Mathematics as a social practice? Antagonisms as a conceptual tool for examining discourses

Dionysia Pitsili-Chatzi, University of Ottawa, 🖃 dpits102@uottawa.ca

This paper discusses the notion of antagonism as a conceptual tool for examining discourses in mathematics education, with a focus on the political dimensions of discourse. Drawing upon a short university classroom interaction about a visual proof, I suggest that the discourse of mathematics as a product and the discourse of mathematics as a social practice are in an antagonistic relation. I illustrate this relation by exploring how each discourse tries to fix the meaning of concepts like proof, formal proof, and mathematics. Finally, I explore how antagonisms can describe political aspects of discourses, by discussing the limits of both hegemonic discourses and counter-discourses, as well as the space for challenging hegemonic discourses.

Mathematics: A product or a social practice?

Mathematics education has been largely framed by what we might call "mathematics as a product", namely the idea that students need to learn the product of academic mathematicians' labour. However, as mathematics education took a social turn (Lerman, 2000), we notice an embracement of epistemological and theoretical frameworks which highlight the social character of learning. At the same time, the focus towards non-academic mathematics (e.g., ethnomathematics) also contributes to the development of a discourse which conceptualises mathematics as a human activity (e.g., Pais, 2013). Thus, we can identify two discourses which conceptualise mathematics in roughly two opposing ways: the discourse of mathematics as the product of mathematicians' labor and the discourse of mathematics as a social practice (of students, mathematicians, or communities).

Valero (2018) maintains that the practices and meanings of mathematics education are not fixed, but there is rather a constant struggle with regards to their validity and legitimisation. The relation between looking at, teaching, and learning mathematics as if it were a social practice versus as if it were a product is an example of this kind of struggles. This paper aims to explore the relation between the two discourses, by discussing how the notion of antagonism (Laclau & Mouffe, 2001) can be used as a conceptual tool, which allows consideration for the political aspects of discourse. I illustrate this idea by using the notion of antagonism to analyse a short interaction in a mathematics university classroom, where a visual proof is presented and discussed. I argue that the discourse of mathematics as a

Please cite as: Pitsili-Chatzi, D. (2021). Mathematics as a social practice? Antagonisms as a conceptual tool for examining discourses. In D. Kollosche (Ed.), *Exploring new ways to connect: Proceedings of the Eleventh International Mathematics Education and Society Conference* (Vol. 3, pp. 777–786). Tredition. https://doi.org/10.5281/zenodo.5416089

product and the discourse of mathematics as a social activity are in an antagonistic relation, each trying to fix the meaning of concepts like proof, formal proof, and mathematics. The paper is organised as follows: I first draw upon the work of Laclau and Mouffe to outline the role of antagonisms in their discourse theory; I then continue with the context, the presentation and the analysis of the interaction; and I finally discuss how looking at the interplay between the two discourses as an antagonistic relationship helps highlight the political aspects of the discourses.

Antagonisms in the Discourse Analysis theory of Laclau and Mouffe

There is a growing body of mathematics education research, which uses discourse analysis as a theoretical, epistemological, and methodological framework. In discourse theories, language is not a mere description of social reality; instead, language plays a role in constituting social reality, or, in other words, the ways in which we talk about things give meaning to them. To illustrate, the case of a student suggesting that 1 is a solution of the equation $x+1=3$ could be described as them experimenting with mathematical equations, not having mastered the equation solving algorithm, lacking previous knowledge, not being careful, learning by making a mistake, or it could even not be the focus of anyone's attention. Following Doxiadis' (2011) Foucauldian conceptualisation of discourse, all these discourses refer to the very same event, but make different identities available for students, produce different kinds of knowledge about the student and their mathematical learning, and guide the teacher's, students', and researchers' actions in different ways, thus creating different power relations.

