



[white paper: pedagogical]

Diamond Open Access

[waiting peer review]

The isomorphism between structures is an equivalence relation

Open Mathematics Collaboration^{*†}

September 5, 2021

Abstract

We present a pedagogical proof that the function of an isomorphism between two structures is an equivalence relation.

keywords: isomorphism, structures, equivalence relation, first-order logic

The most updated version of this white paper is available at

<https://osf.io/7uwnh/download>

<https://zenodo.org/record/5459284>

Introduction

1. This is a pedagogical white paper on *first-order logic*.
2. Our purpose is to discuss a result in [1] which is licensed under [2].
3. We use minimal notation but preserving all relevant mathematical information.

^{*}All *authors* with their *affiliations* appear at the end of this white paper.

[†]Corresponding author: mplobo@uft.edu.br | **Open Mathematics Collaboration**

Meta-linguistic symbols

- 4. $:=$ means that what is on the left is defined by what is on the right.
- 5. \equiv means that the strings on both sides are identical.
- 6. $\mathbf{a}, \mathbf{b} \vdash \mathbf{c}$ means deduction of \mathbf{c} from \mathbf{a}, \mathbf{b} .

Proposition

- 7. *The function of an isomorphism between two structures is an equivalence relation.*

Given

- 8. $\mathfrak{A}, \mathfrak{B} := \mathcal{L}$ -structures
- 9. $\mathfrak{A} \cong \mathfrak{B}$
- 10. $A, B :=$ universes of \mathfrak{A} and \mathfrak{B} , respectively
- 11. Definition: Two \mathcal{L} -structures are *isomorphic*, written $\mathfrak{A} \cong \mathfrak{B}$, if there exists a *bijection* $\iota : A \rightarrow B$ such that
 - (a) $\iota(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ for any constant symbol c of \mathcal{L} ,
 - (b) $\iota(f^{\mathfrak{A}}(a_1, \dots, a_n)) = f^{\mathfrak{B}}(\iota(a_1), \dots, \iota(a_n))$ for any n -ary function symbol f of \mathcal{L} and any $a_1, \dots, a_n \in A$,
 - (c) $\langle a_1, \dots, a_n \rangle \in R^{\mathfrak{A}} \leftrightarrow \langle \iota(a_1), \dots, \iota(a_n) \rangle \in R^{\mathfrak{B}}$ for any n -ary relation symbol R of \mathcal{L} and any $a_1, \dots, a_n \in A$.

Proof of (7)

- 12. We need to prove that \cong is an **equivalence relation**, i.e.,
 - (a) $\mathfrak{A} \cong \mathfrak{A}$, (reflexive)
 - (b) $(\mathfrak{A} \cong \mathfrak{B}) \rightarrow (\mathfrak{B} \cong \mathfrak{A})$, (symmetric)

(c) $(\mathfrak{A} \cong \mathfrak{B} \text{ and } \mathfrak{B} \cong \mathfrak{C}) \rightarrow (\mathfrak{A} \cong \mathfrak{C}), \quad (\text{transitive}).$

Reflexive

13. First, we prove that $\mathfrak{A} \cong \mathfrak{A}$.

14. Let $\iota : A \rightarrow A$ such that $\iota(a) = a$ for any $a \in A$.

15. It is clear that (14) is a bijection.

16. (14) satisfies (11.a).

17. $(9), (11.b), (14) \vdash \iota(f(a_1, \dots, a_n)) = f(\iota(a_1), \dots, \iota(a_n)) = f(a_1, \dots, a_n)$

18. (14) satisfies (11.c).

Symmetric

19. Next, we prove that $(\mathfrak{A} \cong \mathfrak{B}) \rightarrow (\mathfrak{B} \cong \mathfrak{A})$.

20. Assume $\mathfrak{A} \cong \mathfrak{B}$.

21. Due to (20), there is a bijection $\iota : A \rightarrow B$ (an isomorphism) such that (11) is satisfied.

22. Let $j : B \rightarrow A$ such that $j(b) = \iota^{-1}(b)$ for any $b \in B$.

23. (21), (22) $\vdash j$ is a bijection from B into A .

