The Nicolas criterion for the Riemann Hypothesis

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Abstract

For every prime number p_n, we define the sequence $X_n = \prod_{q \leq p_n} \frac{q}{q-1} - e^{\gamma} \times \log \theta(p_n)$, where $\theta(x)$ is the Chebyshev function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The Nicolas theorem states that the Riemann hypothesis is true if and only if the $X_n > 0$ holds for all prime $p_n > 2$. For every prime number p_k, $X_k > 0$ is called the Nicolas inequality. We show if the sequence X_n is strictly decreasing for n big enough, then the Riemann hypothesis must be true. For every prime number $p_n > 2$, we define the sequence $Y_n = \frac{e^{\frac{1}{2 \times \log(p_n)}}}{(1 - \frac{1}{\log(p_n)})}$ and show that Y_n is strictly decreasing for $p_n > 2$. For all prime $p_n > 286$, we demonstrate that the inequality $X_n < e^{\gamma} \times \log Y_n$ is always satisfied. We prove that $\lim_{n \to \infty} X_n = \lim_{n \to \infty} (\log Y_n) = 0$.

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1. Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x [2]. For every prime p_n, we define the sequence

$$X_n = \prod_{q \leq p_n} \frac{q}{q-1} - e^{\gamma} \times \log \theta(p_n).$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. The importance of this property is:

Theorem 1.1. *[3], [4]. $X_n > 0$ holds for all prime $p_n > 2$ if and only if the Riemann hypothesis is true. Moreover, the Riemann hypothesis is false if and only if there are infinitely many prime numbers q_i for which $X_i \leq 0$ and infinitely many prime numbers r_j for which $X_j > 0$.*

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We use the following properties of the Chebyshev function:

Theorem 1.2. *[2].*
$$\lim_{x\to\infty} \frac{\theta(x)}{x} = 1.$$

Theorem 1.3. *[5]. For $x \geq 41$:*

$$\theta(x) > (1 - \frac{1}{\log(x)}) \times x.$$

Besides, we use the following result:

Theorem 1.4. *[5]. For $x \geq 286$:*

$$\prod_{q\leq x} \frac{q}{q-1} < e^{\gamma} \times (\log x + \frac{1}{2 \times \log(x)}).$$

We also use the Mertens' theorem which states:

Theorem 1.5. *[6].*
$$\lim_{x\to\infty} (\frac{1}{\log x} \times \prod_{q\leq x} \frac{q}{q-1}) = e^{\gamma}.$$

We prove if the sequence X_n is strictly decreasing for n big enough, then the Riemann hypothesis must be true. For every prime number $p_n > 2$, we define the sequence $Y_n = \frac{e^{\frac{1}{2\times\log(p_n)}}}{(1-\frac{1}{\log(p_n)})}$ and show that Y_n is strictly decreasing for $p_n > 2$. Finally, for all prime $p_n > 286$, we demonstrate that the inequality $X_n < e^{\gamma} \times \log Y_n$ is always satisfied.

2. Results

Theorem 2.1.
$$\lim_{n\to\infty} X_n = 0.$$

Proof. We know by the theorem 1.5:

$$\lim_{n\to\infty} (\frac{1}{\log p_n} \times \prod_{q\leq p_n} \frac{q}{q-1}) = e^{\gamma},$$

and we have by the theorem 1.2:
$$\lim_{n\to\infty} \frac{\theta(p_n)}{p_n} = 1.$$

Putting all this together yields the proof:

$$\lim_{n\to\infty} \left(\prod_{q\leq p_n} \frac{q}{q-1} - e^{\gamma} \times \log\theta(p_n) \right) = \lim_{n\to\infty} (e^{\gamma} \times \log p_n - e^{\gamma} \times \log p_n) = 0.$$

$\square$

Theorem 2.2. *If X_n is strictly decreasing for n big enough, then the Riemann hypothesis must be true.*

Proof. Suppose that $p_n > 2$ is the smallest prime number such that the Nicolas inequality is false under the assumption that X_i is strictly decreasing (that is $X_i > X_{i+1}$). In this way, we have

$$X_n \leq 0$$

and thus

$$X_{n+1} < X_n \leq 0.$$

This implies

$$\limsup_{n\to\infty} X_n < 0$$

which is a contradiction with the theorem 2.1. By contraposition, the Nicolas inequality would be satisfied for all prime p_n big enough. Consequently, there would be not infinitely many prime numbers for which the Nicolas inequality is unsatisfied. In this way, using the theorem 1.1, we can conclude that the Riemann hypothesis must be true when X_n is strictly decreasing for n big enough. □

For every prime number $p_n > 2$, we define sequence $Y_n = \frac{e^{\frac{1}{2\times\log(p_n)}}}{(1-\frac{1}{\log(p_n)})}$.