Laclau and Mouffe (2001) conceptualize discourses as partial fixations of meaning. They maintain that "any discourse is constituted as an attempt to dominate the field of discursivity, to arrest the flow of differences, to construct a centre" (p. 112). Different discourses struggle to fix the meaning of privileged points called floating signifiers, while points whose meaning has been crystalized within a discourse are called nodal points (Jørgensen & Phillips, 2011). For example, learning is a floating signifier, in the sense that different learning theories, such as behaviourism and social constructivism, struggle to fix its meaning. Learning can also be a nodal point within a particular discourse, such as a constructivist discourse. However, the meaning of "learning" shall never be permanently fixed: even if a discourse (for example, social constructivism) becomes hegemonic, this fixation is never final, as other discourses will eventually strive to contribute to, challenge, or deconstruct this meaning.

This is where the concept of antagonism becomes relevant. Antagonisms occur when discourses collide, struggling to fix the meaning of floating signifiers, or privileged points within the field of discursivity (Jørgensen & Phillips, 2011). For example, in mathematics education discourses, mathematics is a floating signifier, in the sense that different discourses struggle to fix its meaning and its limits. An ethnomathematical discourse and a cognitive theory discourse would (typically) fill the meaning of mathematics differently, thus depicting mathematics education differently. Nevertheless, as antagonistic discourses attempt to dominate the field of discursivity, meaning can never be fixed (Laclau & Mouffe, 2001). While antagonisms can be resolved through a hegemonic intervention, a perspective which has earned social consensus (Jørgensen & Phillips, 2011), this is only a temporary state. Domination of the field of discursivity can never be fully achieved.

The importance of antagonisms as a concept in Laclau and Mouffe's theory lies with the idea that the political is based on antagonisms and is constitutive of human societies (Mouffe, 2005). This is an ontological statement which asserts that it is through discourses and the antagonisms between them that meaning is produced. Stavrakakis (1997) maintains that this theory aims to move from a reductionist logic towards a logic of articulation: "since social identities do not arise from the (class or other) essence of (individual or collective) subjects, they can only be a product of construction, a construction at the level of discourse, a construction that articulates heterogeneous elements and gives them new meaning" (p. 24; my translation). Therefore, the political is constitutive of mathematics and mathematics education: all things mathematical do not exist in an a priori sense; mathematics and mathematics education are/become as a result of antagonisms between discourses.

Antagonism of discourses in a university mathematics classroom Interaction

In this section, I use the notion of antagonism to analyse a short interaction between the professor and a student in a university "Introduction to Proofs" classroom. I collected these data as part of my doctorate thesis, titled "Political Aspects of Mathematics in two Undergraduate Mathematics Courses". The interaction discussed here is about the formality of a proof and I chose this piece of data because of its potential to highlight the notion of antagonism between two discourses.

Proofs and visual proofs in mathematics education literature

Proofs are considered a vital part of academic mathematics. Proofs do not only verify the truthfulness of a mathematical statement, but have a variety of other functions, including explanation, systematisation, discovery, and communication (de Villiers, 1990). From a socio-political perspective, it is important to consider the role of proofs as devices which can have the final say about a mathematical statement: if something has been proven mathematically, there is no doubt that it is true (e.g., Gutiérrez, 2013). In this way, not only proving but also mathematics itself are constructed as a type of knowledge which cannot be challenged.

The proof discussed here is what we may call a visual argument/proof (e.g., Alsina & Nelsen, 2010), although the professor himself does not use the word "visual" to describe it. While there seems to be a consensus that visual representations are heuristic tools both in the philosophy of mathematics and in mathematics education, their status as proofs or even as parts of proofs is contested (Hanna & Sidoli, 2007). Hanna and Sidoli (2007) talk about the different perspectives regarding the role of visualisation in proofs and distinguish between three conceptualisations of visual representations: as adjuncts to proofs, as an integral part of proofs, and as proofs themselves. We can see this tension (or antagonism between discourses about the status of visual proofs) in how Alsina and Nelsen's (2010) paper "An

Invitation to Proofs without Words" refers to "visual proofs" as "pictures or diagrams that help the reader see why a particular mathematical statement may be true, and also to see how one might begin to go about proving it true" (p. 118). Although Alsina and Nelsen name these visual arguments "proofs" and argue about their importance in mathematical practice, they also describe them as a starting point for proving the theorem, indicating that the proof starts after the visual argument has taken place.

