Algebraic translation:

A(a1,a2), B(b1,b2), D(x,y), F (u,v)

c:=(x-a1)^2+(y-a2)^2-((a1-b1)^2+(a2-b2)^2),

d:=(u-b1)^2+(v-b2)^2-((a1-b1)^2+(a2-b2)^2),

g: =(x-a1)*(b1-a1)+ (y-a2)*(b2-a2)

h: =(u-b1)*(b1-a1)+(v-b2)*(b2-a2)

so D is defined by these two equations (x-a1)^2+(y-a2)^2-((a1-b1)^2+(a2-b2)^2)=0, (x-a1)*(b1-a1)+ (y-a2)*(b2-a2)=0

and F is defined by these other two (u-b1)^2+(v-b2)^2-((a1-b1)^2+(a2-b2)^2)=0, (u-b1)*(b1-a1)+(v-b2)*(b2-a2)=0

Thesis: DF and AB are parallel: (u-x)*(b2-a2) -(v-y)*(b1-a1)=0

Thus the hypotheses ideal is H

```
> restart:with(PolynomialIdeals):with(Operators): with(Groebner):H:=<
(x-a1)^2+(y-a2)^2-((a1-b1)^2+(a2-b2)^2),(u-b1)^2+(v-b2)^2-((a1-b1)
^2+(a2-b2)^2),(x-a1)*(b1-a1)+ (y-a2)*(b2-a2),(u-b1)*(b1-a1)+(v-b2)*
(b2-a2)>;
```

$$H := \langle (u-b1)(b1-a1)+(v-b2)(b2-a2), (x-a1)(b1-a1)+(y-a2)(b2-a2), (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2, (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2 \rangle \quad \textbf{(1)}$$

HILBERT DIMENSION OF THE HYPOTHESES IDEAL

Now we consider the hypotheses ideal in all the variables and check the dimension, independent variables (should be 4, the coordinates of A and B) but it turns to be 5!

```
> restart:with(PolynomialIdeals):with(Operators): with(Groebner):H:=<
(x-a1)^2+(y-a2)^2-((a1-b1)^2+(a2-b2)^2),(u-b1)^2+(v-b2)^2-((a1-b1)
^2+(a2-b2)^2),(x-a1)*(b1-a1)+ (y-a2)*(b2-a2),(u-b1)*(b1-a1)+(v-b2)*
```

```
(b2-a2)>; HilbertDimension(H);
```

$$H := \langle (u - b1)\,(b1 - a1) + (v - b2)\,(b2 - a2),\ (x - a1)\,(b1 - a1) + (y - a2)\,(b2 - a2),\ (u - b1)^2 + (v - b2)^2 - (a1 - b1)^2 - (a2 - b2)^2,\ (x - a1)^2 + (y - a2)^2 - (a1 - b1)^2 - (a2 - b2)^2 \rangle$$

$$5 \tag{1.1}$$

```
> EliminationIdeal(H,{a1,a2,b1,b2});
```

$$\langle 0 \rangle \tag{1.2}$$

But all possible inclusion in the above elimination of other variables, such as x,y, u, v yield not zero. So

{a1,a2,b1,b2}

is a maximal set of indept. variables (but NOT maximum size). Let us try with other , less intuitive, set of independent variables

```
> EliminationIdeal(H,{x,y,u,v});
```

$$\langle 0 \rangle \tag{1.3}$$

Moreover, the elimination of {x,y,u,v} is zero, but it is NOT zero if we add a1 or a2 or b1 or b2. So, again, {x,y,u,v} is maximal set but not maximum size. Let us try the other option: to consider a mixture of {a1, a2, u,v}

```
> EliminationIdeal(H,{a1,a2,u,v});
```

$$\langle 0 \rangle \tag{1.4}$$

Now add b1, b2, x, y...all yields not zero. So let us try with {a1,b2,x,v}. It yields zero. Now add b1 and.. BINGO, it is zero too!! Same if we add a2, etc. So it happens that independent variables are, for instance, ONE free point such as A (two coordinates), then one single variable from B, one from D, one from F.

```
> EliminationIdeal(H,{a1,a2,x,u,b1});EliminationIdeal(H,{a1,b2,x,v,
  a2});
```

$$\langle 0 \rangle$$

$$\langle 0 \rangle \tag{1.5}$$

Obviously, this is not geometrically expected...It is hard to understand... Let us try to see if there are some 5-dimensional components of the Hypotheses variety that we can consider as "geometrically meaningful"

```
> PrimaryDecomposition(H);
```

But the computation does not succeed after 22444 seconds and 449.3 M of memory.

CHECKING THE GEOMETRICAL DEGENERACY OF THE 5 DIMENSIONAL PARTS OF THE HYPOTHESES

So now we do the following. We start considering the five independent variables as parameters: a1,a2, b1,x,u and b2,y, v as variables. The result must be a zero dimensional ideal. We can do the primary o prime decomposition and then check for some relation of one of the missing coordinates of A or B in this ideal (assuming the other coordinates are part of the parameter set, we have seen above this is the general way of getting such 5 dimensional components):

```
> restart:with(PolynomialIdeals): H:=<(x-a1)^2+(y-a2)^2-((a1-b1)^2+
  (a2-b2)^2),(u-b1)^2+(v-b2)^2-((a1-b1)^2+(a2-b2)^2),(x-a1)*(b1-a1)+
  (y-a2)*(b2-a2),(u-b1)*(b1-a1)+(v-b2)*(b2-a2), variables={b2,y,v}>;
  HilbertDimension(H); PrimaryDecomposition(H);EliminationIdeal(H,
  {b2});
```

$$H := \langle (u-b1)\,(b1-a1)+(v-b2)\,(b2-a2),\ (x-a1)\,(b1-a1)+(y-a2)\,(b2-a2),\ (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2,\ (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2\rangle$$

$$0$$

$$\langle (u-b1)\,(b1-a1)+(v-b2)\,(b2-a2),\ (x-a1)\,(b1-a1)+(y-a2)\,(b2-a2),\ (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2,\ (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2\rangle$$

$$\langle a1^2-2\,b1\,a1+a2^2-2\,b2\,a2+b1^2+b2^2\rangle \qquad \textbf{(2.1)}$$

The conclusion is that in this component (with the coefficient field with parameters) there lies the polynomial (a1-b1)^2+(a2-b2)^2; thus a product of a polynomial in the five parameters times this polynomial is in the component over the K[a1,a2,b1,b2,x,y,u,v] ring. Since the five parameters are free over the component ..so its zero set over the component is closed, proper, it follows that this polynomial must be identically zero over the component, so that A=B in the component. In particular, being a single component, we can say also that the polynomial is in the component (since a product of polynomials is inside, one is not in the radical, so the other must be in the component), so the component is not free for a1,a2,b1,b2, as expected (because being a prime ideal, maximal set of independent variables= maximum size of sets of independent variables, so if a1,a2,b1,b2 would have been free in the component, we could have added one extra variable to have 5 independent variables=dimension of the component; but we have already checked that this is not possible for a1, a2,b1,b2).

Similar conclusions for other choices of 5 free parameters.

