Stokes' Flow Analytic Solution:

Basic Equations:

The equations governing Stokes' (creeping) flow for an incompressible fluid are given by three sets of equations: The first are Cauchy's equations of motion with no inertial terms:

$$\nabla \cdot \boldsymbol{\tau} + \boldsymbol{f} = 0, \tag{1}$$

where $\boldsymbol{\tau}$ is the fluid total stress tensor and $\boldsymbol{f}$ is the body force on the fluid. The second is the continuity equation for an incompressible fluid:

$$\nabla \cdot \boldsymbol{v} = 0, \tag{2}$$

where $\boldsymbol{v}$ is the fluid velocity. The third is the constitutive law for the stress tensor:

$$\boldsymbol{\tau} = -p\boldsymbol{\delta} + 2\eta\boldsymbol{\epsilon}, \tag{3}$$

where p is the total pressure, $\boldsymbol{\delta}$ is the Kronecker delta tensor, η is the viscosity and $\boldsymbol{\epsilon}$ is the rate of strain tensor. We take the body force, $\boldsymbol{f}$, to act only in the z coordinate direction and is given by

$$\boldsymbol{f} = (0, 0, -\rho). \tag{4}$$

Stokes' Flow Analytic Solution: (solA)

Description:

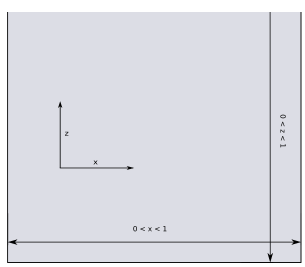
solA is a 2-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The viscosity is constant. The flow is driven by a temperature field represented by the density, ρ, as follows:

$$\rho = -\sigma \sin(k_m z) \cos(k_n x). \tag{5}$$

Parameters:

The variable parameters of this solution are:

- density/temperature parameter: σ.
- wave number in z domain: $k_m = m\pi z$. (m may be non-integral)
- wave number in x domain: $k_n = n\pi x$. (n must be integral)
- constant viscosity: η.



$\rho = -\sigma \sin(k_m z) \cos(k_n x)$

Figure 1: Solution (**SolA**): This solution has a box of density $\rho = -\sigma \sin(k_m z) \cos(k_n x)$. It is isoviscous. The Boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solB)

Description:

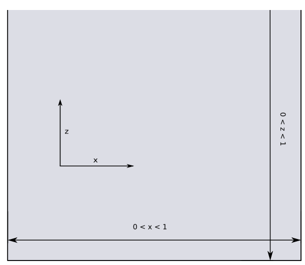
solB is a 2-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The viscosity is constant. The flow is driven by a temperature field represented by the density, ρ, as follows:

$$\rho = -\sigma \sinh(k_m z) \cos(k_n x). \tag{6}$$

Parameters:

The variable parameters of this solution are:

- density/temperature parameter: σ.
- wave number in z domain: $k_m = m\pi z$. (m may be non-integral)
- wave number in x domain: $k_n = n\pi x$. (n must be integral and not equal to m)
- constant viscosity: η.



$\rho = -\sigma \sinh(k_m z) \cos(k_n x)$

Figure 2: Solution (**SolB**): This solution has a box of density $\rho = -\sigma \sinh(k_m z) \cos(k_n x)$. It is isoviscous. The Boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solC)

Description:

solC is a 2-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The viscosity is constant. The flow is driven by a density jump in the x direction.

Parameters:

The variable parameters of this solution are:

- density parameter: σ.
- constant viscosity: η.
- width of dense column: x_c.

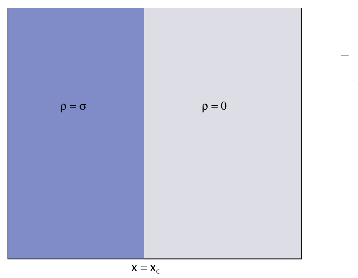


Figure 3: Solution (**SolC**): This solution has a column of density $\rho = \sigma$ from $0 < x < x_c$. It is isoviscous. The boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solCA)

Description:

solCA is a 2-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The viscosity is constant. The flow is driven by a dense column centred at $x = x_0$ of width dx.