The interaction and its context

The interaction happens in a large class of approximately 150 students. The course is an "Introduction to Proofs" course in a University in Ontario (Canada). Introduction to Proofs courses are typical 1st year courses in North America for students who wish to follow a mathematics-related discipline. Most students in this course aspire to follow a computer science program and in order to enter it, they need to achieve a high grade in the proofs course.

The interaction happens in the chapter of cardinality, which is a formal mathematical concept for describing the "size" of a set (including infinite sets). Here, the class focuses on proving that the cardinality of the naturals is the same as the cardinality of the cartesian product of the integers with the integers (i.e., Z×Z). An alternative description of this statement is that the infinite set whose elements are all pairs between the numbers $0, \pm 1, \pm 2, \pm 3, \ldots$ (e.g., $(-2,5), (5,-2), (3,1), \ldots$) has exactly as many elements as the set of natural numbers (i.e., $\{0,1,2,3,\ldots\}$). To prove this statement, the professor presents the visual proof shown in Figure 1, which is based on identifying a spiral representation of all elements of Z×Z, so as to consider them as elements of a sequence (i.e., constructing a bijection from N to Z×Z).

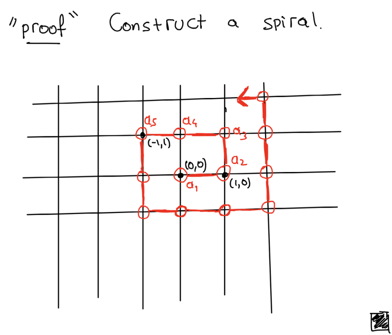

Figure 1: Visual proof (Recreation of the picture that I took of the board)

Before the following interaction happens, the professor has orally suggested that the proof is not a formal one, while – as can be seen in the figure – the word proof itself has been written in quotation marks on the board (indicating that it might not be a "proper" proof). Regarding the reason that this (visual) proof is presented in the class, the professor mentions that "a formal proof of this is annoying" and that the way in which the proof is written in the textbook might be difficult to understand. After the presentation of this (visual) proof has been completed, the professor asks students if they have any questions and the following interaction happens:

Student: Is this considered a formal proof?

Professor: This is not considered a formal proof, uhm, partly because it's- ok. Here's what [inaudible] Could you find the tenth element of this sequence? And to put it another way, if I asked you to find the tenth element and I asked someone else to find the tenth element in this room, would you come up with the same tenth element?

Student: [definitively] Yes.

Professor: So, that's pretty good. If the answer is "yes", that means that there is a pretty good common understanding of what's happening here. Uhm, ah, we could write that a little more formally, like, take one step to the right, then one step up. Then you go left one more than you've gone, go right one more than you've gone. And, like, I could explain that, like, robotically. But it's not clear that it's going to be easy to understand. Or that it will be, like, you have a better understanding of it. Uhm, yeah, we typically avoid presenting the complete formal proof of this, because it hides what's actually going on.

In this interaction, a student asks if the (visual) proof given by the professor is considered a formal proof. The professor categorically responds that it is not. As he proceeds to explain why this is the case, he asks the student back, whether the proposed sequence is presented in a way that any two people would find the same tenth element. The student responds with a definite "yes", which is a response that does not support the professor's argument regarding the informality of the proof. The professor accepts this as there being a "pretty good common understanding" and then suggests that this proof could be written more formally (through a more "robotic" description of the steps) but it then would hinder understanding.

The two discourses and the antagonism between them

The above interaction can be read as two discourses being in antagonism: mathematics as a social practice and mathematics as the product of an established discipline. In this section, I will first describe how I see each of the two discourses as being present in the interaction and I will then discuss the antagonistic character of their relation.

The first of the two discourses conceptualises mathematics as a social practice. This refers to mathematics being done in and through interactions between people. In the example, we can see the discourse of mathematics as a social practice in the following aspects: the discursive construction of a "good" proof as one which facilitates understanding (e.g., as shown in the sentence: "we typically avoid presenting the complete formal proof of this, because it hides what's actually going on"); the very fact that a visual proof is presented in order to make the idea of the argument clear and communicable; and the need for people to have a shared understanding of the argument (e.g., as shown in "would you come up with the same tenth element?"). In the Proofs class, the discourse of mathematics as a social practice does not only appear in this interaction; it is also evident in multiple events, including when the professor encourages collaboration and communication between the students, invites students to share their disagreements and discomforts with the presented ideas, engages students in a "collecting data – making conjectures – proving the conjecture" process, emphasises the existence of a diversity of thinking about mathematics among different people, and emphasises the need for mathematical arguments to be understood by their audience(s). In all these events, mathematics is conceptualised as a social practice.