Theorem 2.3. *For every prime number $p_n > 2$, the sequence Y_n is strictly decreasing.*

Proof. For every real value $x \geq 3$, we state the function

$$f(x) = \frac{e^{\frac{1}{2\times\log(x)}}}{(1 - \frac{1}{\log(x)})}$$

where the derivative of $f(x)$ is

$$f'(x) = -\frac{1.5 \times e^{\frac{1}{2\times\log(x)}} \times (\log(x) - 0.333333)}{x \times (\log(x) - 1)^2 \times \log(x)}.$$

Consequently, the function $f(x)$ is monotonically decreasing for every real value $x \geq 3$ and therefore, the sequence Y_n is monotonically decreasing as well. Indeed, a function $f(x)$ of a real variable x is monotonically decreasing in some interval if the derivative of $f(x)$ is lesser than zero and the function $f(x)$ is continuous over that interval [7]. Certainly, the function $f'(x)$ is lesser than zero for all values $x \geq 3$ where $f(x)$ is continuous. In addition, Y_n is essentially a strictly decreasing sequence, since there is not any natural number $n > 1$ such that $Y_n = Y_{n+1}$. □

We will prove another important result:

Theorem 2.4. *Let $q_1, q_2, \ldots, q_m$ denote the first m consecutive primes such that $q_1 < q_2 < \cdots < q_m$ and $q_m > 286$. Then*

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times \log\left(Y_m \times \theta(q_m)\right).$$

Proof. From the theorem 1.3, we know that

$$\theta(q_m) > (1 - \frac{1}{\log(q_m)}) \times q_m.$$

In this way, we can show that

$$\begin{aligned}\log\left(Y_m \times \theta(q_m)\right) &> \log\left(Y_m \times (1 - \frac{1}{\log(q_m)}) \times q_m\right) \\ &= \log q_m + \log\left(Y_m \times (1 - \frac{1}{\log(q_m)})\right).\end{aligned}$$

We know that

$$\begin{aligned}\log\left(Y_m \times (1 - \frac{1}{\log(q_m)})\right) &= \log\left(\frac{e^{\frac{1}{2\times\log(q_m)}}}{(1 - \frac{1}{\log(q_m)})} \times (1 - \frac{1}{\log(q_m)})\right) \\ &= \log\left(e^{\frac{1}{2\times\log(q_m)}}\right) \\ &= \frac{1}{2 \times \log(q_m)}.\end{aligned}$$

Consequently, we obtain that

$$\log q_m + \log\left(Y_m \times (1 - \frac{1}{\log(q_m)})\right) \geq (\log q_m + \frac{1}{2 \times \log(q_m)}).$$

Due to the theorem 1.4, we prove that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times (\log q_m + \frac{1}{2 \times \log(q_m)}) < e^{\gamma} \times \log\left(Y_m \times \theta(q_m)\right)$$

when $q_m > 286$. □

We finally obtain the main result:

Theorem 2.5. *For all prime $p_n > 286$, we show that the inequality $X_n < e^{\gamma} \times \log Y_n$ is always satisfied.*

Proof. According to the theorem 2.4, we have that for all prime $p_n > 286$:

$$\prod_{q \leq p_n} \frac{q_i}{q_i - 1} < e^{\gamma} \times \log\left(Y_n \times \theta(p_n)\right)$$

which is equivalent to

$$\prod_{q \leq p_n} \frac{q_i}{q_i - 1} - e^{\gamma} \times \log\theta(p_n) < e^{\gamma} \times \log Y_n$$

and thus,

$$X_n < e^{\gamma} \times \log Y_n.$$

□

Theorem 2.6.

$$\lim_{n\to\infty}(\log Y_n) = 0.$$

Proof. We obtain that

$$\begin{aligned}\lim_{n\to\infty}(\log Y_n) &= \lim_{n\to\infty}(\log \frac{e^{\frac{1}{2\times\log(p_n)}}}{(1-\frac{1}{\log(p_n)})})\\ &= \log 1\\ &= 0.\end{aligned}$$

□

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