Parameters:

The variable parameters of this solution are:

- density parameter: σ.
- constant viscosity: η.
- width of dense column: dx.
- centre of dense column: x_0.

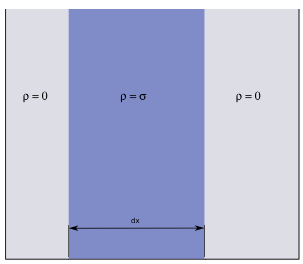


Figure 4: Solution (**SolCA**): This solution has a column of density $\rho = \sigma$ from $x_0 - dx/2 < x < x_0 + dx/2$. It is isoviscous. The boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solCx)

Description:

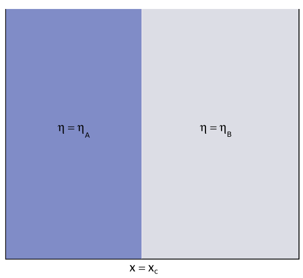
solCx is a 2-dimensional analytical solution to the Stokes' flow equations. The boundary conditions are free-slip everywhere on a unit domain. There is a viscosity jump in the x direction at $x = x_c$. The flow is driven by a temperature field represented by the density, ρ, as follows:

$$\rho = -\sigma \sin(n_z \pi z) \cos(n_x \pi x). \tag{7}$$

Parameters:

The variable parameters of this solution are:

- density parameter: σ.
- viscosities: η_A and η_B.
- viscosity jump location: x_c.
- wave number in z domain: n_z. (n_z may be non-integral)
- wave number in x domain: n_x. (n_x must be integral)



$\rho = -\sigma \sin(n_z \pi z) \cos(n_x \pi x)$

Figure 5: Solution (**solCx**): This solution has a box of density $\rho = -\sigma \sin(n_z \pi z) \cos(n_x \pi x)$. It is has a viscosity jump at $x = x_c$. The boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solD)

Description:

solDA is a 2-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The viscosity is layered with a jump at $z = z_c$. The flow is driven by a dense column centred at $x = x_0$ of width dx.

Parameters:

The variable parameters of this solution are:

- density parameter: σ.
- viscosities: η_A and η_B.
- viscosity jump location: z_c.
- width of dense column: dx.
- centre of dense column: x_0.

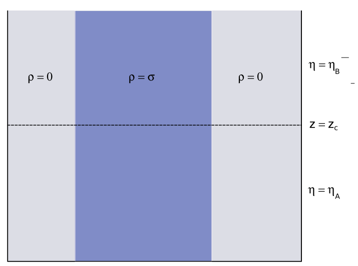


Figure 6: Solution (**SolDA**): This solution has a column of density $\rho = \sigma$ from $x_0 - dx/2 < x < x_0 + dx/2$. The viscosity is layered with a jump at $z = z_c$. The boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solDA)

Description:

solDA is a 2-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The viscosity is layered with a jump at $z = z_c$. The flow is driven by a dense column centred at $x = x_0$ of width dx.

Parameters:

The variable parameters of this solution are:

- density parameter: σ.
- viscosities: η_A and η_B.
- viscosity jump location: z_c.
- width of dense column: dx.
- centre of dense column: x_0.

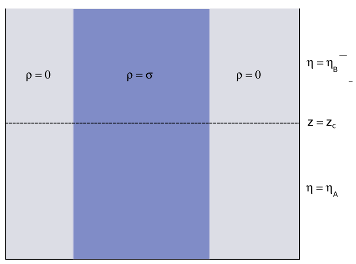


Figure 7: Solution (**SolDA**): This solution has a column of density $\rho = \sigma$ from $x_0 - dx/2 < x < x_0 + dx/2$. The viscosity is layered with a jump at $z = z_c$. The boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solE)

Description:

