

On the Active Nodes of Network Systems

Kaiwen Chen and Alessandro Astolfi

Abstract—This paper studies interconnected systems (nodes) and exploits dissipation inequalities and the structure of the interconnection (the network) to derive analysis and design tools for stabilization. First, systems with quadratic supply rates in the dissipation inequalities are investigated. A stability condition based on the dissipation inequality associated to each node is given. This condition allows checking the sign of the dissipation inequality for the overall network system. By considering the underlying directed graph, the feasibility of controller design is discussed. The design problem is re-formulated into the problem of finding a solution to a system of linear inequalities. This allows the efficient search and computation of the design parameters of what we call *active nodes*. Then, systems with non-quadratic supply rates are considered. A vector of positive definite functions that is used as a basis of the non-quadratic supply rates is constructed: this requires augmenting the underlying directed graph. Similarly to the quadratic supply rates case, a stability condition for analysis and a graph-based criterion for checking the feasibility of controller design are discussed. Finally a design example to demonstrate how to exploit the so-called *active nodes* to design a controller without numerical computation is presented. The proposed method does not presume any stability property of the nodes and therefore can be applied to various scenarios occurring in the study of stability properties for network systems.

I. INTRODUCTION

Network systems have been of interest to researchers since the 1970s. In particular, stability analysis has been one of the main research areas since, as it is well known, even if each node possesses some stability property, such property is not necessarily retained by the interconnected system.

The mainstream method to guarantee that stability properties of the nodes are propagated to the network is based on small-gain-type theorems. These can be intuitively understood, from a signal perspective, as requiring that the signals are not amplified while flowing through the interconnection; and, from an energy perspective, as requiring that the total energy stored in the network does not accumulate because of the interconnection. The first interpretation leads to \mathcal{L}_p small-gain theorems (see *e.g.* [1]). These have been used in early works, see for example [2], on the study

This work has been partially supported by the European Union's Horizon 2020 Research and Innovation Programme under grant agreement No 739551 (KIOS CoE), and by Italian Ministry for Research in the framework of the 2017 Program for Research Projects of National Interest (PRIN), Grant no. 2017YKXYXJ.

K. Chen is with the Department of Electrical and Electronic Engineering, Imperial College London, London, SW7 2AZ, UK. E-mail: kaiwen.chen16@imperial.ac.uk

A. Astolfi is with the Department of Electrical and Electronic Engineering, Imperial College London, London, SW7 2AZ, UK and with the Dipartimento di Ingegneria Civile e Ingegneria Informatica, Università di Roma "Tor Vergata", Rome, 00133, Italy. E-mail: a.astolfi@imperial.ac.uk

of stability properties for large-scale network systems. The second interpretation, which is typically conveyed in terms of Lyapunov-type small-gain theorems (see *e.g.* [3]), exploits the properties of *input-to-state stability* (see *e.g.* [4]) or *integral input-to-state stability* (see *e.g.* [5]) of the nodes to conclude stability properties, with some robustness margin, for the overall network system. Recent works have exploited *weak* small-gain theorems to study stability properties, under relaxed assumptions, of network systems, see *e.g.* [6].

In addition to general stability results for networks with arbitrary structure, some of the works in the literature exploit the structure of the network, which leads to the so-called *cyclic small-gain theorem*, see *e.g.* [7]–[9], and the survey [10]. This type of *small-gain* can be summarized as: given that all nodes in the network satisfy certain *input-to-state stability* or *input-to-output stability* property, and the composition of the *gain functions* along every directed cycle of the underlying graph are less than the identity function, then the corresponding stability property holds for the overall network system.

One may note that the stability properties mentioned above, though indicating the existence of certain type of Lyapunov functions, do not explicitly give the algebraic expression of such Lyapunov functions and it is in general difficult to find them. In the light of this, differently from most of the works in this area that rely on stability properties of the nodes, this paper intends to apply a small-gain-like analysis by directly exploiting algebraic properties of the dissipation inequalities, which are much easier to obtain in practical designs, without making any assumption on the stability properties of the nodes, in the spirit of \mathcal{L}_p stability, *input-to-state stability*, or similar stability notions, yet not restricted by any of these properties during our derivation. We also give guidelines on how to design the parameters of a special type of *controlled nodes*, referred to as *active nodes* to guarantee the desired sign of the overall dissipation inequality.