The second discourse is about mathematics as the product of an established discipline. In the interaction, we can see the manifestation of this discourse in the clarity with which a visual proof does not quite qualify as a formal proof and maybe not even as a proof. This clarity implies that the standards of what constitutes a proof and a formal proof are created outside the walls of the classroom, independently of the professor's and students' interactions (e.g., passive voice in "this is not considered a formal proof"). These standards are the result of the practices and norms of the mathematical community, which are in turn reflected in the academic requirements set for novice mathematicians and computer scientists by mathematics departments. In this context, it is the mathematical result (i.e., theorem, proof, etc.) expressed according to the traditional norms and rules, which is of most importance. A formal expression of this result can verify that the result is correct, can be tested, and is expressed in a way in which no misunderstanding may occur. In this discourse, while mathematics as a product might be socially created, it is also reified, taking, in a sense, a life of its own.

The first point which I wish to highlight about how the two discourses are in an antagonistic relation is that by antagonism, I do not refer to the disagreement between the professor and the student regarding the preciseness of the algorithm's presentation; in other words, here, we do not have two people (the professor and the student) antagonising each other. The two discourses are not each voiced by the professor and student respectively; instead, both discourses are voiced by the professor himself. For example, we can see both discourses present in the question "if I asked you to find the tenth element and I asked someone else to find the tenth element in this room, would you come up with the same tenth element?". Here, the idea that a formal proof leaves no space for ambiguity refers to proof as a product; however, the enunciation of it in terms of "actual" people having an "actual" conversation and agreeing or disagreeing highlights mathematics as a social practice.

By suggesting that the two discourses are in an antagonistic relation, I intend to highlight that the concepts of proof, formal proof, and mathematics become floating signifiers, whose meaning is attempted to be fixed by the two discourses. Regarding "proof" as a floating signifier, we see that the professor orally refers to the visual argument as being a proof, but he uses quotation marks when writing down the word proof in the notes. This difference between the oral and written practice can be interpreted in terms of the presence of the two aforementioned discourses. On the one hand, the discourse of mathematics as a social practice attempts to conceptualise proofs as including well-articulated visual arguments. On the other hand, the discourse of mathematics as an established practice attempts to conceptualise proofs as a formal construct in which each proposition follows from the previous one. The two discourses are differently privileged in the professor's written and oral speech. Mathematics as the product of mathematicians' work is privileged in the professor's written speech, in which the quotation marks indicate that the visual argument is similar to a proof but does not quite qualify as one. Mathematics as a social practice is privileged in the professor's oral speech, in which he never challenges the status of the proof as a proof. The differently privileged manifestation of the two discourses in the oral and written speech can be understood in terms of the different functions of the oral and written speech: in his oral speech, the professor aims to help students understand an argument, while his written speech aims (among other purposes) to act as notes for students to study. In the example, the two discourses collide with each other, while struggling to "fix" the meaning of "proof" in different ways.

With regards to "formal proof" as a floating signifier, the discourse of mathematics as a social activity tries to fix the meaning of the formality of a proof in terms of the existence of a "common understanding". In the interaction, the student suggests and the professor accepts that there is a common understanding regarding the tenth element of the sequence. Yet, this does not result in the recognition of the proof as formal. In terms of collision of discourses, we can argue that there seems to be "something more" about the formality of a proof which the discourse of mathematics as a social practice cannot quite challenge. This "more" is created by the discourse of mathematics as an established product, which is at a privileged position for defining what is formal. Formality – as developed historically – involves much more than the disappearance of the possibility of a mistaken representation, such as compliance with strict linguistic rules. To use Laclau and Mouffe's (2001) concepts, the discourse of mathematics as an established practice is *hegemonic*, when it comes to formality of proofs. Playing a bit more with the metaphor of collision in a visual way, I would suggest that as the discourse of mathematics as a social practice attempts to fix the meaning of formality, it hits the "wall" of mathematics as an established practice. The wall is not unbreakable, but it is extremely hard to break.