Let us consider a1,a2,b2,x,u as parameters and b1,y, v as variables.

```
> restart:with(PolynomialIdeals): H:=<(x-a1)^2+(y-a2)^2-((a1-b1)^2+
  (a2-b2)^2),(u-b1)^2+(v-b2)^2-((a1-b1)^2+(a2-b2)^2),(x-a1)*(b1-a1)+
  (y-a2)*(b2-a2),(u-b1)*(b1-a1)+(v-b2)*(b2-a2), variables={b1,y,v}>;
  HilbertDimension(H); PrimaryDecomposition(H);EliminationIdeal(H,
  {b1});
```

$$H := \langle (u-b1)\,(b1-a1)+(v-b2)\,(b2-a2),\ (x-a1)\,(b1-a1)+(y-a2)\,(b2$$

$- a2), (u - b1)^2 + (v - b2)^2 - (a1 - b1)^2 - (a2 - b2)^2, (x - a1)^2 + (y - a2)^2 - (a1 - b1)^2 - (a2 - b2)^2\rangle$

$$0$$

$$\langle (u - b1)(b1 - a1) + (v - b2)(b2 - a2), (x - a1)(b1 - a1) + (y - a2)(b2 - a2), (u - b1)^2 + (v - b2)^2 - (a1 - b1)^2 - (a2 - b2)^2, (x - a1)^2 + (y - a2)^2 - (a1 - b1)^2 - (a2 - b2)^2\rangle$$

$$\langle a1^2 - 2\, b1\, a1 + a2^2 - 2\, b2\, a2 + b1^2 + b2^2\rangle \quad \textbf{(2.2)}$$

Let us consider a1,a2,b1,y,u as parameters and b2,x, v as variables.

```
> restart:with(PolynomialIdeals): H:=<(x-a1)^2+(y-a2)^2-((a1-b1)^2+
  (a2-b2)^2),(u-b1)^2+(v-b2)^2-((a1-b1)^2+(a2-b2)^2),(x-a1)*(b1-a1)+
  (y-a2)*(b2-a2),(u-b1)*(b1-a1)+(v-b2)*(b2-a2), variables={b2,x,v}>;
  HilbertDimension(H); PrimaryDecomposition(H);EliminationIdeal(H,
  {b2});
```

$$H := \langle (u - b1)(b1 - a1) + (v - b2)(b2 - a2), (x - a1)(b1 - a1) + (y - a2)(b2 - a2), (u - b1)^2 + (v - b2)^2 - (a1 - b1)^2 - (a2 - b2)^2, (x - a1)^2 + (y - a2)^2 - (a1 - b1)^2 - (a2 - b2)^2\rangle$$

$$0$$

$$\langle (u - b1)(b1 - a1) + (v - b2)(b2 - a2), (x - a1)(b1 - a1) + (y - a2)(b2 - a2), (u - b1)^2 + (v - b2)^2 - (a1 - b1)^2 - (a2 - b2)^2, (x - a1)^2 + (y - a2)^2 - (a1 - b1)^2 - (a2 - b2)^2\rangle$$

$$\langle a1^2 - 2\, b1\, a1 + a2^2 - 2\, b2\, a2 + b1^2 + b2^2\rangle \quad \textbf{(2.3)}$$

Let us consider a1,a2,b1,y,v as parameters and b2,x, u as variables.

```
> restart:with(PolynomialIdeals): H:=<(x-a1)^2+(y-a2)^2-((a1-b1)^2+
  (a2-b2)^2),(u-b1)^2+(v-b2)^2-((a1-b1)^2+(a2-b2)^2),(x-a1)*(b1-a1)+
  (y-a2)*(b2-a2),(u-b1)*(b1-a1)+(v-b2)*(b2-a2), variables={b2,x,u}>;
  HilbertDimension(H); PrimaryDecomposition(H);EliminationIdeal(H,
  {b2});
```

$$H := \langle (u - b1)(b1 - a1) + (v - b2)(b2 - a2), (x - a1)(b1 - a1) + (y - a2)(b2 - a2), (u - b1)^2 + (v - b2)^2 - (a1 - b1)^2 - (a2 - b2)^2, (x - a1)^2 + (y - a2)^2 - (a1 - b1)^2 - (a2 - b2)^2\rangle$$

$$0$$

$$\langle (u - b1)(b1 - a1) + (v - b2)(b2 - a2), (x - a1)(b1 - a1) + (y - a2)(b2 - a2), (u - b1)^2 + (v - b2)^2 - (a1 - b1)^2 - (a2 - b2)^2, (x - a1)^2 + (y - a2)^2 - (a1 - b1)^2 - (a2 - b2)^2\rangle$$

$$\langle a1^2 - 2\, b1\, a1 + a2^2 - 2\, b2\, a2 + b1^2 + b2^2\rangle \quad \textbf{(2.4)}$$

Let us consider a1,a2,b2,x,u as parameters and b1,y, v as variables.

```
> restart:with(PolynomialIdeals): H:=<(x-a1)^2+(y-a2)^2-((a1-b1)^2+
  (a2-b2)^2),(u-b1)^2+(v-b2)^2-((a1-b1)^2+(a2-b2)^2),(x-a1)*(b1-a1)+
  (y-a2)*(b2-a2),(u-b1)*(b1-a1)+(v-b2)*(b2-a2), variables={b1,y,v}>;
  HilbertDimension(H); PrimaryDecomposition(H);EliminationIdeal(H,
```

```
{b1});
```

$H := \langle (u-b1)(b1-a1)+(v-b2)(b2-a2), (x-a1)(b1-a1)+(y-a2)(b2-a2), (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2, (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2\rangle$

$$0$$

$\langle (u-b1)(b1-a1)+(v-b2)(b2-a2), (x-a1)(b1-a1)+(y-a2)(b2-a2), (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2, (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2\rangle$

$$\langle a1^2-2\,b1\,a1+a2^2-2\,b2\,a2+b1^2+b2^2\rangle \qquad \textbf{(2.5)}$$

Let us consider a1,a2,b2,y,u as parameters and b1,x, v as variables.

```
> restart:with(PolynomialIdeals): H:=<(x-a1)^2+(y-a2)^2-((a1-b1)^2+
  (a2-b2)^2),(u-b1)^2+(v-b2)^2-((a1-b1)^2+(a2-b2)^2),(x-a1)*(b1-a1)+
  (y-a2)*(b2-a2),(u-b1)*(b1-a1)+(v-b2)*(b2-a2), variables={b1,x,v}>;
  HilbertDimension(H); PrimaryDecomposition(H);EliminationIdeal(H,
  {b1});
```

$H := \langle (u-b1)(b1-a1)+(v-b2)(b2-a2), (x-a1)(b1-a1)+(y-a2)(b2-a2), (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2, (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2\rangle$

$$0$$

$\langle (u-b1)(b1-a1)+(v-b2)(b2-a2), (x-a1)(b1-a1)+(y-a2)(b2-a2), (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2, (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2\rangle$

$$\langle a1^2-2\,b1\,a1+a2^2-2\,b2\,a2+b1^2+b2^2\rangle \qquad \textbf{(2.6)}$$

Let us consider a1,a2,b2,x,v as parameters and b1,x, u as variables.

```
> restart:with(PolynomialIdeals): H:=<(x-a1)^2+(y-a2)^2-((a1-b1)^2+
  (a2-b2)^2),(u-b1)^2+(v-b2)^2-((a1-b1)^2+(a2-b2)^2),(x-a1)*(b1-a1)+
  (y-a2)*(b2-a2),(u-b1)*(b1-a1)+(v-b2)*(b2-a2), variables={b1,x,u}>;
  HilbertDimension(H); PrimaryDecomposition(H);EliminationIdeal(H,
  {b1});
```