Preliminaries

The paper uses standard notation unless stated otherwise. For an n -dimensional vector $v \in \mathbb{R}^n$, $v > 0$ means that v is element-wise positive and similarly for other inequality signs. An all-one vector of proper dimension is denoted by 1 . For a matrix M , $(M)_i$ denotes the i -th column and $(M)_{ij}$ denotes the i -th element on the j -th column, $M > 0$ means that M is element-wise positive and similarly for other inequality signs.

The following definition and theorem are recalled since they are useful in the remainder of the paper.

Definition 1 (Z-matrix): A matrix M is called a *Z-matrix* (or negated *Metzler matrix*) if all of its off-diagonal elements are non-positive, that is, $(M)_{ij} \leq 0$, $i \neq j$. \diamond

Definition 2 (M-matrix): A matrix $M = N + sI$, where N is a square *Z-matrix* and s is a real number not smaller than the spectral radius of N , is called an *M-matrix*. \diamond

Theorem 1: Let M be a *Z-matrix*. Then the following conditions¹ are equivalent.

- 1) M is a non-singular *M-matrix*.
- 2) All principal minors of M are positive.
- 3) All leading principal minors of M are positive.
- 4) M^{-1} exists and $M^{-1} \geq 0$.
- 5) There exists a vector $v > 0$ such that $Mv > 0$.

\diamond

II. SYSTEMS WITH QUADRATIC SUPPLY RATES

Consider a network of n interconnected dynamical systems in which each node, a dynamical system denoted as Σ_i , $i = 1, \dots, n$, has $n-1$ inputs $u_{ji}(t) \in \mathbb{R}$, $j = 1, \dots, i-1, i+1, \dots, n$, one output $y_i(t) \in \mathbb{R}$, and satisfies the dissipation inequality

$$\dot{V}_i \leq -a_i y_i^2 + \sum_{j=1}^{i-1} b_{ji} u_{ji}^2 + \sum_{j=i+1}^n b_{ji} u_{ji}^2, \quad (1)$$

with respect to a storage function $V_i: \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$ of class \mathcal{C}^1 , where n_i is the dimension of the state vector of Σ_i , $a_i > 0$, and $b_{ji} \geq 0$. The nodes are interconnected via the equations $u_{ji} = y_j$, for all $b_{ji} \neq 0$, whereas $b_{ji} = 0$ means that Σ_j is not connected to Σ_i , that is $u_{ji} = 0$. The structure of the network can be described by a directed graph, see Fig. 1 for an example, in which the directed edges represent input/output relations.

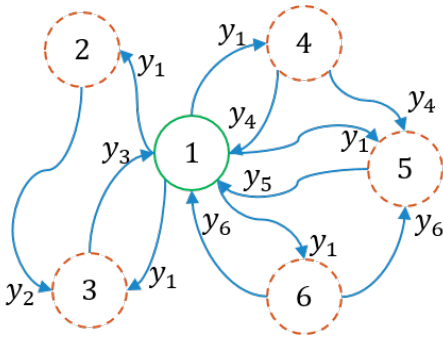


Fig. 1. The underlying directed graph of a network system.

In practice one is interested in the dissipation inequality for the overall network system. More specifically, one may wonder whether there exist positive scaling coefficients c_1, \dots, c_n such that the storage function of the overall network system, defined by $V = \sum_{i=1}^n c_i V_i$, satisfies the dissipation inequality

¹These are extracted from 40 equivalent conditions listed in [11], in which the proof of this result is available.

$\dot{V} \leq 0$. In addition, in the case in which the V_i 's are positive definite and radially unbounded one may require that

$$\dot{V} \leq -W(y) \leq 0, \quad (2)$$

where $W(\cdot)$ is a positive definite function of $y = [y_1, \dots, y_n]^T$. From the dissipation inequality (2) one can conclude Lyapunov stability of the equilibrium of the overall system and convergence of all y_i 's to 0 by means of some invariance-like analysis.

This, however, is not the focus on the paper, although it reveals a possible area of application of the forthcoming results. The aim of the paper is to study how to use the structure of the network to make the dissipation inequality (2) hold, in other words, we do not impose any condition on V and simply focus on \dot{V} . To this end, the next result provides a condition for the existence of the scaling coefficients c_i , $i = 1, \dots, n$, mentioned above.