As far as mathematics as a floating signifier is concerned, it is notable that the word mathematics is never mentioned in the above incident. Nevertheless, I argue that there is an implicit antagonism about its meaning in the above interaction. As proof has developed to be the golden standard of mathematics and formal proof has developed to be the golden standard of proofs, antagonisms regarding the meaning of "proof" and "formal proof" are also antagonisms about mathematics. In other words, what constitutes "mathematics", "academic mathematics", or "formal mathematics" are at stake here. There is no a priori meaning of these concepts; instead their meaning is constantly negotiated and produced through antagonisms between discourses, which are historically, culturally, and politically situated.

As the two discourses are in an antagonistic relation in the professor's articulation, we can see the limits of each discourse. Laclau and Mouffe (2001) write that "[a]ntagonism, far from being an objective relation, is a relation wherein the limits of every objectivity are shown – in the sense in which Wittgenstein used to say that what cannot be said can be shown" (p. 125). In the example above, we can see the limits of the discourse about mathematics being a social practice when it comes to the universality of understanding: if universality of understanding fails, then the discourse of mathematics as a social activity fails, too. We can also see the limits of the same discourse in the non-negotiability of whether a visual argument (in the context of cardinality) constitutes a formal proof. On the other side of the antagonistic relation, mathematics as an established practice fails when its formal presentation hinders the meaning which it attempts to purify.

Mathematics as a product vs. a social activity: An antagonistic relation

Besides this course, the antagonism between the discourse of mathematics as a social practice and mathematics as a product is also present more generally in mathematics education without being necessarily framed as antagonistic. In this last section of this paper, I argue that the concept of antagonism is able to capture the relation between the two discourses, in a way which highlights its political aspects. My argument runs as follows: I describe how the two discourses (social practice and product) are evident in mathematics education; I continue by examining and critiquing an alternative narrative which captures their relation as a relation of progress; I then explore how this relation can be understood as an antagonistic one; and I conclude by suggesting that the notion of antagonism is useful for highlighting political aspects of discourses.

For quite some time, in mathematics education, the idea that mathematics is what mathematicians do has been hegemonic. Within what might be called "the social turn" in mathematics education, we see the gradual prevalence of a counter-discourse that suggests that mathematics is a social practice; a discourse that conceptualises meaning, thinking, and reasoning (both in the classroom and in mathematicians' work) as products of a social activity (Lerman, 2001). Contrary to Platonic views which claim that mathematics is somewhere out there (not in our tools, actions, or discussions), the discourse of mathematics as a social practice conceptualises mathematics as happening through people's communication and actions. At the same time, however, the discourse of mathematics as a product preserves much of its hegemony in mathematics education. We can see this discourse in two aspects: first, academic mathematics bears a hegemonic status: its truth, once produced, is not negotiable (Gutiérrez, 2013). Second, as schooling has a function of distributing students into different social positions (e.g., Pais, 2013), the institutional framework is instrumental in reinforcing the discourse of mathematics as a product. For example, "the kinds of mathematics that we teach" is described in curricula across the world through specific outcomes and objectives that the students are required to meet (Abtahi, 2020). Similarly, standardized assessment is enabled through a discourse of mathematics as a product: every student is assessed on the same piece of knowledge, which has been created outside the classroom walls.

One way of describing the relation between the two discourses is through a story of progress: mathematics education used to focus on mathematics as a product, but it has gradually turned towards focusing on the social elements of doing mathematics. In this story, mathematics as a social practice is framed as a set of more recent moments in a linear progressive path followed by a mathematics education which constantly becomes better, more participatory, and more equitable. In this way, mathematics as a social practice is seen as contributing to mathematics education constantly becoming "better", more participatory, and more inclusive, mitigating or challenging the impersonal, inequitable character of ("old") mathematics. This narrative of progress runs through mathematics education which, both as research and practice, appears to be committed to straightforward progress (Lllewellyn, 2015). Deriving from Laclau and Mouffe's (2001) view on the impossibility of meaning fixation, we can see the fragility of this approach: even if we agreed that making mathematics more activity based contributes to a betterment of mathematics education, there is no reason to assume that at some point, through this practice, mathematics education will become fully participatory or equitable. In my view, this narrative of progress fails to address how the "old practice" is still here, hegemonic, trying to absorb the "new" practice.