24. Now we need to prove that j satisfies (11).

25. We now prove (11.a), i.e., that

$$j(c^{\mathfrak{B}}) = c^{\mathfrak{A}}$$

for any constant symbol c of \mathcal{L} .

26. $(21), (22) \vdash j(c^{\mathfrak{B}}) = \iota^{-1}(c^{\mathfrak{B}}) = \iota^{-1}(\iota(c^{\mathfrak{A}})) = c^{\mathfrak{A}}$

27. We now prove (11.b), i.e., that

$$j(f^{\mathfrak{B}}(b_1, \dots, b_n)) = f^{\mathfrak{A}}(j(b_1), \dots, j(b_n))$$

for any n -ary function symbol f of \mathcal{L} and any $b_1, \dots, b_n \in B$.

28. Hereafter we assume $n = 1$, one can easily generalize the proof to $n > 1$.

29. Pick an arbitrary $b \in B$.

30. Since (21), $\exists! a : \iota(a) = b$.

31. ι is an isomorphism from A to $B \vdash \iota(f^{\mathfrak{A}}(a)) = f^{\mathfrak{B}}(\iota(a))$

$$\begin{aligned} 32. \quad j(f^{\mathfrak{B}}(b)) &= \iota^{-1}(f^{\mathfrak{B}}(b)) = \iota^{-1}(f^{\mathfrak{B}}(\iota(a))) = \iota^{-1}(\iota(f^{\mathfrak{A}}(a))) = f^{\mathfrak{A}}(a) = \\ &= f^{\mathfrak{A}}(\iota^{-1}(b)) = f^{\mathfrak{A}}(j(b)) \end{aligned}$$

33. We now prove (11.c), i.e., that

$$\langle b_1, \dots, b_n \rangle \in R^{\mathfrak{B}} \leftrightarrow \langle j(b_1), \dots, j(b_n) \rangle \in R^{\mathfrak{A}}$$

for any n -ary relation symbol R of \mathcal{L} and any $b_1, \dots, b_n \in A$.

34. Again, assume $n = 1$ like in (28).

$$35. \quad (11.c), (21) \vdash a \in R^{\mathfrak{A}} \leftrightarrow \iota(a) \in R^{\mathfrak{B}}$$

36. Pick an arbitrary $b \in B$ such that $\iota(a) = b$.

$$37. \quad b \in R^{\mathfrak{B}} \leftrightarrow \iota(a) \in R^{\mathfrak{B}} \leftrightarrow a \in R^{\mathfrak{A}} \leftrightarrow \iota^{-1}(b) \in R^{\mathfrak{A}} \leftrightarrow j(b) \in R^{\mathfrak{A}}$$

38. (26), (32), (37) $\vdash \cong$ is a **symmetric** relation.

Transitive

39. Let $\iota : A \rightarrow B$ be an isomorphism from \mathfrak{A} to \mathfrak{B} .

40. Let's define $\iota : A \rightarrow B$ by $\iota(a) = b$.

$$41. \quad \iota(f^{\mathfrak{A}}(a)) = f^{\mathfrak{B}}(\iota(a))$$

$$42. \langle a \rangle \in R^{\mathfrak{A}} \leftrightarrow \langle j(a) \rangle \in R^{\mathfrak{B}}$$

43. Let $j : B \rightarrow C$ be an isomorphism from \mathfrak{B} to \mathfrak{C} .

44. Let's define $j : B \rightarrow C$ by $j(b) = c$.

$$45. j(f^{\mathfrak{B}}(b)) = f^{\mathfrak{C}}(j(b))$$

$$46. \langle b \rangle \in R^{\mathfrak{B}} \leftrightarrow \langle j(b) \rangle \in R^{\mathfrak{C}}$$

47. Let's define $k : A \rightarrow C$ by $k(a) = j(\iota(a))$.

48. We now prove (11.a), i.e., that

$$k(c^{\mathfrak{A}}) = c^{\mathfrak{C}}$$

for any constant symbol c of \mathcal{L} .

$$49. k(a) = j(\iota(a)) = j(b) = c$$

50. We now prove (11.b), i.e., that

$$k(f^{\mathfrak{A}}(a_1, \dots, a_n)) = f^{\mathfrak{C}}(k(a_1), \dots, k(a_n))$$

for any n -ary function symbol f of \mathcal{L} and any $a_1, \dots, a_n \in A$.