Proposition 1: There exists a vector of scaling coefficients $c = [c_1, \dots, c_n]^T > 0$ such that V satisfies the dissipation inequality (2) if

$$M = \begin{bmatrix} a_1 & -b_{12} & \cdots & -b_{1n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ -b_{n1} & \cdots & -b_{n(n-1)} & a_n \end{bmatrix} \quad (3)$$

is a non-singular *M-matrix*. \diamond

Remark 1: There are several variants of Proposition 1 available in the literature, for example the criteria for \mathcal{L}_p -stability based on the so-called *test matrix*, that is essentially the matrix (3) written in terms of \mathcal{L}_p gains, see [2, Section 6.2]. Proposition 1, however, provides purely an algebraic result which does not require any assumption on the stability properties of each node. \diamond

It is worth noting that the condition expressed by Proposition 1 is not generic: it is straightforward to build networks for which it is not satisfied. One such a network is a simple-loop² containing two scalar nodes that violates the small-gain condition, that is either the condition $\frac{b_{21}}{a_1} \frac{b_{12}}{a_2} < 1$ or equivalently, the condition $a_1 a_2 - b_{12} b_{21} > 0$. (The counterpart of this small-gain condition for more complex networks is precisely condition 3) of Theorem 1.) The small-gain analysis for the considered example reveals the fact that if one is allowed to adjust the coefficients a_i 's arbitrarily one can always enforce the dissipation inequality (2), provided that there is a distributed controller on each of the nodes of the network to make the a_i 's tunable design parameters. In practice, however, this is not always feasible, for example because of dynamics that cannot be controlled, or economical concerns do not allow using as many distributed controllers as nodes. This highlights the fact that, even if Proposition 1

²The definition of loop is different in control theory and in graph theory (in which "loop" specifically means "self-loop"). The underlying directed graph of the considered example is a single-cycle graph. In this example we use the *control convention*.

provides a tool for network analysis, we need to answer the following questions from a design perspective. How many controllers are needed to enforce the dissipation inequality (2) and where these should be placed considering the structure of the network? How to tune the design parameters a_i 's of those nodes that can be actively controlled?

To answer the first question we define a special class of nodes.

Definition 3: A node Σ_{i_a} is called an *active node* if it satisfies the dissipation inequality (1) with an adjustable $a_{i_a} \in [\underline{a}_{i_a}, +\infty)$, with $\underline{a}_{i_a} > 0$, where i_a is the node index of the *active node*. \diamond

We now make a convention for graphic representation: an *active node* is represented by a solid green circle (e.g. node Σ_1 in Fig. 1) and non-active nodes are represented by red dashed circles. Exploiting the concept of *active nodes* we now present a feasibility condition for the considered design problem.

Proposition 2: The matrix (3) can be made a non-singular *M-matrix* by adjusting the parameters a_{i_a} (recall that the indices i_a denote the indices of the *active nodes*) if every directed cycle of the underlying directed graph describing the network contains at least one vertex corresponding to an *active node*. \diamond

To illustrate the result expressed by Proposition 2 consider the example illustrated in Fig. 1. The matrix M related to this graph is a non-singular *M-matrix* with a_1 sufficiently large, as node Σ_1 is an *active node*, and its corresponding vertex is contained in every directed cycle of the graph. Proposition 2 and Proposition 1 therefore reveal that as long as there is at least one *active node* in every directed cycle, there exist positive scaling coefficients c_1, \dots, c_n and design parameters a_{i_a} such that the dissipation inequality (2) holds, where i_a represents the set of indices of the *active nodes*. Note that this condition simply guarantees the feasibility of the underlying design problem yet does not provide an approach to carry out the design computationally.

In what follows we show how to formulate the set of admissible scaling coefficients and design parameters using linear inequalities to make the search of such parameters efficient. To this end, recall that

$$\dot{V} \leq -\phi^\top(y)Mc. \quad (4)$$

As a result, the dissipation inequality (2) holds if $Mc > 0$, for some $c > 0$. Note that M can be decomposed as $M = \underline{M} + \tilde{M}$, where

$$(\underline{M})_{ij} = \begin{cases} \underline{a}_i, & \text{if node } \Sigma_i \text{ is active,} \\ (M)_{ij}, & \text{otherwise,} \end{cases} \quad (5)$$

and therefore

$$(\tilde{M})_{ij} = \begin{cases} \tilde{a}_i, & \text{if node } \Sigma_i \text{ is active,} \\ 0, & \text{otherwise,} \end{cases} \quad (6)$$

where $\tilde{a}_i = a_i - \underline{a}_i$. Now re-write $\tilde{M}c$ as $\tilde{M}c = Nd$, where N is a constant matrix and $d > 0$ is the vector of the decision