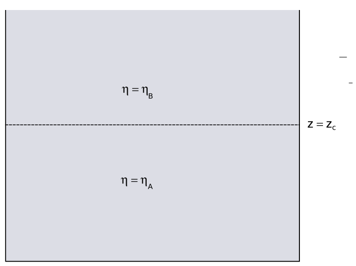
solE is a 2-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The viscosity is layered with a jump at $z = z_c$. The flow is driven by a temperature field represented by the density, ρ, as follows:

$$\rho = -\sigma \sin(k_m z) \cos(k_n x). \tag{8}$$

Parameters:

The variable parameters of this solution are:

- density parameter: σ.
- viscosities: η_A and η_B.
- viscosity jump location: z_c.
- wave number in z domain: $k_m = m\pi z$. (m may be non-integral)
- wave number in x domain: $k_n = n\pi x$. (n must be integral)



$$\rho = -\sigma \sin(k_m z) \cos(k_n x)$$

Figure 8: Solution (**SolE**): This solution has a column of density $\rho = -\sigma \sin(k_m z) \cos(k_n x)$. The viscosity is layered with a jump at $z = z_c$. The boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solF)

Description:

solF is a 2-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The viscosity is layered with a jump at $z = z_c$. The flow is driven by a dense block.

Parameters:

The variable parameters of this solution are:

- density parameter: σ.
- viscosities: η_A and η_B.
- width of dense block: x_c.
- bottom of dense block: z_c.

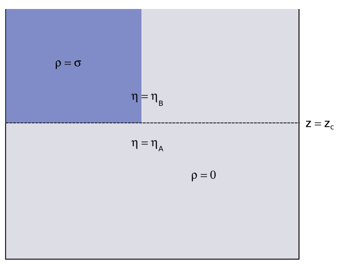


Figure 9: Solution (**SolF**): This solution has a block of density $\rho = \sigma$ from $0 < x < x_c$ above $z = z_c$. The viscosity is layered with a jump at $z = z_c$. The boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solG)

Description:

solG is a 2-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The viscosity is layered with a jump at $z = z_c$. The flow is driven by a dense block.

Parameters:

The variable parameters of this solution are:

- density parameter: σ.
- viscosities: η_A and η_B.
- width of dense block: dx.
- centre of dense block: x_0.
- bottom of dense block: z_c.

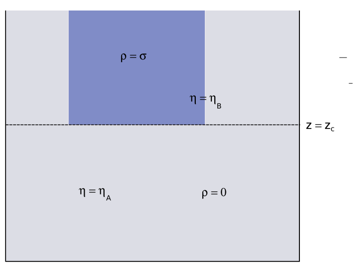


Figure 10: Solution (**SolG**): This solution has a block of density $\rho = \sigma$ from $x_0 - dx/2 < x < x_0 + dx/2$ above $z = z_c$. The viscosity is layered with a jump at $z = z_c$. The boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solH)

Description:

solH is a 3-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The flow is driven by a dense column in one corner. The output from the code is a two-dimensional slice at a given height z.

Parameters:

The variable parameters of this solution are:

- density parameter: σ.
- viscosity: η.
- x width of dense block: dx.
- y width of dense block: dy.
- z height of 2D slice: z.

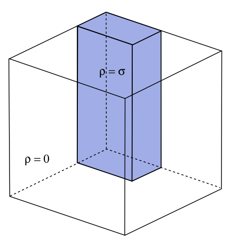


Figure 11: Solution (**SolH**): This solution has a block of density $\rho = \sigma$ from $0 < x < dx$ and $0 < y < dy$ extending in the z direction. The boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solHA)

Description:

solHA is a 3-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The flow is driven by a dense column in the interior. The output from the code is a two-dimensional slice at the given height z.

Parameters:

The variable parameters of this solution are:

- density parameter: σ.
- viscosity: η.
- x width of dense column: dx.
- y width of dense column: dy.
- x centre of dense column: $x0$.
- y centre of dense column: $y0$.
- z height of 2D slice: z.

