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ANALYSIS OF MULTI-COMPONENT NON-STATIONARY SIGNALS USING FOURIER-BESSEL TRANSFORM AND WIGNER DISTRIBUTION

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ABSTRACT

We present a new method for time-frequency representation (TFR), which combines the Fourier-Bessel (FB) transform and the Wigner-Ville distribution (WVD). The FB transform decomposes a multi-component signal into a number of mono-component signals, and then the WVD technique is applied on each component of the composite signal to analyze its time-frequency distribution (TFD). The simulation results show that the proposed technique based on the FB decomposition is a powerful tool for analyzing multi-component non-stationary signals and for obtaining the TFR of the signal without cross terms.

1. INTRODUCTION

In many engineering applications such as speech analysis, speech synthesis, radar, sonar and telecommunications, the signals under considerations are known to be non-stationary, for which the signal parameters are time-varying. For spectral analysis of such type of signals, the discrete Fourier transform (DFT) can not be employed. The time-frequency analysis technique [1], among other methods, was proposed to deal with such signals.

The short-time Fourier transform (STFT) is one of the earliest methods used for time-frequency analysis. A moving window cuts out a slice of the signal, and the Fourier transform of this slice gives the local properties of the signal. The spectrogram, which is the squared magnitude of the STFT, is used for the analysis of non-stationary signals. The result of analysis depends on the choice of the window function leading to a trade off between time localization and frequency resolution [1].

Another commonly used TFD is the Wigner-Ville distribution (WVD) [1, 2]. Theoretically the WVD has an infinite resolution in time due to the absence of averaging over any finite time interval. Moreover for infinite lag length, it has an infinite frequency resolution. The WVD being quadratic in nature introduces cross terms for a multi-component signal. The cross terms can have significant amplitudes and they can corrupt the transform space. A particular application where the cross terms can have serious implications is speech analysis, because speech can be modeled as a sum of amplitude modulated (AM) and frequency modulated (FM) signals corresponding to the formant frequencies [3, 4].

In the last two decades, the research has been carried out for effective suppression of the cross terms and improvement of the frequency resolution, preserving the desired properties of the quadratic TFDs [1]. The Choi-Williams distribution [5] has a tradeoff between cross term suppression and the frequency resolution. On the other hand, in the cone shaped kernel [6], the cross term suppression and the frequency resolution are achieved without much importance for the TFR properties. The reduced kernel [7] is an improved and generalized version of Choi-Williams distribution. In this direction, an application specific signal-dependent optimal kernel design [8], useful for different class of signals, is a major step. Based on orthogonal expansion like Gabor expansion [9], a decomposition of the WVD achieves a balance between cross terms and useful properties. A denoising approach [10] based on the shift-invariant wavelet packet (WP) decomposition has been proposed for adaptive suppression of the cross terms.

The WVD approach based on signal decomposition realized by a perfect reconstruction filter bank (PRFB) has been proposed [11]. The PRFB decomposes the multi-component signal into its components. The summation of the WVDs of the individual components results in the WVD of the composite signal, where cross terms and noise are significantly reduced [11].

In this paper, a new technique based on the FB transform has been proposed. The present technique combines the FB transform and the WVD. The FB transform decomposes a multi-component signal into its constituents like the PRFB in [11]. However, the major difference between the two approaches is that while using the PRFB-based technique we must know a priori the frequency-band of the signal, whereas no such information will be required for the proposed technique based on the FB transform.

2. THE WIGNER-VILLE DISTRIBUTION

The WVD of a signal $x(t)$ is defined in the time domain as

$$W_x(t,\omega) = \int_{-\infty}^{\infty} x(t+\tau/2)\, x^*(t-\tau/2)\, e^{-j\omega\tau} d\tau \qquad (1)$$

where $x^*(t)$ is the complex conjugate of $x(t)$. In the frequency domain, the WVD is defined as follows:

$$W_x(t,\omega) = \int_{-\infty}^{\infty} X(\omega+\xi/2)\, X^*(\omega-\xi/2)\, e^{j\xi t} d\xi \qquad (2)$$