$H := \langle (u-b1)(b1-a1)+(v-b2)(b2-a2), (x-a1)(b1-a1)+(y-a2)(b2-a2), (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2, (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2\rangle$

$$0$$

$\langle (u-b1)(b1-a1)+(v-b2)(b2-a2), (x-a1)(b1-a1)+(y-a2)(b2-a2), (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2, (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2\rangle$

$$\langle a1^2-2\,b1\,a1+a2^2-2\,b2\,a2+b1^2+b2^2\rangle \qquad \textbf{(2.7)}$$

Let us consider a1,b1,b2,x,u as parameters and a2,y, v as variables.

```
> restart:with(PolynomialIdeals): H:=<(x-a1)^2+(y-a2)^2-((a1-b1)^2+
```

```
(a2-b2)^2),(u-b1)^2+(v-b2)^2-((a1-b1)^2+(a2-b2)^2),(x-a1)*(b1-a1)+
(y-a2)*(b2-a2),(u-b1)*(b1-a1)+(v-b2)*(b2-a2), variables={a2,y,v}>;
HilbertDimension(H); PrimaryDecomposition(H);EliminationIdeal(H,
{a2});
```

$$H := \langle (u-b1)\,(b1-a1) + (v-b2)\,(b2-a2),\ (x-a1)\,(b1-a1) + (y-a2)\,(b2-a2),\ (u-b1)^2 + (v-b2)^2 - (a1-b1)^2 - (a2-b2)^2,\ (x-a1)^2 + (y-a2)^2 - (a1-b1)^2 - (a2-b2)^2 \rangle$$

$$0$$

$$\langle (u-b1)\,(b1-a1) + (v-b2)\,(b2-a2),\ (x-a1)\,(b1-a1) + (y-a2)\,(b2-a2),\ (u-b1)^2 + (v-b2)^2 - (a1-b1)^2 - (a2-b2)^2,\ (x-a1)^2 + (y-a2)^2 - (a1-b1)^2 - (a2-b2)^2 \rangle$$

$$\langle a1^2 - 2\,b1\,a1 + a2^2 - 2\,b2\,a2 + b1^2 + b2^2 \rangle \qquad \textbf{(2.8)}$$

Let us remark that the PrimaryDEcomposition above and the PrimeDecomposition commands yields the same results.

CHECKING THE GEOMETRICAL DEGENERACY OF SOME 4 DIMENSIONAL PARTS OF THE HYPOTHESES

In conclusion: we can forget about any geometric interest for the five dimensional components and let us see what happens, following the same protocol, with 4 dimensional components where, say, x,y, u, v are free. Now we consider x,y, u, v as parameters and a1, a2, b1, b2 as variables. We compute over the coefficient field of parameters the primary decomposition, yielding 5 componentes. For all of them we consider the reduced, leading monic monomial, Gbasis for some ordering.

What follows is a toy example of the next computations. We consider a simple polynomial x*y^2 in the variables x, y, each single one is free, so the dimension is 1, now we consider x as parameter and y as varialbe and compute the components where x is free by considering the decompositoin in k(x)[y], yielding <x*y^2> , and the contraction to k[x,y] is <y^2>, so here the component where x is free is just one where y is NOT free.

We will see that, in our example, with x,y, u, v as parameters, the primary decomposition over k(x,y, u, v)[a1,a2,b1,b2] yields five components where x,y, u, v are free AND the first four have also a1,a2,b1,b2 free,but not the fifth one!!

```
> <x*y^2,variables={x,y}>; with(Groebner):PrimaryDecomposition(%);<x*
y^2,variables={y}>; PrimaryDecomposition(%);Basis(%,tdeg(y));
```

$$\langle x\,y^2 \rangle$$

$$\langle x \rangle,\ \langle y^2 \rangle$$

$$\langle x\,y^2 \rangle$$

$$\langle x\,y^2\rangle$$

$$[y^2] \quad \textbf{(3.1)}$$

```
> restart:with(PolynomialIdeals): with(Groebner):H:=<(x-a1)^2+(y-a2)
^2-((a1-b1)^2+(a2-b2)^2),(u-b1)^2+(v-b2)^2-((a1-b1)^2+(a2-b2)^2),
(x-a1)*(b1-a1)+ (y-a2)*(b2-a2),(u-b1)*(b1-a1)+(v-b2)*(b2-a2),
variables={a1,a2,b1,b2}>; HilbertDimension(H);PP:=
{PrimaryDecomposition(H)};for i from 1 to nops(PP) do
HilbertDimension(PP[i]),IdealInfo[KnownGroebnerBases](PP[i]), B[i]=
Basis(PP[i],tdeg(b2,b1,a2,a1)) end do;
```

$H := \langle (u-b1)\,(b1-a1)+(v-b2)\,(b2-a2),\ (x-a1)\,(b1-a1)+(y-a2)\,(b2-a2),\ (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2,\ (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2\rangle$

$$0$$

$PP := \{\langle (u-b1)\,(b1-a1)+(v-b2)\,(b2-a2),\ (x-a1)\,(b1-a1)+(y-a2)\,(b2-a2),\ a1-v-x+y,\ (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2,\ (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2\rangle,\ \langle (u-b1)\,(b1-a1)+(v-b2)\,(b2-a2),\ (x-a1)\,(b1-a1)+(y-a2)\,(b2-a2),\ a1+v-x-y,\ (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2,\ (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2\rangle,\ \langle (u-b1)\,(b1-a1)+(v-b2)\,(b2-a2),\ (x-a1)\,(b1-a1)+(y-a2)\,(b2-a2),\ (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2,\ (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2,\ 5\,a1-2\,u-v-3\,x+y\rangle,\ \langle (u-b1)\,(b1-a1)+(v-b2)\,(b2-a2),\ (x-a1)\,(b1-a1)+(y-a2)\,(b2-a2),\ (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2,\ (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2,\ 5\,a1-2\,u+v-3\,x-y\rangle,\ \langle (u-b1)\,(b1-a1)+(v-b2)\,(b2-a2),\ (x-a1)\,(b1-a1)+(y-a2)\,(b2-a2),\ (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2,\ (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2,\ 4\,a1^2-4\,a1\,u-4\,x\,a1+u^2+2\,u\,x+v^2-2\,y\,v+x^2+y^2\rangle\}$

$0,\ \{plex(b2,b1,a2,a1),\ tdeg(b2,b1,a2,a1)\},\ B_1=[a1-v-x+y,\ a2+u-x-y,\ b1-u-v+y,\ b2+u-v-x]$

$0,\ \{plex(b2,b1,a2,a1),\ tdeg(b2,b1,a2,a1)\},\ B_2=[a1+v-x-y,\ a2-u+x-y,\ b1-u+v-y,\ b2-u-v+x]$

$0,\ \{plex(b2,b1,a2,a1),\ tdeg(b2,b1,a2,a1)\},\ B_3=[5\,a1-2\,u-v-3\,x+y,\ 5\,a2+u-2\,v-x-3\,y,\ 5\,b1-3\,u+v-2\,x-y,\ 5\,b2-u-3\,v+x-2\,y]$