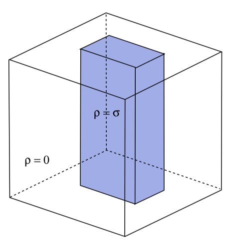


Figure 12: Solution (**SolHA**): This solution has a block of density $\rho = \sigma$ of width $dx \times dy$ centred at $x = x0$ and $y = y0$ extending in the z direction. The boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solHAy)

Description:

solHAy is a 3-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The flow is driven by a dense column in the interior. The output from the code is a two-dimensional slice in the $z-x$ plane at a given coordinate y.

Parameters:

The variable parameters of this solution are:

- density parameter: σ.
- viscosity: η.
- x width of dense column: dx.
- y width of dense column: dy.
- x centre of dense column: $x0$.
- y centre of dense column: $y0$.
- y coordinate of 2D slice: y.

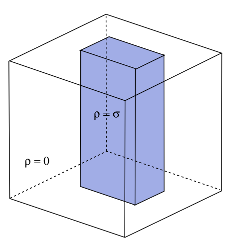


Figure 13: Solution (**SolHAy**): This solution has a block of density $\rho = \sigma$ of width $dx \times dy$ centred at $x = x0$ and $y = y0$ extending in the z direction. The boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solHy)

Description:

solHy is a 3-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The flow is driven by a dense column in one corner. The output from the code is a two-dimensional slice in the $z - x$ plane at a given coordinate y.

Parameters:

The variable parameters of this solution are:

- density parameter: σ.
- viscosity: η.
- x width of dense block: dx.
- y width of dense block: dy.
- y coordinate of 2D slice: y.

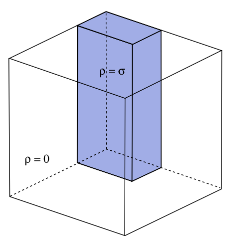


Figure 14: Solution (**SolHy**): This solution has a block of density $\rho = \sigma$ from $0 < x < dx$ and $0 < y < dy$ extending in the z direction. The boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solI)

Description:

solI is a 2-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The viscosity varies exponentially in the z direction. The flow is driven by a density jump in the x direction.

Parameters:

The variable parameters of this solution are:

- density parameter: σ.
- viscosity parameter: B.
- width of dense column: x_c.

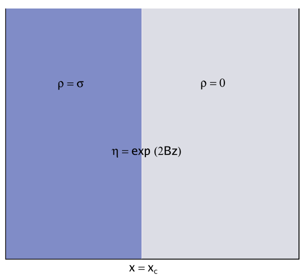


Figure 15: Solution (**SolI**): This solution has a column of density $\rho = \sigma$ from $0 < x < x_c$. The viscosity varies exponentially in the z direction and is given by $\eta = \exp(2Bz)$. The boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solIA)

Description:

solIA is a 2-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The viscosity varies exponentially in the z direction. The flow is driven by a dense column centred at $x = x_0$ of width dx.

Parameters:

The variable parameters of this solution are:

- density parameter: σ.
- viscosity parameter: B.
- width of dense column: dx.
- centre of dense column: x_0.

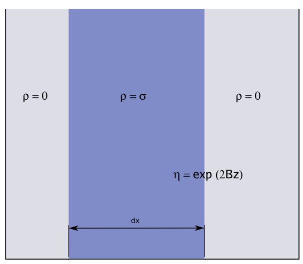


Figure 16: Solution (**SolIA**): This solution has a column of density $\rho = \sigma$ from $x_0 - dx/2 < x < x_0 + dx/2$. The viscosity varies exponentially in the z direction and is given by $\eta = \exp(2Bz)$. The boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solJ)

Description:

solJ is a 2-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The viscosity is layered with a jump at $z = z_c$. The flow is driven by two dense blocks in different layers.

Parameters:

The variable parameters of this solution are:

- density parameters: σ_B and σ_A.
- viscosities: η_A and η_B.
- width of upper dense block: dx_B.
- width of lower dense block: dx_A.
- centre of upper dense block: x_{0B}.
- centre of lower dense block: x_{0A}.
- bottom of upper dense block: z_c.