where $X(\omega)$ is the Fourier transform of $x(t)$. The various desirable properties of the WVD such as preservation of time and frequency support, infinite time and frequency resolutions etc., make the WVD a useful tool for signal analysis [2]. The main drawback of this distribution is that it is quadratic and the method based on the WVD introduces the cross terms in the time-frequency domain making the transform space difficult to interpret [12]. The WVD of the sum of M signals

$$x(t) = \sum_{i=1}^{M} x_i(t) \tag{3}$$

is given by

$$W_x(t,\omega) = \underbrace{\sum_{i=1}^{M} W_{x_i}(t,\omega)}_{autocomponents} + \underbrace{\sum_{k=1}^{M-1}\sum_{l=k+1}^{M} 2\Re\left[W_{x_k x_l}(t,\omega)\right]}_{crosscomponent} \tag{4}$$

which shows that the WVD of the composite signal $x(t)$ has M autocomponents and $\binom{M}{2}$ cross-components, i.e., a cross term for every pair of autocomponents. The geometry of these cross terms on the time-frequency plane has been well defined in [13]. Let the base signal $x_0(t)$ be a linear chirp, $x_0(t) = e^{j(\omega_0 t + \frac{1}{2}\beta t^2)}$ and let $x_1(t) = x_0(t-t_1)e^{j\omega_1 t}$ and $x_2(t) = x_0(t-t_2)e^{j\omega_2 t}$ be the time and frequency shifted versions of the base signal. We form the composite signal $x(t) = x_1(t) + x_2(t)$, then, the WVD of $x(t)$ is given by

$$\begin{aligned} W_x(t,\omega) &= 2\pi\delta(\omega - (\omega_1+\omega_0) - \beta(t-t_1) \\ &+ 2\pi\delta(\omega - (\omega_2+\omega_0) - \beta(t-t_2) \\ &+ 4\pi\delta(\omega - (\omega_m+\omega_0) - \beta(t-t_m) \\ &\times \cos[\omega_d(t-t_m) - t_d(\omega-\omega_m) + \omega_d t_m] \end{aligned} \tag{5}$$

where $\delta(\omega)$ is the Dirac delta function that is zero everywhere except at the origin, $\omega_d = (\omega_2 - \omega_1)$, $t_d = (t_2 - t_1)$, $\omega_m = (\frac{\omega_1+\omega_2}{2})$, $t_m = (\frac{t_1+t_2}{2})$. It is observed from the above equation that the cross term (i) occurs mid-time, mid-frequency, (ii) oscillates at a frequency proportional to the difference in frequency- and time-shifts of the signals, (iii) oscillates in the direction orthogonal to the line that connects the autocomponents, and (iv) can have an amplitude twice as large as the amplitude of the WVD of each signal under consideration.

3. FOURIER-BESSEL (FB) TRANSFORM

A signal exhibiting characteristics of amplitude or frequency modulation or both may be more compactly represented by the Bessel function bases rather than by pure sinusoids [14]. The FB transform of order zero of a signal $x(t)$ is given by [15],

$$F_0(\alpha) = \int_0^\infty t\, x(t)\, J_0(\alpha t)\, dt \tag{6}$$

where J_0 is the Bessel function of the first kind of order zero. The inverse FB transform is defined as:

$$x(t) = \int_0^\infty \alpha\, F_0(\alpha)\, J_0(\alpha t)\, d\alpha \tag{7}$$

The Bessel functions are orthogonal with respect to the weighting functions t and α. The orthogonality relations are given by

$$\int_0^\infty t\, J_0(\alpha t)\, J_0(\alpha' t)\, dt = \frac{\delta(\alpha-\alpha')}{\alpha} \tag{8}$$

and

$$\int_0^\infty \alpha\, J_0(\alpha t)\, J_0(\alpha t')\, d\alpha = \frac{\delta(t-t')}{t} \tag{9}$$