$0,\ \{plex(b2,b1,a2,a1),\ tdeg(b2,b1,a2,a1)\},\ B_4=[5\,a1-2\,u+v-3\,x-y,\ 5\,a2-u-2\,v+x-3\,y,\ 5\,b1-3\,u-v-2\,x+y,\ 5\,b2+u-3\,v-x-2\,y]$

(3.2)

$0, \{plex(b2, b1, a2, a1), tdeg(b2, b1, a2, a1)\}, B_5 = [a1\,(2u - 2x) + (2v - 2y)\,a2 - u^2 - v^2 + x^2 + y^2, b1 - a1, a1\,(2u - 2x) + (2v - 2y)\,b2 - u^2 - v^2 + x^2 + y^2, 4\,a1^2 + (-4u - 4x)\,a1 + u^2 + 2ux + v^2 - 2yv + x^2 + y^2]$ **(3.2)**

We observe that there are 5 components in the extended ideal and that the first four of them can be easily contracted to the ring K[a1,a2,b1,b2,x,y,u,v], since the basis has coeff =1. The result is that the contracted ideal has the same basis, but now will all the variables. And we see that a1,a1,b1,b2 are free on them, too.

```
> EliminationIdeal(<a1 - v - x + y, a2 + u - x - y, b1 - u - v + y,
  b2 + u - v - x,variables={a1,a2,b1,b2,x,y,u,v}>,{a1,a2,b1,b2});
  EliminationIdeal(<(u - b1)*(b1 - a1) + (v - b2)*(b2 - a2), (x - a1)
  *(b1 - a1) + (y - a2)*(b2 - a2), a1 - v - x + y, (u - b1)^2 + (v -
  b2)^2 - (a1 - b1)^2 - (a2 - b2)^2, (x - a1)^2 + (y - a2)^2 - (a1 -
  b1)^2 - (a2 - b2)^2,variables={a1,a2,b1,b2,x,y,u,v}>,{a1,a2,b1,b2})
  ;
```

$$\langle 0 \rangle$$

$$\langle 0 \rangle \qquad (3.3)$$

```
> EliminationIdeal(<a1 + v - x - y, a2 - u + x - y, b1 - u + v - y,
  b2 - u - v + x,variables={a1,a2,b1,b2,x,y,u,v}>,{a1,a2,b1,b2});
  EliminationIdeal(<(u - b1)*(b1 - a1) + (v - b2)*(b2 - a2), (x - a1)
  *(b1 - a1) + (y - a2)*(b2 - a2), a1 + v - x - y, (u - b1)^2 + (v -
  b2)^2 - (a1 - b1)^2 - (a2 - b2)^2, (x - a1)^2 + (y - a2)^2 - (a1 -
  b1)^2 - (a2 - b2)^2,variables={a1,a2,b1,b2,x,y,u,v}>,{a1,a2,b1,b2})
  ;
```

$$\langle 0 \rangle$$

$$\langle 0 \rangle \qquad (3.4)$$

```
> EliminationIdeal(<5*a1 - 2*u - v - 3*x + y, 5*a2 + u - 2*v - x - 3*
  y, 5*b1 - 3*u + v - 2*x - y, 5*b2 - u - 3*v + x - 2*y,variables=
  {a1,a2,b1,b2,x,y,u,v}>,{a1,a2,b1,b2}); EliminationIdeal(<(u - b1)*
  (b1 - a1) + (v - b2)*(b2 - a2), (x - a1)*(b1 - a1) + (y - a2)*(b2 -
  a2), (u - b1)^2 + (v - b2)^2 - (a1 - b1)^2 - (a2 - b2)^2, (x - a1)
  ^2 + (y - a2)^2 - (a1 - b1)^2 - (a2 - b2)^2, 5*a1 - 2*u - v - 3*x +
  y,variables={a1,a2,b1,b2,x,y,u,v}>,{a1,a2,b1,b2});
```

$$\langle 0 \rangle$$

$$\langle 0 \rangle \qquad (3.5)$$

```
> EliminationIdeal(<5*a1 - 2*u + v - 3*x - y, 5*a2 - u - 2*v + x - 3*
  y, 5*b1 - 3*u - v - 2*x + y, 5*b2 + u - 3*v - x - 2*y,variables=
  {a1,a2,b1,b2,x,y,u,v}>,{a1,a2,b1,b2}); EliminationIdeal(<(u - b1)*
  (b1 - a1) + (v - b2)*(b2 - a2), (x - a1)*(b1 - a1) + (y - a2)*(b2 -
  a2), (u - b1)^2 + (v - b2)^2 - (a1 - b1)^2 - (a2 - b2)^2, (x - a1)
  ^2 + (y - a2)^2 - (a1 - b1)^2 - (a2 - b2)^2, 5*a1 - 2*u + v - 3*x -
  y,variables={a1,a2,b1,b2,x,y,u,v}>,{a1,a2,b1,b2});
```

$$\langle 0 \rangle$$

$$\langle 0 \rangle \qquad (3.6)$$

The only different behavior is with the fifht component, that in order to contract it to the initial ring has to be saturated by the coeff. (2*u-2*x). The output is an ideal that clearly has b1-a1 as restriction, so it is a DEGENERATE component, but this does not seem to be great degeneracy from the geometric point of view.????

CHECKING THE TRUTH OF THE STATEMENT (TRADITIONAL WAY)

So, let us try to see if the the statement holds

Thesis: DF is parallel to AB, ie. (u-x)*(b2-a2) -(v-y)*(b1-a1)=0

First, let us see if it is generally true (considering a1a2b1b2 as the set of independent variables, even we know it is not maximal).

```
> EliminationIdeal(<(x-a1)^2+(y-a2)^2-((a1-b1)^2+(a2-b2)^2),(u-b1)^2+
  (v-b2)^2-((a1-b1)^2+(a2-b2)^2),(x-a1)*(b1-a1)+ (y-a2)*(b2-a2),(u-
  b1)*(b1-a1)+(v-b2)*(b2-a2), ((u-x)*(b2-a2) -(v-y)*(b1-a1))*t-1>,
  {a1,a2,b1,b2});
```

$$\langle 0 \rangle \qquad \textbf{(4.1)}$$

So it is not generally true.

Let us see if there is a difference using the other 5 independent variables

```
> EliminationIdeal(<(x-a1)^2+(y-a2)^2-((a1-b1)^2+(a2-b2)^2),(u-b1)^2+
  (v-b2)^2-((a1-b1)^2+(a2-b2)^2),(x-a1)*(b1-a1)+ (y-a2)*(b2-a2),(u-
  b1)*(b1-a1)+(v-b2)*(b2-a2), ((u-x)*(b2-a2) -(v-y)*(b1-a1))*t-1>,
  {a1,b2,x,v,a2});
```

$$\langle a1^2 - 2\, a1\, x - a2^2 + 2\, a2\, b2 - b2^2 + x^2 \rangle \qquad \textbf{(4.2)}$$

So here it is true,except when $a1^2 - 2\, a1\, x - a2^2 + 2\, a2\, b2 - b2^2 + x^2 = 0$, that is, when (x-a1)^2-(a2-b2)^2=0 which implies (y-a2)^2-(a1-b1)^2=0 too. Notice that both equalities hold, for instance, when (a1,a2)=(0,0) and (b1,b2) = (1,0)...so truth must be a rare situation from the geometric point of view... because, probably, the irreducible components where the five variables {a1,b2,x,v,a2} are free are geometrically rare...