variables. To do this, define $d_j = c_{i_{aj}}\tilde{a}_{i_{aj}}$, where i_{aj} is the node index of the j -th active node, and $d = [d_1, \dots, d_{n_a}]^\top$, where n_a is the total number of *active nodes*. This yields

$$(N)_{ij} = \begin{cases} 1, & \text{if } i = i_{aj}, \\ 0, & \text{otherwise,} \end{cases} \quad (7)$$

which is a constant matrix such that $\tilde{M}c = Nd$ and therefore $Mc = \underline{M}c + Nd$. Thus, the resulting system of linear inequalities is described by the conditions

$$\begin{aligned} \underline{M}c + Nd &> 0, \\ c &> 0, \\ d &> 0, \end{aligned} \quad (8)$$

which allows searching for c and d efficiently. This is especially important for large-scale network systems, for which criteria based on computing principal minors are too costly. Note that the actual design parameters $a_{i_{aj}}$ of the j -th *active node* can be obtained from d by using the relation $a_{i_{aj}} = \frac{d_j}{c_{i_{aj}}} + \underline{a}_{i_{aj}}$. We complete our discussion giving a sufficient condition for the existence of a non-empty solution set for the linear inequalities (8).

Proposition 3: The solution set of (8) is non-empty if there is at least one *active node* in every direct cycle of the graph describing the network system. \diamond

Remark 2: Proposition 3 provides a sufficient condition in the sense that even if there are directed cycles without *active nodes*, the condition of Proposition 1 can still be satisfied by the original system parameters without any adjustment. \diamond

Remark 3: One can replace the first inequality of (8) with

$$\underline{M}c + Nd \geq \sigma > 0. \quad (9)$$

This yields $\dot{V} \leq -\phi^\top(y)\sigma \leq 0$, which allows creating a margin of supply rate specified by $\sigma > 0$. \diamond

III. SYSTEMS WITH NON-QUADRATIC SUPPLY RATES

The systems discussed in Section II have dissipation inequalities with quadratic supply rates. The advantage of this formulation is that each term in the dissipation inequality is one-to-one related to a node subsystem, which allows carrying out analysis and design via the underlying directed graph. This, however, puts restrictions that do not allow many common nonlinear control design techniques, e.g. the use of nonlinear damping. Therefore we start this section by considering a more general class of network systems with dissipation inequalities

$$\dot{V} \leq -\alpha_i(y_i) + \sum_{j=1}^{i-1} \beta_{ji}(u_{ji}) + \sum_{j=i+1}^n \beta_{ji}(u_{ji}), \quad (10)$$

where $\alpha_i(\cdot)$ and $\beta_{ji}(\cdot)$ are positive definite functions. In this case we can write the overall dissipation inequality as

$$\dot{V} \leq -I^\top M_\phi(y)c = -I^\top w(y), \quad (11)$$

where

$$M_\phi(y) = \begin{bmatrix} \alpha_1(y_1) & -\beta_{12}(y_1) & \cdots & -\beta_{1n}(y_1) \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ -\beta_{n1}(y_n) & -\beta_{n2}(y_n) & \cdots & \alpha_n(y_n) \end{bmatrix} \quad (12)$$

and $W(y) = I^\top w(y)$. Note that $w(y)$ is not unique for each given $W(y)$. Without loss of generality, assume that each element of $w(y)$ is a positive definite function. Then one can use the following proposition to check whether the dissipation inequality (2) holds for some scaling coefficients.

Proposition 4: There exists a vector of scaling coefficients $c > 0$ such that V satisfies the dissipation inequality (2) if both of the conditions below are satisfied.

- 1) $M_\phi(y)$ is a non-singular M -matrix for all $y \neq 0$.
- 2) There exists $w(y)$ such that $M_\phi^{-1}(y)w(y)$ is independent of y .