The integral in (6) is also known as Hankel transform. We note that the FB transform coefficients $F_0(\alpha)$ are unique for a given signal $x(t)$, similar to the Fourier coefficients. Unlike the sinusoidal basis functions in the Fourier transform, the Bessel functions decay within the range of the signal, similar to the rise and fall of speech within a pitch interval [16, 17]. Let the signal $x(t)$ be a damped cosine signal given by

$$x(t) = e^{-\sigma t}\cos(\omega_0 t), \quad \sigma > 0 \tag{10}$$

Then the 0^{th} order FB transform can be computed as [18]

$$F_0(\alpha) = \frac{\alpha}{2}\frac{\left[\alpha^2 + (\sigma + j\omega_0)^2\right]^{\frac{3}{2}} + \left[\alpha^2 + (\sigma - j\omega_0)^2\right]^{\frac{3}{2}}}{\left[(\alpha^2+\sigma^2-\omega_0^2)^2 + 4\sigma^2\omega_0^2\right]^{\frac{3}{2}}} \tag{11}$$

After simplification (11) can be rewritten as

$$F_0(\alpha) = \frac{\alpha\cos\left(\frac{3}{2}\theta\right)}{r^{\frac{3}{2}}} \tag{12}$$

where

$$\theta = \tan^{-1}\left(\frac{-2\sigma\omega_0}{\alpha^2+\sigma^2-\omega_0^2}\right)$$

and

$$r = \left[(\alpha^2+\sigma^2-\omega_0^2)^2 + 4\sigma^2\omega_0^2\right]^{\frac{1}{2}}$$

It can be shown with a little effort that for small damping constant σ, the FB transform $F_0(\alpha)$ will have peak-amplitude occurring at $\alpha \approx \omega_0$, and far away from that point $F_0(\alpha)$ will tend to become zero. When $x(t)$ is the superposition of M sub-signals $x_i(t)$, (3) expressed as

$$x(t) = \sum_{i=1}^{M} e^{-\sigma_i t}\cos(\omega_i t) \tag{13}$$

then the FB transform,

$$F_0(\alpha) = \sum_{i=1}^{M} \frac{\alpha\cos\left(\frac{3}{2}\theta_i\right)}{r_i^{\frac{3}{2}}} \tag{14}$$

where

$$\theta_i = \tan^{-1}\left(\frac{-2\sigma_i\omega_i}{\alpha^2+\sigma_i^2-\omega_i^2}\right)$$

and

$$r_i = \left[(\alpha^2+\sigma_i^2-\omega_i^2)^2 + 4\sigma^2\omega_i^2\right]^{\frac{1}{2}}$$

It can be identified now that for small damping constants σ_i and well-separated circular frequencies ω_i, each term on the right-hand side of (14) will represent a region of the FB variable α where the coefficients are non-zero corresponding to a sub-signal of the composite signal $x(t)$. Since coefficients are real, each component $x_i(t)$ can be directly reconstructed from the FB coefficient plot.

When the signal $x(t)$ is available only for a finite duration of time, we use the Fourier-Bessel series expansion of the signal to separate the signal components [19].

4. TECHNIQUE BASED ON FB TRANSFORM

In order to carry out the time-frequency analysis of a multi-component non-stationary signal, the components of the signal are separated by using the FB transform. First, the FB coefficients are calculated for multi-component signal from (6). Every component of multi-component signal has non-overlapping coefficients. Since coefficients are real; each component is directly reconstructed from FB coefficient plot. Next, each reconstructed component is converted into an analytic component signal. The analytic signal of $x(n)$ is defined as, $x_a(n) = x(n) + j\hat{x}(n)$, where $\hat{x}(n)$ is the Hilbert transform of $x(n)$. The use of an analytic signal ensures that the spectrum of $x_a(n)$ has nonzero values only for positive frequencies, and the corresponding WVD has no spurious cross term at zero frequency. We apply WVD for each analytic component to analyze its time-frequency distribution, and finally summation of these distributions gives the WVD of composite signal. The block diagram of the proposed technique based on FB transform and WVD is shown in Figure 1.