Now let us see if it is generally false (considering a1a2b1b2 as the set of independent variables, even we know it is not maximal).

```
> EliminationIdeal(<(x-a1)^2+(y-a2)^2-((a1-b1)^2+(a2-b2)^2),(u-b1)^2+
  (v-b2)^2-((a1-b1)^2+(a2-b2)^2),(x-a1)*(b1-a1)+ (y-a2)*(b2-a2),(u-
  b1)*(b1-a1)+(v-b2)*(b2-a2), ((u-x)*(b2-a2) -(v-y)*(b1-a1))>,{a1,a2,
  b1,b2});
```

$$\langle 0 \rangle \qquad \textbf{(4.3)}$$

So we do not know if it is not generally false (because the statement generally false ==> elimination ≠ 0, requires restrictions in dim(H) = number of free variables). Thus we have to look specifically for a non degenerate component where this statement is false.

CHECKING THE TRUTH WITH SATURATION (1)

We extend H to the ring K(a1,a2,b1,b2)[x,y,uv], let He be the extended ideal . The primary components of the extended ideal are four PPe[1...4], of dimension 0 and, computing a GBasis for each of them under the tdeg(x,y,u,v) order yields a basis that has leading coefficients constant, so the saturation is not needed, except reconsidering the same components onn the base ring K[a1,a2,b1,b2,x,y,uv], say, PPec[1..4]. Over each of these components we apply our criterion

EliminationIdeal(PPe1c+<((u-x)*(b2-a2) -(v-y)*(b1-a1))*t-1>,{a1,a2,b1,b2});

for generally true (if not zero)

and

EliminationIdeal(PPe1c+<((u-x)*(b2-a2) -(v-y)*(b1-a1))>,{a1,a2,b1,b2});

for generally false (if not zero). We get that the result is true over the first and last components and false on the two middle ones.

```
> He:=<(x-a1)^2+(y-a2)^2-((a1-b1)^2+(a2-b2)^2),(u-b1)^2+(v-b2)^2-((a1
  -b1)^2+(a2-b2)^2),(x-a1)*(b1-a1)+ (y-a2)*(b2-a2),(u-b1)*(b1-a1)+(v-
  b2)*(b2-a2), variables={x,y,u,v}>;HilbertDimension(He);PPe:=
  {PrimaryDecomposition(He)};nops(PPe);for i from 1 to nops(PPe) do
  HilbertDimension(<PPe[i], variables={x,y,u,v}>)end do;
```

$$He := \langle (u-b1)\,(b1-a1)+(v-b2)\,(b2-a2),\ (x-a1)\,(b1-a1)+(y-a2)\,(b2-a2),\ (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2,\ (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2 \rangle$$

$$0$$

$$PPe := \{\langle (u-b1)\,(b1-a1)+(v-b2)\,(b2-a2),\ (x-a1)\,(b1-a1)+(y-a2)\,(b2-a2),\ a1-a2+b2-x,\ a2-b1-b2+u,\ (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2,\ (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2 \rangle,\ \langle (u-b1)\,(b1-a1)+(v-b2)\,(b2-a2),\ (x-a1)\,(b1-a1)+(y-a2)\,(b2-a2),\ a1-a2+b2-x,\ a2+b1-b2-u,\ (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2,\ (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2 \rangle,\ \langle (u-b1)\,(b1-a1)+(v-b2)\,(b2-a2),\ (x-a1)\,(b1-a1)+(y-a2)\,(b2-a2),\ a1+a2-b2-x,\ a2-b1-b2+u,\ (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2,\ (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2 \rangle,\ \langle (u-b1)\,(b1-a1)+(v-b2)\,(b2-a2),\ (x-a1)\,(b1-a1)+(y$$

$-a2)(b2-a2), a1+a2-b2-x, a2+b1-b2-u, (u-b1)^2+(v-b2)^2$
$-(a1-b1)^2-(a2-b2)^2, (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2\rangle\}$

$$4$$
$$0$$
$$0$$
$$0$$
$$0 \quad (5.1)$$

```
> PPe[1];
```

$$\langle (u-b1)(b1-a1)+(v-b2)(b2-a2), (x-a1)(b1-a1)+(y-a2)(b2-a2), a1-a2+b2-x, a2-b1-b2+u, (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2, (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2\rangle \quad (5.2)$$

```
> PPe1:=<(u - b1)*(b1 - a1) + (v - b2)*(b2 - a2), (x - a1)*(b1 - a1)
  + (y - a2)*(b2 - a2), a1 - a2 + b2 - x, a2 - b1 - b2 + u, (u - b1)
  ^2 + (v - b2)^2 - (a1 - b1)^2 - (a2 - b2)^2, (x - a1)^2 + (y - a2)
  ^2 - (a1 - b1)^2 - (a2 - b2)^2,variables={x,y,u,v}>;
  HilbertDimension(%);Basis(PPe1,tdeg(x,y,u,v));
```

$$PPe1 := \langle (u-b1)(b1-a1)+(v-b2)(b2-a2), (x-a1)(b1-a1)+(y-a2)(b2-a2), a1-a2+b2-x, a2-b1-b2+u, (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2, (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2\rangle$$

$$0$$

$$[-a1+b1-b2+v, a2-b1-b2+u, -a2-a1+b1+y, -a1+a2-b2+x] \quad (5.3)$$

```
> PPe1c:=<-a1 + b1 - b2 + v, a2 - b1 - b2 + u, -a2 - a1 + b1 + y, -a1
  + a2 - b2 + x, variables={a1,a2,b1,b2,x,y,u,v}>; HilbertDimension
  (PPe1c); EliminationIdeal(PPe1c,{a1,a2,b1,b2});
```

$$PPe1c := \langle -a1+a2-b2+x, -a1+b1-b2+v, -a2-a1+b1+y, a2-b1-b2+u\rangle$$

$$4$$

$$\langle 0\rangle \quad (5.4)$$

```
> PrimaryDecomposition(PPe1c);EliminationIdeal(PPe1c+<((u-x)*(b2-a2)
  -(v-y)*(b1-a1))*t-1>,{a1,a2,b1,b2});
```

$$\langle -a1+a2-b2+x, -a1+b1-b2+v, -a2-a1+b1+y, a2-b1-b2+u\rangle$$

$$\langle 1\rangle \quad (5.5)$$

TRUE on first component

```
> PPe[2];
```

$$\langle (u-b1)(b1-a1)+(v-b2)(b2-a2), (x-a1)(b1-a1)+(y-a2)(b2-a2), a1-a2+b2-x, a2+b1-b2-u, (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2, \quad (5.6)$$