◇

Remark 4: Proposition 1 is a special case of Proposition 4 in the sense that $M_\phi(y) = \Phi(y)M$ and $w(y) = \Phi(y)\sigma$, where $\Phi(y) = \text{diag}(\phi(y))$ and $\sigma > 0$. Given that the condition of Proposition 1 is satisfied, that is, M is a non-singular M -matrix, considering $M_\phi(y)c = w(y)$ yields

$$c = M^{-1}\Phi^{-1}(y)\Phi(y)\sigma = M^{-1}\sigma > 0, \quad (13)$$

by invoking condition 4) of Theorem 1, which guarantees the existence of positive constant scaling coefficients such that the dissipation inequality (2) holds. ◇

Remark 5: It is not necessary to require the scaling coefficients c to be independent of y . This restriction can be relaxed by using the so-called *state-dependent scaling* technique, see e.g. [12]. ◇

In general, it is not always possible to find a $w(y)$ such that $M_\phi^{-1}(y)w(y)$ is independent of y . Even if this is possible, it requires to compute $M_\phi^{-1}(y)$, which is not desirable when dealing with large-scale systems. In what follows, we focus on a class of systems such that condition 2) of Proposition 4 can be naturally satisfied. One way to do so is to augment $\phi(y)$ in Section II with non-quadratic positive definite terms. For example, consider a two-node system with dissipation inequalities

$$\begin{aligned} \dot{V}_1 &\leq -a_1y_1^2 + b_{21}y_2^2 - a_3y_1^4, \\ \dot{V}_2 &\leq b_{12}y_1^2 - a_2y_2^2 + b_{32}y_1^4. \end{aligned} \quad (14)$$

Obviously, the selection $\phi(y) = [y_1^2, y_2^2]^\top$ in the spirit of Section II does not work for this system since it does not take the y_1^4 -terms into account. Instead, we can augment $\phi(y)$ with an extra positive definite term, say, y_1^4 , and define the augmented vector as $\hat{\phi}(y) = [y_1^2, y_2^2, y_1^4]^\top$. Compared to the case in Section II, in which each term of the supply rate is one-to-one related to a node, now y_1^2 and y_1^4 are both related to the same node, node Σ_1 . In other words, node Σ_1 has two

supply rate basis functions (basis functions for short), y_1^2 and y_1^4 , while node Σ_2 has only one basis function y_2^2 .

To generalize this idea, suppose that there are \hat{n} basis functions in total and define $\hat{\phi}(y) = [\varphi_1^1(y_1), \varphi_2^1(y_2), \dots, \varphi_n^1(y_n), \varphi_1^2(y_1), \varphi_2^2(y_2), \dots, \varphi_n^2(y_n), \dots]^\top \in \mathbb{R}^{\hat{n}}$, where $\phi_i^j(y_i)$ is the j -th basis function of node Σ_i . Under this definition of $\phi(y)$, the overall dissipation inequality can be written as

$$\dot{V} \leq -\hat{\phi}^\top \bar{M}c, \quad (15)$$

where \bar{M} is defined such that $-\hat{\phi}^\top(\bar{M})_i$ is the supply rate of the dissipation inequality of node Σ_i . It is not difficult to note that the $n \times n$ leading principal submatrix of \bar{M} is exactly the matrix M in Section II: in other words \bar{M} is augmented from M . For example, we have

$$\bar{M} = \begin{bmatrix} a_1 & -b_{12} \\ -b_{21} & a_2 \\ a_3 & -b_{32} \end{bmatrix}, \quad (16)$$

for the two-node system with dissipation inequalities (14). Since \bar{M} is not a Z -matrix, if we want to use the results established in Section II, it is better to restore the one-to-one relation between each basis function and each matrix dimension by augmenting \bar{M} so that we have a Z -matrix to analyze. From a graph theoretic perspective, this requires adding some virtual vertices in the underlying directed graph so that the total number of vertices is the same as the basis functions rather than the number of nodes/dissipation inequalities. To this end, rewrite the overall dissipation inequality as

$$\dot{V} \leq -\hat{\phi}^\top \hat{M}\hat{c}, \quad (17)$$

where \hat{M} is an $\hat{n} \times \hat{n}$ Z -matrix defined by keeping the $n \times n$ leading principal submatrix of \bar{M} unchanged, moving the positive element (only one per row) of the $(n+1)$ -th to \hat{n} -th row to the diagonal entry of the same row, and filling the empty entries with zeros, and $\hat{c} \in \mathbb{R}^{\hat{n}}$ is augmented from c by repeating the corresponding scaling coefficients at the augmented entries. Consider the previous two-node system as an example again, the augmented version of (16) is

$$\hat{M} = \begin{bmatrix} a_1 & -b_{12} & 0 \\ -b_{21} & a_2 & 0 \\ 0 & -b_{32} & a_3 \end{bmatrix}. \quad (18)$$

Since the third column of \hat{M} is coming from the first dissipation inequality, which should be multiplied by c_1 , we also need to define $\hat{c} = [c_1, c_2, c_3]^\top$ with the constraint $c_1 = c_3$ or in general

$$L\hat{c} = 0, \quad (19)$$

where in this example $L = [1, 0, -1]$.

