)^{(1-r)/r}y^r\ f\left[\left\{x(wr+y^r)^{1/r}\right\}/y,(wr+y^r)^{1/r}\right]$$

(53)

Now consider the following generating relation;

$$G(x,w)=\sum_{n=0}^{\infty}a_n Z_n^{(\alpha)}(x,r,p,k)\,w^n \qquad (54)$$

replacing w by wz and then multiplying both sides of (54) by y^{α}; we get

$$G(x,wz)\,y^{\alpha}=y^{\alpha}\sum_{n=0}^{\infty}a_n Z_n^{(\alpha)}(x,r,p,k)(wx)^n \qquad (55)$$

operating both sides of (55) by exp (w Δ), we get

$$\exp(w\Delta)\left[G(x,wz)\,y^{\alpha}\right]=\exp(w\Delta)\left[y^{\alpha}\sum_{n=0}^{\alpha}a_nZ_n^{(\alpha)}(x,r,p,k)(wz)^n\right] \quad (56)$$

now, using (51), the left hand member of (54), becomes:

$$(rw+y^r)^{(1-r)/r}\ y^{\alpha+r}\ G\left[\left\{x(wr+y^r)^{1/r}\right\}/y,(wr+y^r)^{1/r}\right]$$

(57)

also using (40), we see that the right hand member of (44) becomes

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_n w^{n+m}z^n\Delta^m\left[Z_n^{(\alpha)}(x,r,p;k)y^{\alpha}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_n(1/m!)w^{n+m}z^n(-r)^m(1-\{1+\alpha+kn\}/r)m\ Z_n^{(\alpha-mr)}(x,r,p;k)\,y^{\alpha-mr}$$

(58)

$$=y^{\alpha}\sum_{n=0}^{\infty}(wz)^n\sum_{m=0}^{n}a_{n-m}(1/m!)(1-\{1+\alpha+k(n-m)/r\})_m Z_{n-m}^{(\alpha-mr)}(x,r,p;k)$$

$$*\left(-1/zy^t\right)^m$$

equation (45) and (46), we get;

$$(rw+y^r)^{(1-r)/r}\ y^rG\left[\left\{x(wr+y^r)^{1/r}\right\}/y,(wr+y^r)^{1/r}\right]$$

$$=\sum_{n=0}^{\infty}(wz)^n\sum_{m=0}^{n}a_{n-m}(1/m!)(1-\{1+\alpha+k(n-m)\}/r)_m\ Z_{n-m}^{(\alpha-mr)}(x,r,p;k)$$

$$*\left(-r/zy^t\right)^m \qquad (59)$$

finally putting wz=t and $-r/zy^t=v$ in (59), we get

$$;\left[rw+(-vt/wr)^{1/r}\right]^{(1/r)-k}(-vt/wr)G\left[\frac{x(r^2w^2-vt)^{1/t}}{(-vt)^{1/r}},\frac{(r^2w^2-vt)^{1/r}}{wr}\right]$$

$$=\sum_{n=0}^{\infty}t^n\sigma_n(x,v)$$

where $\sigma_n(x,v)=\sum_{m=0}^{n}(a_m/m!)(1-\{1+\alpha+km\}/r)\,Z_m^{(\alpha-mr)}(x,r,p;k)\,v^m$

This completes proof of the Theorem.

Theorem-4

$$F(x,w)=\sum_{n=0}^{\infty}a_n\gamma_n^{(\alpha)}(x,r,p;k)\,w^n \qquad (60)$$

then there exists a generating relation of the form,

$$\exp(ut)F\left[\left\{(-ut/p)+x^1\right\}^{1/r},t\right]=\sum_{n=0}^{\infty}t^n\sigma_n(x,u)$$

Where

$$\sigma_n(x,u)=\sum_{j=0}^{n}a_n(1/j!)\,\gamma_n^{(\alpha+rj)}(x,r,p;k)\,u^j \qquad (61)$$

Proof:

Consider the linear partial differential operator ϕ as follow

Such that $\phi=y^r\left(x^{-r+1}\partial/\partial x-pr\right)$ (62)

$\varphi=\left[\gamma_n^{(\alpha)}(x,r,p;k)y^{\alpha}\right]=-pr\gamma_n^{(\alpha+r)}(x,r,p;k)\,y^{\alpha+r}$ the multiplier representation of this operator is given by

$$\exp(w\phi)f(x,y)=\exp(-pry^rw)f\left[(ry^rw+x^r)^{1/r},y\right] \qquad (63)$$

let us now consider the following generating relation

$$F(x,w)=\sum_{n=0}^{\infty}a_n\gamma_n^{(\alpha)}(x,r,p;k)\,w^n \qquad (64)$$

replacing w by wz and then multiplying both side of (49) by γ^{α}, we get

$$\gamma^{\alpha}F(x,wz)=\gamma^{\alpha}\sum_{n=0}^{\infty}a_n\gamma_n^{(\alpha)}(x,r,p;k)(wz)^n \qquad (65)$$

operating both sides of (49) by exp $(w\phi)$, we get:

$$\exp(w\phi)\left[y^{\alpha}F(x,wz)\right]=\exp(w\phi)\left[\gamma^{\alpha}\sum_{n=0}^{\infty}a_n\gamma_n^{(\alpha)}(x,r,p;k)(wz)^n\right]$$

(66)

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now using (47), we see that the left hand member of (50) becomes

$$\exp\left(-pry^{r}w\right)\gamma^{\alpha}\ F\left[\left(ry^{r}w+x^{r}\right)^{1/r},\ wz\right] \quad (67)$$

also using (46), the right hand member of (50) becomes,

$$\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} a_{n}\ (1/j!)\ w^{n+j}z^{n}\phi^{j}\left[\gamma_{n}^{(\alpha)}\ (x,r,p;ky)y^{\alpha}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} a_{n-j}\,(1/j!)\,w^{n}z^{n-j}(-1)^{j}\,(pr)^{j}\left[\gamma_{n}^{(\alpha+r)}(x,r,p;k)\,y^{\alpha+rj}\right]$$

(68) $$=\sum_{n=0}^{\infty}(wt)^{n}\left[\sum_{j=0}^{\infty} a_{n-j}(1/j!)(-rpy^{r}/z)^{j}\left[\gamma_{n}^{(\alpha+r)}(x,r,p;k)\,y^{\alpha}\right]\right]$$

(69)

equating (51) and (52), we get:

$$\exp\left(-pry^{r}w\right)y^{\alpha}\ F\left[\left(ry^{r}w+x^{r}\right)^{1/r},wz\right] \quad (53)$$

$$=\sum_{n=0}^{\infty}(wt)^{n}\left[\sum_{j=0}^{\infty} a_{n-j}(1/j!)\left(-rpy^{r}/z\right)^{j}\left[\gamma_{n}^{(\alpha+rj)}(x,r,p;k)\,y^{\alpha}\right]\right]$$

finally, putting wz = t and $\left(-rpy^{r}/z\right)=u$ in (53), we get

$$\exp\ (ut)F\left[\left\{(-ut/P)+x^{r}\right\}^{1/r},t\right]=\sum_{n=0}^{\infty} t^{n}\sigma_{n}(x,u)$$ where

$$\sigma_{n}\ (x,u)=\sum_{j=0}^{n} a^{n}\ (1/j!)\ \gamma_{n}^{(\alpha+r)}\ (x,r,p,k)\,u^{j}$$

This completes proof of the theorem

VIII. CONCLUSION

As we can see, the results are different in many conditions of generalization for one member of the pair of Konhauser biorthogonal polynomials, generating relation and its really get multiple representation of the such kind operators which are discuss in the above results.

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