$(x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2\rangle$

```
> PPe2:=<(u - b1)*(b1 - a1) + (v - b2)*(b2 - a2), (x - a1)*(b1 - a1)
  + (y - a2)*(b2 - a2), a1 - a2 + b2 - x, a2 + b1 - b2 - u, (u - b1)
  ^2 + (v - b2)^2 - (a1 - b1)^2 - (a2 - b2)^2, (x - a1)^2 + (y - a2)
  ^2 - (a1 - b1)^2 - (a2 - b2)^2,variables={x,y,u,v}>;
  HilbertDimension(%);Basis(PPe2,tdeg(x,y,u,v));
```

$$PPe2 := \langle (u-b1)(b1-a1)+(v-b2)(b2-a2), (x-a1)(b1-a1)+(y-a2)(b2-a2), a1-a2+b2-x, a2+b1-b2-u, (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2, (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2\rangle$$

$$0$$

$$[a1-b1-b2+v, -a2-b1+b2+u, -a2-a1+b1+y, -a1+a2-b2+x] \quad (5.7)$$

```
> PPe2c:=<a1 - b1 - b2 + v, -a2 - b1 + b2 + u, -a2 - a1 + b1 + y, -a1
  + a2 - b2 + x, variables={a1,a2,b1,b2,x,y,u,v}>; HilbertDimension
  (PPe2c); EliminationIdeal(PPe2c,{a1,a2,b1,b2});
```

$$PPe2c := \langle -a1+a2-b2+x, a1-b1-b2+v, -a2-a1+b1+y, -a2-b1+b2+u\rangle$$

$$4$$

$$\langle 0\rangle \quad (5.8)$$

```
> PrimaryDecomposition(PPe2c);EliminationIdeal(PPe2c+<((u-x)*(b2-a2)
  -(v-y)*(b1-a1))*t-1>,{a1,a2,b1,b2});EliminationIdeal(PPe2c+<((u-x)*
  (b2-a2) -(v-y)*(b1-a1))>,{a1,a2,b1,b2});
```

$$\langle -a1+a2-b2+x, a1-b1-b2+v, -a2-a1+b1+y, -a2-b1+b2+u\rangle$$

$$\langle 0\rangle$$

$$\langle a1^2-2\,b1\,a1+a2^2-2\,b2\,a2+b1^2+b2^2\rangle \quad (5.9)$$

FALSE, IT IS TRUE ONLY IF (a1-b1)^2+(a2-b2)^2 IS ZERO

```
> PPe[3];
```

$$\langle (u-b1)(b1-a1)+(v-b2)(b2-a2), (x-a1)(b1-a1)+(y-a2)(b2-a2), a1+a2-b2-x, a2-b1-b2+u, (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2, (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2\rangle \quad (5.10)$$

```
> PPe3:=<(u - b1)*(b1 - a1) + (v - b2)*(b2 - a2), (x - a1)*(b1 - a1)
  + (y - a2)*(b2 - a2), a1 + a2 - b2 - x, a2 - b1 - b2 + u, (u - b1)
  ^2 + (v - b2)^2 - (a1 - b1)^2 - (a2 - b2)^2, (x - a1)^2 + (y - a2)
  ^2 - (a1 - b1)^2 - (a2 - b2)^2,variables={x,y,u,v}>;
  HilbertDimension(%);Basis(PPe3,tdeg(x,y,u,v));
```

$$PPe3 := \langle (u-b1)(b1-a1)+(v-b2)(b2-a2), (x-a1)(b1-a1)+(y-a2)(b2-a2), a1+a2-b2-x, a2-b1-b2+u, (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2, (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2\rangle$$

$$0$$

$$[-a1+b1-b2+v, a2-b1-b2+u, -a2+a1-b1+y, -a1-a2+b2+x] \quad (5.11)$$

```
> PPe3c:=<-a1 + b1 - b2 + v, a2 - b1 - b2 + u, -a2 + a1 - b1 + y, -a1
  - a2 + b2 + x, variables={a1,a2,b1,b2,x,y,u,v}>; HilbertDimension
  (PPe3c); EliminationIdeal(PPe3c,{a1,a2,b1,b2});
```

$$PPe3c := \langle -a1-a2+b2+x, -a1+b1-b2+v, -a2+a1-b1+y, a2-b1-b2+u\rangle$$

$$4$$

$$\langle 0 \rangle \quad \mathbf{(5.12)}$$

```
> PrimaryDecomposition(PPe3c);EliminationIdeal(PPe3c+<((u-x)*(b2-a2)
  -(v-y)*(b1-a1))*t-1>,{a1,a2,b1,b2});EliminationIdeal(PPe3c+<((u-x)*
  (b2-a2) -(v-y)*(b1-a1))>,{a1,a2,b1,b2});
```

$$\langle -a1 - a2 + b2 + x,\ -a1 + b1 - b2 + v,\ -a2 + a1 - b1 + y,\ a2 - b1 - b2 + u \rangle$$

$$\langle 0 \rangle$$

$$\langle a1^2 - 2\, b1\, a1 + a2^2 - 2\, b2\, a2 + b1^2 + b2^2 \rangle \quad \mathbf{(5.13)}$$

FALSE, IT IS TRUE ONLY IF (a1-b1)^2+(a2-b2)^2 IS ZERO

```
> PPe[4];
```

$$\langle (u - b1)\,(b1 - a1) + (v - b2)\,(b2 - a2),\ (x - a1)\,(b1 - a1) + (y - a2)\,(b2 - a2),\ a1 + a2 - b2 - x,\ a2 + b1 - b2 - u,\ (u - b1)^2 + (v - b2)^2 - (a1 - b1)^2 - (a2 - b2)^2,\ (x - a1)^2 + (y - a2)^2 - (a1 - b1)^2 - (a2 - b2)^2 \rangle \quad \mathbf{(5.14)}$$

```
> PPe4:=<(u - b1)*(b1 - a1) + (v - b2)*(b2 - a2), (x - a1)*(b1 - a1)
  + (y - a2)*(b2 - a2), a1 + a2 - b2 - x, a2 + b1 - b2 - u, (u - b1)
  ^2 + (v - b2)^2 - (a1 - b1)^2 - (a2 - b2)^2, (x - a1)^2 + (y - a2)
  ^2 - (a1 - b1)^2 - (a2 - b2)^2,variables={x,y,u,v}>;
  HilbertDimension(%);Basis(PPe4,tdeg(x,y,u,v));
```

$$PPe4 := \langle (u - b1)\,(b1 - a1) + (v - b2)\,(b2 - a2),\ (x - a1)\,(b1 - a1) + (y - a2)\,(b2 - a2),\ a1 + a2 - b2 - x,\ a2 + b1 - b2 - u,\ (u - b1)^2 + (v - b2)^2 - (a1 - b1)^2 - (a2 - b2)^2,\ (x - a1)^2 + (y - a2)^2 - (a1 - b1)^2 - (a2 - b2)^2 \rangle$$

$$0$$

$$[a1 - b1 - b2 + v,\ -a2 - b1 + b2 + u,\ -a2 + a1 - b1 + y,\ -a1 - a2 + b2 + x] \quad \mathbf{(5.15)}$$

```
> PPe4c:=<a1 - b1 - b2 + v, -a2 - b1 + b2 + u, -a2 + a1 - b1 + y, -a1
  - a2 + b2 + x, variables={a1,a2,b1,b2,x,y,u,v}>; HilbertDimension
  (PPe4c); EliminationIdeal(PPe4c,{a1,a2,b1,b2});
```

$$PPe4c := \langle -a1 - a2 + b2 + x,\ a1 - b1 - b2 + v,\ -a2 + a1 - b1 + y,\ -a2 - b1 + b2 + u \rangle$$

$$4$$

$$\langle 0 \rangle \quad \mathbf{(5.16)}$$

```
> PrimaryDecomposition(PPe4c);EliminationIdeal(PPe4c+<((u-x)*(b2-a2)
  -(v-y)*(b1-a1))*t-1>,{a1,a2,b1,b2});EliminationIdeal(PPe4c+<((u-x)*
  (b2-a2) -(v-y)*(b1-a1))>,{a1,a2,b1,b2});
```

$$\langle -a1 - a2 + b2 + x,\ a1 - b1 - b2 + v,\ -a2 + a1 - b1 + y,\ -a2 - b1 + b2 + u \rangle$$

$$\langle 1 \rangle$$

$$\langle 0 \rangle \quad \mathbf{(5.17)}$$

TRUE

CHECKING THE TRUTH WITH SATURATION (2)

Here we do not even compute a primary decomposition of He. We just do the extension and contraction and verify over the non-degenerate components of the initial ideal if it is true on parts. Saturation here was needed because of the coefficients of the Gbasis included (a1-b1) as coefficient. We took the basis of He and performed the saturation with respect to (a1-b1). The result is SS., dimension 4 and with a1,a2,b1,b2 free, as expected. Now we apply the standard protocol to SS yielding that is is true on parts.