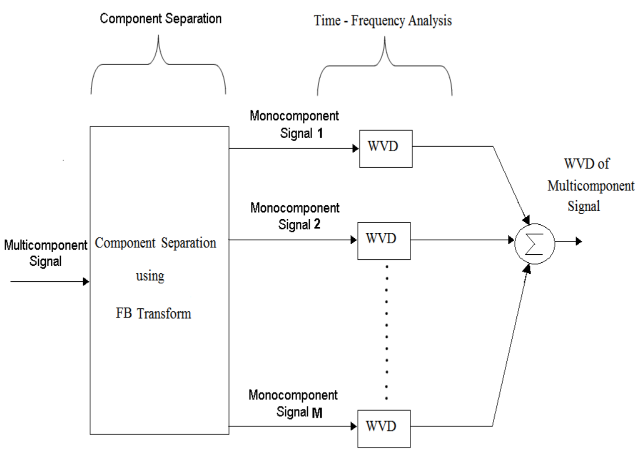

Figure 1: Block diagram of the proposed technique

5. SIMULATION RESULTS

We now discuss a few representative examples that are chosen to study the performance of this proposed technique based on the FB transform and the WVD.

5.1 Gaussian modulated signal

For the test signal we use the Gaussian modulated (GM) signal given by

$$x[n] = \frac{1}{\tau} exp\left(-\frac{\pi n^2}{\tau^2}\right)\sin(2\pi f_c n) \qquad (15)$$

where τ is a variable parameter and f_c is the center frequency of the modulated signal [20]. A set of 512 samples of the GM signal with center frequency 100 Hz is processed. Figures 2–4 and Figures 5–7 show the original and regenerated signals, power spectral density (PSD) plot of original and regenerated signals and plot of the FB coefficients at two different values of τ.

Table 1 shows the range of the FB variable where the coefficients are non-zero. Note that for smaller value of τ, the coefficients over larger range of variable are required in the reconstruction of the signal.

Table 1: Results for Gaussian modulated (GM) signal

Value of τ	Required Fourier-Bessel (FB) variables
0.1	550-700 (150)
0.3	600-660 (60)

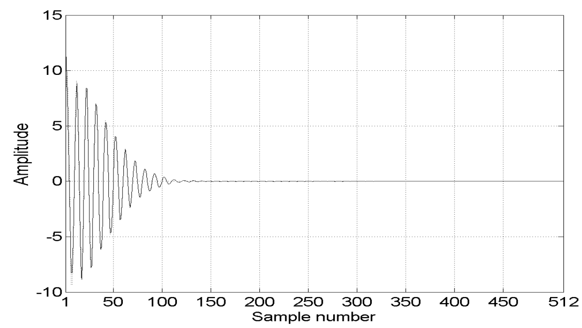

Figure 2: GM signal: original (Dotted line), regenerated (Solid line), $\tau = 0.1$

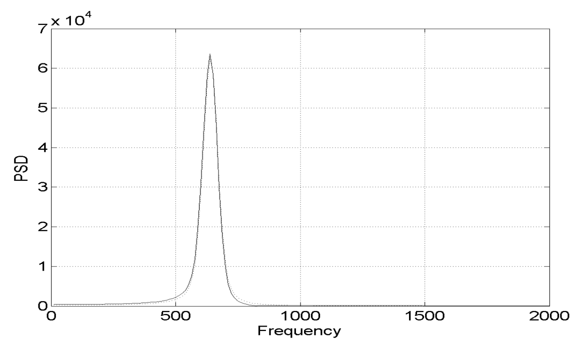

Figure 3: PSD of the GM signal: original (Dotted line), regenerated (Solid line), $\tau = 0.1$

5.2 Multicomponent signals

This section presents numerical examples of the proposed method with the multicomponent signals which are synthetically generated. The signals being considered are as follows [5]:

$$g_1(n) = 4.\cos\left[2\pi\left(\frac{n}{8}\right).\frac{n}{256}\right]$$

$$+4.\cos\left[2\pi\left(\frac{(512-n)}{8}+40\right).\frac{n}{256}\right]$$

$$g_2(n) = 4.\cos\left[\left(2\pi.32+70.\left(\frac{n}{500}\right)\right)\frac{n}{500}\right]$$

$$+4.\cos\left[\left(2\pi.80+100.\left(\frac{n}{500}\right)\right)\frac{n}{500}\right]$$