To derive the counterpart of Proposition 1 we have to first answer two questions: 1) how to determine whether \hat{M} is a non-singular M -matrix by checking the original matrix M and 2) how to use condition 5) of Theorem 1 considering the additional constraint (19). The answer to the first question is given by the result below.

Lemma 1: \hat{M} is a non-singular M -matrix if and only if its $n \times n$ leading principal submatrix is a non-singular M -matrix. \diamond

To answer the second question we need first to clarify the problem we are trying to deal with. Condition 5) of Theorem 1 guarantees the existence of $\hat{c} > 0$ such that $\hat{M}\hat{c} > 0$ if \hat{M} is a non-singular M -matrix, which, however, is not sufficient in this case as there is the additional constraint (19) on \hat{c} . Thus, we need to add an additional condition to \hat{M} such that \hat{c} also satisfies the constraint (19). This leads to the following result.

Lemma 2: Consider the M -matrix \hat{M} . There exists $\hat{c} > 0$ such that $\hat{M}\hat{c} > 0$ and $L\hat{c} = 0$ if and only if there exists $\sigma > 0$ such that σ is in the kernel of $L\hat{M}^{-1}$. \diamond

Having proved Lemma 1 and Lemma 2 we can proceed to give a criterion for the existence of scaling coefficients such that the dissipation inequality (2) holds.

Proposition 5: There exists a vector of scaling coefficients $c > 0$ such that V satisfies the dissipation inequality (2) if both of the conditions below are satisfied.

- 1) The $n \times n$ leading principal submatrix of \hat{M} is a non-singular M -matrix.
- 2) There exists $\sigma > 0$ such that σ is in the kernel of $L\hat{M}^{-1}$. \diamond

Remark 6: Similarly to Proposition 1, Proposition 5 is also a special case of Proposition 4, with a different vector of basis functions $\hat{\phi}(y)$. \diamond

Proposition 5 reveals the fact that the $n \times n$ leading principal submatrix of \hat{M} plays an important role. It is easier to understand this from a graph perspective. Since all the augmented columns in \hat{M} have only one non-zero element on the diagonal entries, the corresponding vertices in the underlying directed graph have only outgoing edges but no incoming edges (see Fig. 2), which guarantees that no directed cycle contains these augmented vertices. In other words, the augmentation of the graph does not create new directed cycles and all directed cycles in the graph are specified by the $n \times n$ leading principal submatrix of \hat{M} .

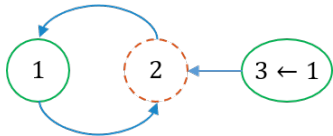


Fig. 2. The underlying directed graph specified by (18). Note that “ $3 \leftarrow 1$ ” means that Vertex 3 is an augmented vertex which originates from node Σ_1 .

Before moving on to make an argument based on *active nodes*, note that now each node can have more than one basis function, and therefore we need to slightly extend Definition 3, allowing i_a to be the indices of all vertices (including the augmented vertices) in the underlying directed graph that originate from this *active node*. For instance, in

the underlying directed graph of the two-node example, both Vertex 1 and Vertex 3 originates from node Σ_1 , and thus both a_1 and a_3 are adjustable if node Σ_1 is an *active node*. Having clarified this, we are ready to see how to enforce the dissipation inequality (2) with *active nodes*.

Proposition 6: There exist a selection of a_{i_a} and a vector of scaling coefficients $c > 0$ such that V satisfies the dissipation inequality (2) if both of the conditions below are satisfied.

- 1) Every directed cycle of the underlying directed graph describing the network contains at least one vertex that originates from an *active node*.
- 2) Every augmented vertex originates from an *active node*. \diamond

Remark 7: It is not difficult to see that condition 1) of Proposition 6 allows enforcing condition 1) of Proposition 5 and condition 2) of Proposition 6 allows enforcing condition 2) of Proposition 5. In this sense, Proposition 5 and Proposition 6 view the same fact from an analysis perspective and a design perspective, respectively. Similar remark also applies to Proposition 1 and Proposition 3. \diamond

It should also be noted that the design of the adjustable parameters a_{i_a} can also be re-formulated into the solution of a system of linear inequalities, the feasibility of which is given by Proposition 6. Although, as we have discussed, the conditions of feasibility are different, the re-formulation of this problem is essentially the same as in Section II, hence it is not repeated.