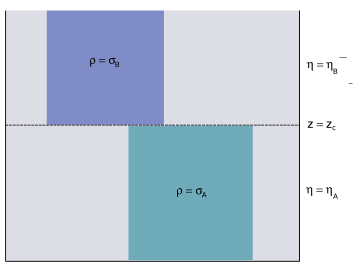


Figure 17: Solution (**SolJ**): This solution has a block of density $\rho = \sigma_B$ from $x_{0B} - dx_B/2 < x < x_{0B} + dx_B/2$ above $z = z_c$ and a block of density $\rho = \sigma_A$ from $x_{0A} - dx_A/2 < x < x_{0A} + dx_A/2$ below $z = z_c$. The viscosity is layered with a jump at $z = z_c$. The boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solJA)

Description:

solJ is a 2-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The viscosity is layered with a jump at $z = z_c$. The flow is driven by a dense block in the upper layer.

Parameters:

The variable parameters of this solution are:

- density parameters: σ.
- viscosities: η_A and η_B.
- width of upper dense block: dx.
- centre of upper dense block: x_0.
- bottom of upper dense block: z_c.

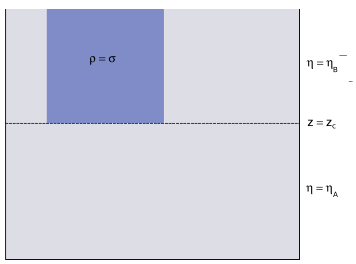


Figure 18: Solution (**SolJA**): This solution has a block of density $\rho = \sigma$ from $x_0 - dx/2 < x < x_0 + dx/2$ above $z = z_c$. The viscosity is layered with a jump at $z = z_c$. The boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solKx)

Description:

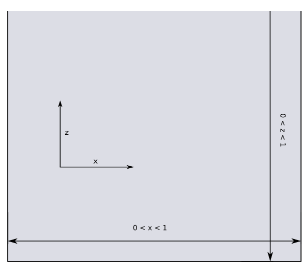
solKx is a 2-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The viscosity varies exponentially in the x direction and is given by $\eta = \exp(2Bx)$. The flow is driven by a temperature field represented by the density, ρ, as follows:

$$\rho = -\sigma \sin(k_m z) \cos(k_n x). \tag{9}$$

Parameters:

The variable parameters of this solution are:

- density/temperature parameter: σ.
- wave number in z domain: $k_m = m\pi z$. (m may be non-integral)
- wave number in x domain: $k_n = n\pi x$. (n must be integral)
- viscosity parameter: B.



$$\rho = -\sigma \sin(k_m z) \cos(k_n x)$$
$$\eta = \exp(2Bx)$$

Figure 19: Solution (**SolKx**): This solution has a box of density $\rho = -\sigma \sin(k_m z) \cos(k_n x)$. The viscosity varies exponentially in the x direction and is given by $\eta = \exp(2Bx)$. The Boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solKz)

Description:

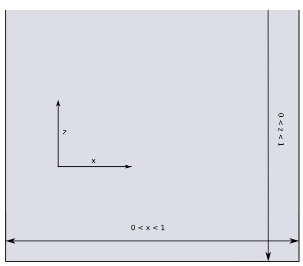
solKz is a 2-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. The viscosity varies exponentially in the z direction and is given by $\eta = \exp(2Bz)$. The flow is driven by a temperature field represented by the density, ρ, as follows:

$$\rho = -\sigma \sin(k_m z) \cos(k_n x). \tag{10}$$

Parameters:

The variable parameters of this solution are:

- density/temperature parameter: σ.
- wave number in z domain: $k_m = m\pi z$. (m may be non-integral)
- wave number in x domain: $k_n = n\pi x$. (n must be integral)
- viscosity parameter: B.



$$\rho = -\sigma \sin(k_m z) \cos(k_n x)$$
$$\eta = \exp(2Bz)$$

Figure 20: Solution (**SolKz**): This solution has a box of density $\rho = -\sigma \sin(k_m z) \cos(k_n x)$. The viscosity varies exponentially in the z direction and is given by $\eta = \exp(2Bz)$. The Boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solL)

Description:

solL is a 2-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The boundary conditions are free-slip everywhere on a unit domain. This solution has an upper triangular block of density $\rho = \sigma_B$ above a lower block of density $\rho = \sigma_A$.