If interested we could even decompose SS but it is not necessary as the four components have been already computed before.

```
> He:=<(x-a1)^2+(y-a2)^2-((a1-b1)^2+(a2-b2)^2),(u-b1)^2+(v-b2)^2-((a1-b1)^2+(a2-b2)^2),(x-a1)*(b1-a1)+ (y-a2)*(b2-a2),(u-b1)*(b1-a1)+(v-b2)*(b2-a2), variables={x,y,u,v}>;
```

$$He := \langle (u-b1)(b1-a1)+(v-b2)(b2-a2), (x-a1)(b1-a1)+(y-a2)(b2-a2), (u-b1)^2+(v-b2)^2-(a1-b1)^2-(a2-b2)^2, (x-a1)^2+(y-a2)^2-(a1-b1)^2-(a2-b2)^2 \rangle \quad (6.1)$$

```
> Basis(He,tdeg(x,y,u,v));
```

$$[(a1-b1)u+(a2-b2)v-b1\,a1-b2\,a2+b1^2+b2^2, (a1-b1)x+(a2-b2)y-a1^2+b1\,a1-a2^2+b2\,a2, -a1^2+2\,b1\,a1-b1^2+b2^2-2\,v\,b2+v^2, -a1^2+2\,b1\,a1+a2^2-2\,y\,a2-b1^2+y^2] \quad (6.2)$$

```
> SS:=Saturate(<(a1 - b1)*u + (a2 - b2)*v - b1*a1 - b2*a2 + b1^2 + b2^2, (a1 - b1)*x + (a2 - b2)*y - a1^2 + b1*a1 - a2^2 + b2*a2, -a1^2 + 2*a1*b1 - b1^2 + b2^2 - 2*b2*v + v^2, -a1^2 + 2*a1*b1 + a2^2 - 2*a2*y - b1^2 + y^2, variables={a1,a2,b1,b2,x,y,u,v}>, (a1-b1));
HilbertDimension(SS);EliminationIdeal(SS,{a1,a1,b1,b2});
```

$$SS := \langle -a1^2+2\,b1\,a1+a2^2-2\,a2\,y-b1^2+y^2, a1^2-2\,a1\,x-a2^2+2\,b2\,a2-b2^2+x^2, -a2^2+2\,b2\,a2+b1^2-2\,u\,b1-b2^2+u^2, a1\,a2-a1\,b2-a2\,b1+2\,b1\,b2-v\,b1-u\,b2+u\,v, 2\,a1\,a2-a1\,b2-y\,a1-a2\,b1-x\,a2+b1\,b2+x\,y, b1\,a1-a1\,u+b2\,a2-a2\,v-b1^2+u\,b1-b2^2+v\,b2, a1^2-b1\,a1-a1\,x+a2^2-b2\,a2-a2\,y+x\,b1+y\,b2, -a1\,b2+v\,a1+a2\,b1-u\,a2-y\,b1+x\,b2+y\,u-x\,v, -a1^2+4\,b1\,a1-2\,a1\,u+2\,b2\,a2-2\,a2\,v-3\,b1^2+2\,u\,b1-b2^2+v^2 \rangle$$

$$4$$

$$\langle 0 \rangle \quad (6.3)$$

```
> EliminationIdeal(<-a1^2 + 2*a1*b1 + a2^2 - 2*a2*y - b1^2 + y^2, a1^2 - 2*a1*x - a2^2 + 2*a2*b2 - b2^2 + x^2, -a2^2 + 2*a2*b2 + b1^2 - 2*b1*u - b2^2 + u^2, a1*a2 - a1*b2 - a2*b1 + 2*b1*b2 - b1*v - b2*u + u*v, 2*a1*a2 - a1*b2 - a1*y - a2*b1 - a2*x + b1*b2 + x*y, a1*b1 - a1*u + a2*b2 - a2*v - b1^2 + b1*u - b2^2 + b2*v, a1^2 - a1*b1 - a1*x + a2^2 - a2*b2 - a2*y + b1*x + b2*y, -a1*b2 + a1*v + a2*b1 - a2*u - b1*y + b2*x + u*y - v*x, -a1^2 + 4*a1*b1 - 2*a1*u + 2*a2*b2 - 2*a2*v - 3*b1^2 + 2*b1*u - b2^2 + v^2, ((u-x)*(b2-a2) -(v-y)*(b1-a1))*t-1>,{a1,a2,b1,b2});
```

$$\langle 0 \rangle \quad (6.4)$$

```
> EliminationIdeal(<-a1^2 + 2*a1*b1 + a2^2 - 2*a2*y - b1^2 + y^2,
  a1^2 - 2*a1*x - a2^2 + 2*a2*b2 - b2^2 + x^2, -a2^2 + 2*a2*b2 + b1^2
  - 2*b1*u - b2^2 + u^2, a1*a2 - a1*b2 - a2*b1 + 2*b1*b2 - b1*v - b2*
  u + u*v, 2*a1*a2 - a1*b2 - a1*y - a2*b1 - a2*x + b1*b2 + x*y, a1*b1
  - a1*u + a2*b2 - a2*v - b1^2 + b1*u - b2^2 + b2*v, a1^2 - a1*b1 -
  a1*x + a2^2 - a2*b2 - a2*y + b1*x + b2*y, -a1*b2 + a1*v + a2*b1 -
  a2*u - b1*y + b2*x + u*y - v*x, -a1^2 + 4*a1*b1 - 2*a1*u + 2*a2*b2
  - 2*a2*v - 3*b1^2 + 2*b1*u - b2^2 + v^2, ((u-x)*(b2-a2) -(v-y)*(b1-
  a1))>,{a1,a2,b1,b2});
```

$$\langle 0 \rangle \tag{6.5}$$

```
> PrimaryDecomposition(SS);
```

$$\langle -a1 - a2 + b2 + x, -a1 + b1 - b2 + v, -a2 + a1 - b1 + y, a2 - b1 - b2 + u\rangle, \langle -a1 - a2 + b2 + x, a1 - b1 - b2 + v, -a2 + a1 - b1 + y, -a2 - b1 + b2 + u\rangle, \langle -a1 + a2 - b2 + x, -a1 + b1 - b2 + v, -a2 - a1 + b1 + y, a2 - b1 - b2 + u\rangle, \langle -a1 + a2 - b2 + x, a1 - b1 - b2 + v, -a2 - a1 + b1 + y, -a2 - b1 + b2 + u\rangle \tag{6.6}$$

CHECKING THE TRUTH DIRECTLY ZERO DIMENSIONAL CASE (3)

Here we do not even compute a primary decomposition of He. We just do the extension and contraction and verify over the non-degenerate components of the initial ideal if it is true on parts. Saturation here was needed because of the coefficients of the Gbasis included (a1-b1) as coefficient. We took the basis of He and performed the saturation with respect to (a1-b1). The result is SS., dimension 4 and with a1,a2,b1,b2 free, as expected. Now we apply the standard protocol to SS yielding that is is true on parts.