51. To improve the readability, we will

(a) omit the parenthesis;

(b) define (temporarily) $A := \mathfrak{A}$, $B := \mathfrak{B}$, and $C := \mathfrak{C}$.

52. Let's write in the following what we know from both isomorphisms, (39) and (43), using the notation in (51).

$$53. \iota a = b, \quad j b = c; \quad j^{-1} c = b, \quad \iota^{-1} b = a$$

$$54. j a = c, \quad \iota^{-1} j^{-1} c = a$$

$$55. k a = j a, \quad k a = k \iota^{-1} b = k \iota^{-1} j^{-1} c$$

56. $k^{-1}c = k^{-1}jb = k^{-1}ja$

57. $\iota f^A a = f^B \iota a, \quad j f^B b = f^C j b$

58. $j^{-1} f^C c = f^B j^{-1} c, \quad \iota^{-1} f^B b = f^A \iota^{-1} b$

59. We need to prove that

$$k f^A a = f^C k a.$$

60. $f^C k a = j j^{-1} f^C k a = j f^B j^{-1} k a = j \iota^{-1} f^B j^{-1} k a = j f^A \iota^{-1} j^{-1} k a = j f^A a = k f^A a$

61. We now prove (11.c), i.e., that

$$\langle a_1, \dots, a_n \rangle \in R^{\mathfrak{A}} \leftrightarrow \langle k(a_1), \dots, k(a_n) \rangle \in R^{\mathfrak{C}}$$

for any n -ary relation symbol R of \mathcal{L} and any $a_1, \dots, a_n \in A$.

62. Suppose $a \in R^{\mathfrak{A}}$.

63. $(40), (44), (47) \vdash k(a) = j(\iota(a)) \in R^{\mathfrak{C}}$

64. Suppose $k(a) \in R^{\mathfrak{C}}$.

65. Let's define $k^{-1} : C \rightarrow A$ by $k^{-1}(c) = \iota^{-1}(j^{-1}(c))$.

66. $(39), (40), (43), (44), (65) \vdash k^{-1}(c) = \iota^{-1}(j^{-1}(c)) = a \in R^{\mathfrak{A}} \quad \square$

Final Remarks

67.

$$(\mathfrak{A} \cong \mathfrak{B}) \leftrightarrow (\cong \text{ is an equivalence relation})$$

Open Invitation

Review, add content, and co-author this white paper [3, 4].

*Join the **Open Mathematics Collaboration**.*

Send your contribution to mplobo@uft.edu.br.

Open Science

The **latex file** for this *white paper* together with other *supplementary files* are available in [5, 6].

How to cite this paper?

<https://doi.org/10.31219/osf.io/7uwnh>

<https://zenodo.org/record/5459284>

Acknowledgements

+ Center for Open Science

<https://cos.io>

+ Open Science Framework

<https://osf.io>

+ Zenodo

<https://zenodo.org>

Agreement

All authors agree with [4].

References

[1] Leary, Christopher C., and Lars Kristiansen. *A friendly introduction to mathematical logic*, 2nd edition, 2015.

<https://knightscholar.geneseo.edu/geneseo-authors/6>

[2] CC. Creative Commons. *CC-By Attribution 4.0 International*.

<https://creativecommons.org/licenses/by/4.0>

- [3] Lobo, Matheus P. “Microarticles.” *OSF Preprints*, 28 Oct. 2019.
<https://doi.org/10.31219/osf.io/ejrct>
- [4] Lobo, Matheus P. “Simple Guidelines for Authors: Open Journal of Mathematics and Physics.” *OSF Preprints*, 15 Nov. 2019.
<https://doi.org/10.31219/osf.io/fk836>
- [5] Lobo, Matheus P. “Open Journal of Mathematics and Physics (OJMP).” *OSF*, 21 Apr. 2020. <https://osf.io/6hzyf/files>
- [6] <https://zenodo.org/record/5459284>

The Open Mathematics Collaboration

Matheus Pereira Lobo (lead author, mplobo@uft.edu.br)^{1,2}
<https://orcid.org/0000-0003-4554-1372>

¹Federal University of Tocantins (Brazil)

²Universidade Aberta (UAb, Portugal)