Parameters:

The variable parameters of this solution are:

- density parameters: σ_B and σ_A.
- viscosity: η .

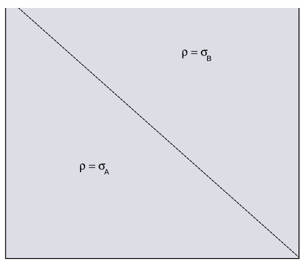


Figure 21: Solution (**SolL**): This solution has an upper triangular block of density $\rho = \sigma_B$ above a lower block of density $\rho = \sigma_A$. The boundary conditions are free slip everywhere on the surfaces of the unit box.

Stokes' Flow Analytic Solution: (solS)

Description:

solS is a 2-dimensional analytical solution to the Cauchy equations with the acceleration term set to zero to represent creeping flow. The domain is given by $\Omega = [0,1] \times [0,1]$. The flow is driven by the velocity field

$$\boldsymbol{v} = \sin(n\pi x)\boldsymbol{i}, \qquad \text{on } y = 1. \tag{11}$$

applied on the upper horizontal boundary. No body forces are present so $\boldsymbol{f} = \boldsymbol{0}$. The boundary conditions on the remaining walls are free-slip on the vertical sides and no slip on the lower horizontal boundary. The viscosity is set to 1.0.

Parameters:

The variable parameters of this solution are:

- wave number in x domain: $n \in \mathbb{Z} \setminus \{0\}$.

Solution:

This problem has a simple analytic solution and is thus provided here:

$$v_x = \sin(n\pi x)\Big((An\pi + C + Cn\pi y)\exp(n\pi y) - (Bn\pi - D + Dn\pi y)\exp(-n\pi y)\Big) \tag{12}$$

$$v_y = -n\pi\cos(n\pi x)\Big((A + Cy)\exp(n\pi y) + (B + Dy)\exp(-n\pi y)\Big) \tag{13}$$

$$p = -2n\pi\cos(n\pi x)\Big(C\exp(n\pi y) + D\exp(-n\pi y)\Big) \tag{14}$$

The strain rate and deviatoric stress can be assembled from the velocity gradients given below;

$$\frac{\partial v_x}{\partial x} = n\pi\cos(n\pi x)\Big((An\pi + C + Cn\pi y)\exp(n\pi y) - (Bn\pi - D + Dn\pi y)\exp(-n\pi y)\Big) \tag{15}$$

$$\frac{\partial v_y}{\partial x} = n^2\pi^2\sin(n\pi x)\Big((A + Cy)\exp(n\pi y) + (B + Dy)\exp(-n\pi y)\Big) \tag{16}$$

$$\frac{\partial v_x}{\partial y} = \sin(n\pi x)\Big((An^2\pi^2 + 2Cn\pi + Cn^2\pi^2 y)\exp(n\pi y) + (Bn^2\pi^2 - 2Dn\pi + Dn^2\pi^2 y)\exp(-n\pi y)\Big) \tag{17}$$

$$\frac{\partial v_y}{\partial y} = -n\pi\cos(n\pi x)\Big((An\pi + C + Cn\pi y)\exp(n\pi y) + (-Bn\pi + D - Dn\pi y)\exp(-n\pi y)\Big) \tag{18}$$

$$\tag{19}$$

The constants A, B, C, D and E are given by

$$E = (4n^2\pi^2 + 2)\exp^2(n\pi) - \exp^4(n\pi) - 1 \tag{20}$$

$$A = \frac{1}{E}(\exp^2(n\pi) - 1)\exp(n\pi) \tag{21}$$

$$B = -A \tag{22}$$

$$C = \frac{1}{E}\Big(2n\pi - \exp^2(n\pi) + 1\Big)\exp(n\pi) \tag{23}$$

$$D = -\frac{1}{E}\Big(2n\pi\exp^2(n\pi) - \exp^2(n\pi) + 1\Big)\exp(n\pi) \tag{24}$$