If interested we could even decompose SS but it is not necessary as the four components have been already computed before.

```
> He:=<(x-a1)^2+(y-a2)^2-((a1-b1)^2+(a2-b2)^2),(u-b1)^2+(v-b2)^2-((a1
  -b1)^2+(a2-b2)^2),(x-a1)*(b1-a1)+ (y-a2)*(b2-a2),(u-b1)*(b1-a1)+(v-
  b2)*(b2-a2), variables={x,y,u,v}>;
```

$$He := \langle (u - b1)\,(b1 - a1) + (v - b2)\,(b2 - a2), (x - a1)\,(b1 - a1) + (y - a2)\,(b2 - a2), (u - b1)^2 + (v - b2)^2 - (a1 - b1)^2 - (a2 - b2)^2, (x - a1)^2 + (y - a2)^2 - (a1 - b1)^2 - (a2 - b2)^2\rangle \tag{7.1}$$

```
> Basis(He,tdeg(x,y,u,v));
```

$$[(a1 - b1)\,u + (a2 - b2)\,v - b1\,a1 - b2\,a2 + b1^2 + b2^2, (a1 - b1)\,x + (a2 - b2)\,y - a1^2 + b1\,a1 - a2^2 + b2\,a2, -a1^2 + 2\,b1\,a1 - b1^2 + b2^2 - 2\,v\,b2 + v^2, -a1^2 + 2\,b1\,a1 + a2^2 - 2\,y\,a2 - b1^2 + y^2] \tag{7.2}$$

```
> SS:=Saturate(<(a1 - b1)*u + (a2 - b2)*v - b1*a1 - b2*a2 + b1^2 +
  b2^2, (a1 - b1)*x + (a2 - b2)*y - a1^2 + b1*a1 - a2^2 + b2*a2, -
  a1^2 + 2*a1*b1 - b1^2 + b2^2 - 2*b2*v + v^2, -a1^2 + 2*a1*b1 + a2^2
  - 2*a2*y - b1^2 + y^2, variables={a1,a2,b1,b2,x,y,u,v}>, (a1-b1));
  HilbertDimension(SS);EliminationIdeal(SS,{a1,a1,b1,b2});
```

$$SS := \langle -a1^2 + 2\,b1\,a1 + a2^2 - 2\,a2\,y - b1^2 + y^2, a1^2 - 2\,a1\,x - a2^2 + 2\,b2\,a2 - b2^2 + x^2, -a2^2 + 2\,b2\,a2 + b1^2 - 2\,u\,b1 - b2^2 + u^2, a1\,a2 - a1\,b2 - a2\,b1 + 2\,b1\,b2 - v\,b1 - u\,b2 + u\,v, 2\,a1\,a2 - a1\,b2 - y\,a1 - a2\,b1 - x\,a2 + b1\,b2 + x\,y, b1\,a1 - a1\,u + b2\,a2 - a2\,v - b1^2 + u\,b1 - b2^2 + v\,b2, a1^2 - b1\,a1 - a1\,x + a2^2 - b2\,a2 - a2\,y + x\,b1 + y\,b2, -a1\,b2 + v\,a1 + a2\,b1 - u\,a2 - y\,b1 + x\,b2 + y\,u - x\,v, -a1^2 + 4\,b1\,a1 - 2\,a1\,u + 2\,b2\,a2 - 2\,a2\,v - 3\,b1^2 + 2\,u\,b1 - b2^2 + v^2\rangle$$

$$4$$

$$\langle 0\rangle \qquad (7.3)$$

```
> EliminationIdeal(<-a1^2 + 2*a1*b1 + a2^2 - 2*a2*y - b1^2 + y^2,
  a1^2 - 2*a1*x - a2^2 + 2*a2*b2 - b2^2 + x^2, -a2^2 + 2*a2*b2 + b1^2
  - 2*b1*u - b2^2 + u^2, a1*a2 - a1*b2 - a2*b1 + 2*b1*b2 - b1*v - b2*
  u + u*v, 2*a1*a2 - a1*b2 - a1*y - a2*b1 - a2*x + b1*b2 + x*y, a1*b1
  - a1*u + a2*b2 - a2*v - b1^2 + b1*u - b2^2 + b2*v, a1^2 - a1*b1 -
  a1*x + a2^2 - a2*b2 - a2*y + b1*x + b2*y, -a1*b2 + a1*v + a2*b1 -
  a2*u - b1*y + b2*x + u*y - v*x, -a1^2 + 4*a1*b1 - 2*a1*u + 2*a2*b2
  - 2*a2*v - 3*b1^2 + 2*b1*u - b2^2 + v^2, ((u-x)*(b2-a2) -(v-y)*(b1-
  a1))*t-1>,{a1,a2,b1,b2});
```

$$\langle 0\rangle \qquad (7.4)$$

```
> EliminationIdeal(<-a1^2 + 2*a1*b1 + a2^2 - 2*a2*y - b1^2 + y^2,
  a1^2 - 2*a1*x - a2^2 + 2*a2*b2 - b2^2 + x^2, -a2^2 + 2*a2*b2 + b1^2
  - 2*b1*u - b2^2 + u^2, a1*a2 - a1*b2 - a2*b1 + 2*b1*b2 - b1*v - b2*
  u + u*v, 2*a1*a2 - a1*b2 - a1*y - a2*b1 - a2*x + b1*b2 + x*y, a1*b1
  - a1*u + a2*b2 - a2*v - b1^2 + b1*u - b2^2 + b2*v, a1^2 - a1*b1 -
  a1*x + a2^2 - a2*b2 - a2*y + b1*x + b2*y, -a1*b2 + a1*v + a2*b1 -
  a2*u - b1*y + b2*x + u*y - v*x, -a1^2 + 4*a1*b1 - 2*a1*u + 2*a2*b2
  - 2*a2*v - 3*b1^2 + 2*b1*u - b2^2 + v^2, ((u-x)*(b2-a2) -(v-y)*(b1-
  a1))>,{a1,a2,b1,b2});
```

$$\langle 0\rangle \qquad (7.5)$$

```
> PrimaryDecomposition(SS);
```

$$\langle -a1 - a2 + b2 + x, -a1 + b1 - b2 + v, -a2 + a1 - b1 + y, a2 - b1 - b2 + u\rangle, \langle -a1 - a2 + b2 + x, a1 - b1 - b2 + v, -a2 + a1 - b1 + y, -a2 - b1 + b2 + u\rangle, \langle -a1 + a2 - b2 + x, -a1 + b1 - b2 + v, -a2 - a1 + b1 + y, a2 - b1 - b2 + u\rangle, \langle -a1 + a2 - b2 + x, a1 - b1 - b2 + v, -a2 - a1 + b1 + y, -a2 - b1 + b2 + u\rangle \qquad (7.6)$$