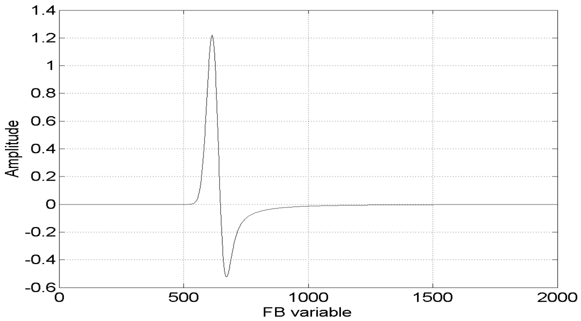

Figure 4: FB transform of the GM signal, $\tau = 0.1$

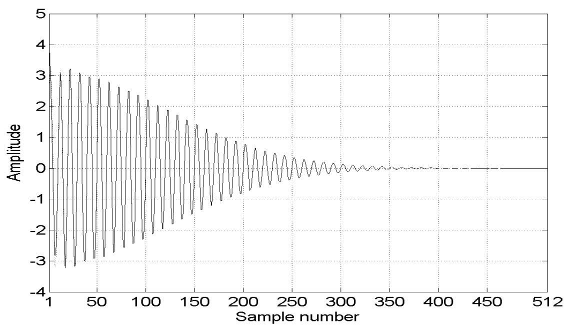

Figure 5: GM signal: original (Dotted line), regenerated (Solid line), $\tau = 0.3$

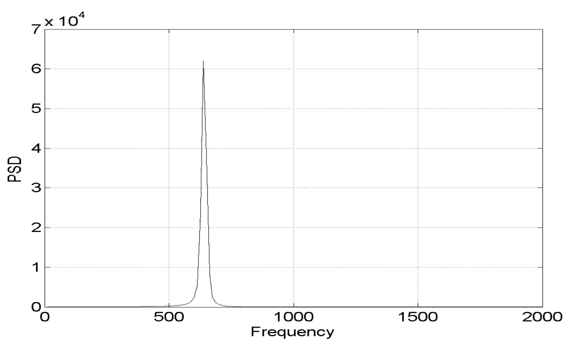

Figure 6: PSD of the GM signal: original (Dotted line), regenerated (Solid line), $\tau = 0.3$