IV. A DESIGN EXAMPLE

In this section we demonstrate the convenience brought by the idea of *active nodes* in control design for network systems. Consider the three-node nonlinear network system given by

$$\begin{aligned} \Sigma_1 : \dot{y}_1 &= y_3 + u, \\ \Sigma_2 : \dot{y}_2 &= y_1 - y_2 + y_1^2, \\ \Sigma_3 : \dot{y}_3 &= y_2 - y_3, \end{aligned} \quad (20)$$

and a controller given by

$$u = -k_1 y_1 - k_2 y_1^3, \quad (21)$$

where $k_1 > 0$ and $k_2 > 0$ are adjustable parameters. The task is to regulate y_1 , y_2 and y_3 to 0. First, we need to find the dissipation inequality for each node. Defining $V_i = \frac{1}{2}y_i^2$, for $1 \leq i \leq 3$ taking the time derivative along the trajectories of the system, and invoking Young’s inequality yields the dissipation inequalities $\dot{V}_1 \leq -(k_1 - \frac{1}{2})y_1^2 + \frac{1}{2}y_3^2 - k_2y_1^4$, $\dot{V}_2 \leq y_1^2 - \frac{1}{2}y_2^2 + y_1^4$, and $\dot{V}_3 \leq \frac{1}{2}y_2^2 - \frac{1}{2}y_3^2$. Define $\hat{\phi}(y) = [y_1^2, y_2^2, y_3^2, y_1^4]^T$ and note that

$$\hat{M}(y) = \begin{bmatrix} k_1 - \frac{1}{2} & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & k_2 \end{bmatrix}. \quad (22)$$

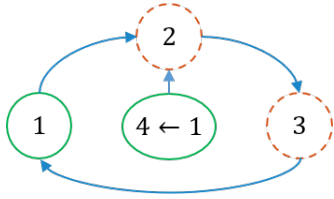


Fig. 3. The underlying directed graph specified by (22).

Note now that node Σ_1 is an *active node*, and as shown by Fig. 3 the conditions in Proposition 6 are satisfied. Therefore we can conclude that for some selection of k_1 and k_2 , there exist positive constants c_1, c_2, c_3 and $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ such that $V = c_1V_1 + c_2V_2 + c_3V_3$ satisfies the dissipation inequality

$$\dot{V} \leq -\sigma_1 y_1^2 - \sigma_2 y_2^2 - \sigma_3 y_3^2 - \sigma_4 y_1^4 \leq 0. \quad (23)$$

Instead of computing k_1 and k_2 explicitly, we take a shortcut by using the adaptive controller given by

$$\hat{k}_1 = \gamma_1 y_1^2, \quad \hat{k}_2 = \gamma_2 y_1^4, \quad u = -\hat{k}_1 y_1 - \hat{k}_2 y_1^3, \quad (24)$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$. Note that replacing controller (21) with controller (24) and re-defining $V_1 = \frac{1}{2}y_1^2 + \frac{1}{2\gamma_1}(k_1 - \hat{k}_1)^2 + \frac{1}{2\gamma_2}(k_2 - \hat{k}_2)^2$ does not change the overall dissipation inequality (23). Therefore by standard stability analysis we can conclude that all closed-loop signals are bounded and y_1, y_2, y_3 converge to 0 asymptotically, as shown in Fig. 4.

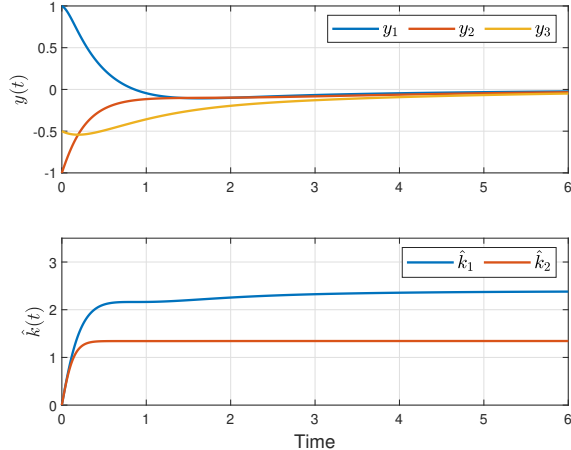


Fig. 4. Time histories of state variables under the conditions $y_1(0) = 1, y_2(0) = -1, y_3(0) = -0.5, \hat{k}_1(0) = \hat{k}_2(0) = 0, \gamma_1 = \gamma_2 = 10$.