The concept of antagonisms (Laclau & Mouffe, 2001) offers an alternative reading of this relation which suggests that both discourses co-exist in mathematics education, while each of them tries to fix the meaning of mathematics; the result of this collision is unfixed. As attempted with the analysis of the short interaction about the visual proof, the conceptualisation of antagonism can be helpful for addressing the limits of both hegemonic discourses and counter-discourses, as well as the space for challenging hegemonic discourses. In the example of the visual proof, the discourse of mathematics as an established practice was hegemonic but we can recognise its limits (e.g., it sometimes hinders understanding) as well as the ways in which the discourse might be challenged, through discourses which value people's understanding and interactions. At the same time, an antagonism-oriented analysis makes visible the limits of the ability of the counter-discourse (i.e., mathematics as a social practice) to challenge the established practice of mathematics as a product.

While there seems to be a wide agreement in mathematics education about mathematics being a "social practice", this discourse seems to often oversee the hegemony and impact of mathematics as a product. In other words, I suggest that the discourse of mathematics as a social practice (within institutions in which students are expected to master a specific material, be examined in it, and be accepted or not in a program based on their performance) falls very short in terms of challenging the existence of a knowledge which will not change, regardless of what the people talking about it think or do. In the example from the Proofs course, the discourse of mathematics as a social practice cannot be fully actualised as it collides with the discourse of mathematics as a product. Looking at mathematics education research and practice more generally, I want to conclude this paper with the following questions. The discourses of mathematics as a product and a social practice both exist in literature and there is a general urge to move from the former to the latter. What are the ideological functions of mathematics education discourses which highlight the social without addressing its shortcomings? Can an antagonism-oriented view of the two discourses' relation highlight how mathematics education research is political and how its constructs (e.g., the discourses about what mathematics is) are products of and agents in power relations?

Many thanks

Many thanks to Richard Barwell, Yasmine Abtahi, and Karli Bergquist for their helpful thoughts on some of the ideas in this paper. I am also required to acknowledge that part of this work has been conducted while on a scholarship by the Onassis Foundation, which I also wish to thank.

References

Abtahi, Y. (2020). Clever or wise: The kinds of mathematics we teach are preparing us for what? The story of my worry. *Philosophy of Mathematics Education Journal, 36.*

Alsina, C., & Nelsen, R. B. (2010). An invitation to proofs without words. *European Journal of Pure and Applied Mathematics, 3*(1), 118–127.

de Villiers, M. (1990). The role and function of proofs in mathematics. *Pythagoras,* Unknown volume, 17–24.

Doxiadis, K. (2011). *Discourse analysis: A social-philosophical grounding.* Common Ground.

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education, 44*(1), 37–68.

Hanna, G., & Sidoli, N. (2007). Visualisation and proof: A brief survey of philosophical perspectives. *ZDM Mathematics Education, 39,* 73–78.

Jørgensen, M., & Phillips, L. (2011). *Discourse analysis as theory and method.* Sage.

Laclau, E., & Mouffe, C. (2001). *Hegemony and socialist strategy: Towards a radical democratic politics* (2nd ed.). Verso.

Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 19–44). Ablex.

Llewellyn, A. (2015). Problematising the pursuit of progress in mathematics education. *Educational Studies in Mathematics, 92*(3), 299–314.

Mouffe, C. (2005). *On the political.* Routledge.

Pais, A. (2013). Ethnomathematics and the limits of culture. *For the Learning of Mathematics, 33*(3), 2–6.

Stavrakakis, G. (1997). Introduction. In G. Stavrakakis (Ed.), *Ernesto Laclau: Για την επανάσταση της εποχής μας* [Ernesto Laclau: New reflections on the revolution of our time]. Νήσος.

Valero, P. (2018). Human capitals: School mathematics and the making of homus oeconomicus. *Journal of Urban Mathematics Education, 11*(1-2), 103–117.