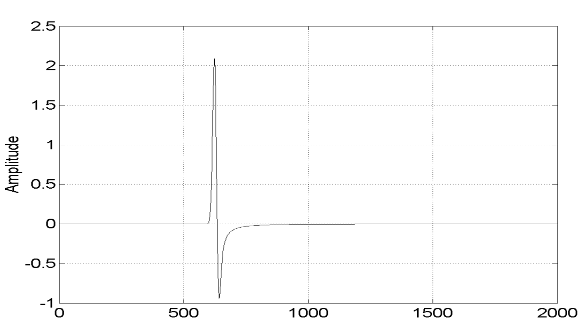

Figure 7: FB transform of the GM signal, $\tau = 0.3$

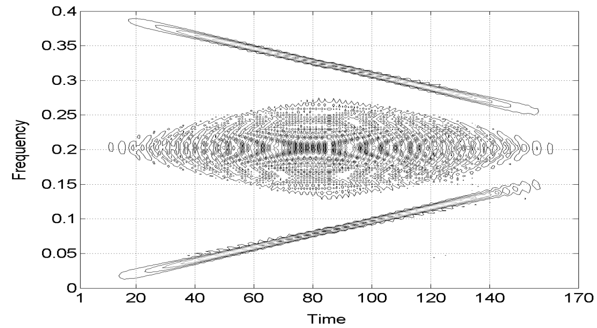

Figure 8: WVD of the signal $g_1(n)$

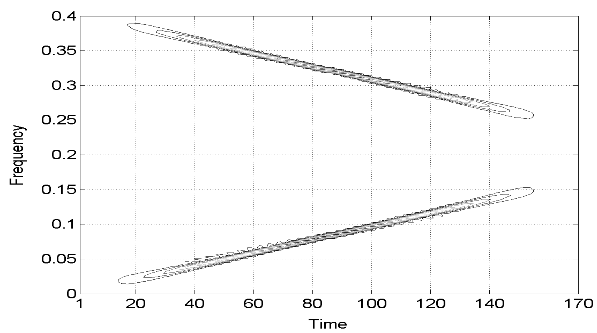

Figure 9: WVD of the signal $g_1(n)$ after separation using FB transform

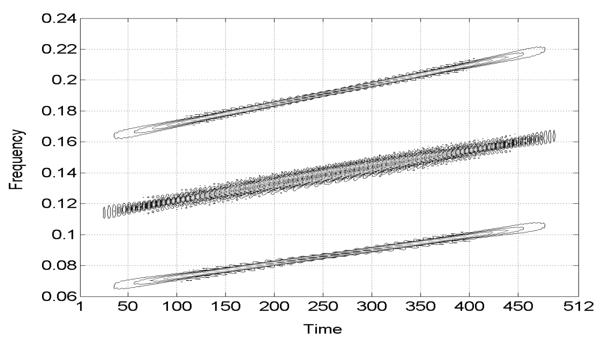

Figure 10: WVD of the signal $g_2(n)$

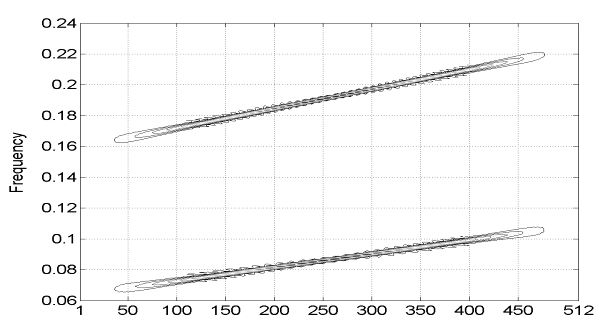

Figure 11: WVD of the signal $g_2(n)$ after separation using FB transform

The signal $g_1(n)$ is a chirp signal which has two frequency components at each time instant; the frequency of one component is increasing with time and the frequency of the other is decreasing with time, but the rate of the change is same in both cases. For the simulation, we have considered 170 sample points of the signal $g_1(n)$.

The signal $g_2(n)$ is another chirp signal which has two frequency components at each time instant, the frequencies of both components of $g_2(n)$ are increasing with time and also the rate of change is different. For the simulation, we have considered 512 sample points of the signal $g_2(n)$.

The WVD of the signals $g_1(n)$ and $g_2(n)$ are shown in the Figures 8 and 10 respectively. Observe that due to presence of strong cross terms, the magnitude of an individual signal does not remain same. This may mislead us to interpret that the test signal is a three component signal. However, the ambiguity is resolved when the components of the signal are separated by using the FB transform. The WVD of the separated components are computed, and summation of these distributions gives the WVD of the composite signals $g_1(n)$ and $g_2(n)$ with out cross terms, as shown in Figures 9 and 11 respectively.

6. CONCLUSION

In this paper, we have demonstrated that undesirable cross terms in the WVD of a multi-component signal can be effectively removed by the method based on the FB transform. The proposed method decomposes the signal into its constituent components, and the components can be processed individually.

It has been illustrated through simulation that by using the FB decomposition technique, the frequencies of the linear chirp signals of a multi-component signal can be estimated accurately.

A particular application where this method will be useful is speech analysis, because speech can be modeled as a sum of AM and FM signals corresponding to formant frequencies, and one of the main objectives in the analysis of speech signals is estimating the formant frequencies.

The method for decomposition of the signal as presented in this paper is advantageous over the technique based on the filter bank approach, because here we do not need any prior information about the frequency-band of the signal.

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