Note that by using the adaptive controller (24) we augment the state of Σ_1 with \hat{k}_1 and \hat{k}_2 , and this subsystem is not *input-to-state stable*. The proposed method, unlike classical method based on *input-to-state stability*, is still valid for this system as it only exploits the algebraic properties of the dissipation inequalities. From this example we can see that by checking the locations of *active nodes* in the graph one can easily check the feasibility of the controller design problem without computation and even design the controller without computation using adaptive control techniques.

V. CONCLUSIONS AND FUTURE WORK

This paper proposes an approach based on dissipation inequalities for analysis and design of network systems. First, network systems with dissipation inequalities with quadratic supply rates are investigated. A matrix-based criterion is discussed for system analysis, and a graph-based criterion is proposed for the *allocation* of the distributed controllers. The admissible set for the design parameters is characterized via a system of linear inequalities, which allows efficient search and computation of these parameters. Then, the same idea is extended to network systems with non-quadratic supply rates by augmenting the underlying directed graph with respect to the so-called supply rate basis functions, for which a matrix-based criterion for analysis and a graph-based criterion for design have been discussed. Finally, a design example to show how to exploit the idea of *active nodes* to design a controller for a network system without computation is presented.

Future work will consider the case in which some nodes are *partially active*, that is, only some of the design parameters related to a node are adjustable and some of these adjustable parameters are not only lower-bounded as discussed in this paper, but also upper-bounded. Also, the algorithm for efficient *active node* placement, that is, to use a minimum number of *active nodes* for control synthesis will be further investigated.

REFERENCES

- [1] A. van der Schaft, *L₂-Gain and Passivity Techniques in Nonlinear Control*. Springer, 1996.
- [2] M. Vidyasagar, *Input-Output Analysis of Large-Scale Interconnected Systems: Decomposition, Well-posedness and Stability*. Springer Berlin Heidelberg, 1981.
- [3] Z.-P. Jiang, I. M. Mareels, and Y. Wang, "A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems," *Automatica*, vol. 32, no. 8, pp. 1211–1215, 1996.
- [4] S. N. Dashkovskiy, B. S. Rüffer, and F. R. Wirth, "Small gain theorems for large scale systems and construction of ISS Lyapunov functions," *SIAM J. on Control and Optimization*, vol. 48, no. 6, pp. 4089–4118, 2010.
- [5] H. Ito, Z.-P. Jiang, S. N. Dashkovskiy, and B. S. Rüffer, "Robust stability of networks of iISS systems: construction of sum-type Lyapunov functions," *IEEE Trans. Automat. Contr.*, vol. 58, no. 5, pp. 1192–1207, 2012.
- [6] G. Scariotti, L. Praly, and A. Astolfi, "Invariance-like theorems and "lim inf" convergence properties," *IEEE Trans. Automat. Contr.*, vol. 61, no. 3, pp. 648–661, 2015.
- [7] Z.-P. Jiang and Y. Wang, "A generalization of the nonlinear small-gain theorem for large-scale complex systems," in *Proc. 7th World Congr. on Intell. Control and Automation*. IEEE, 2008, pp. 1188–1193.
- [8] T. Liu, D. J. Hill, and Z.-P. Jiang, "Lyapunov formulation of ISS cyclic-small-gain in continuous-time dynamical networks," *Automatica*, vol. 47, no. 9, pp. 2088–2093, 2011.
- [9] T. Liu, P. Zhang, and Z.-P. Jiang, "Basic stability and small-gain tools for system synthesis," in *Robust Event-Triggered Control of Nonlinear Systems*. Springer, 2020, pp. 19–40.
- [10] Z.-P. Jiang and T. Liu, "Small-gain theory for stability and control of dynamical networks: A survey," *Ann. Rev. in Control*, vol. 46, pp. 58–79, 2018.
- [11] R. J. Plemmons, "M-matrix characterizations. I-nonsingular M-matrices," *Linear Algebra and its Applications*, vol. 18, no. 2, pp. 175–188, 1977.
- [12] H. Ito, "State-dependent scaling problems and stability of interconnected iISS and ISS systems," *IEEE Trans. Automat. Contr.*, vol. 51, no. 10, pp. 1626–1643, 2006.