

Some General Solutions for Linear Bragg-Hawthorne Equation

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Abstract: Linear cases of Bragg-Hawthorne equation for steady axisymmetric incompressible ideal flows are systematically discussed. The equation is converted to a more convenient form in spherical coordinate system. A new vorticity decomposition is derived. General solutions for 12 out of 16 cases of linear equation are obtained. These solutions can be specified to gain new vortex flows, as examples demonstrate. A lot of well-known solutions like potential flow past a sphere, Hill's vortex with and without swirl, are included in these solutions as specific cases. Special relations between some solutions are also explored.

Keywords: axisymmetric flow; Bragg-Hawthorne equation; Grad-Shafranov equation; Euler equations; Beltrami flow; spherical vortex

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1. Introduction

Bragg-Hawthorne equation ^[1] (or “B-H equation” in short) plays a central role in study of axisymmetric steady flow of incompressible ideal fluids. Due to complexity of the equation however, analytically solvable cases are rare. With special assumptions/restrictions for Bernoulli function and the azimuthal velocity, the equation reduces to a linear one. Most of the known analytic solutions so far are for such linear cases.

In this paper, an attempt is made to explore the linear cases in a systematic way. First, the equation is rewritten in spherical coordinate system and the variable of polar angle is changed from θ to $\cos \theta$. This gives a new form of the equation that is more convenient to solve in most cases. Following that, series of assumptions for Bernoulli function and the azimuthal velocity that make the equation linear are listed. These assumptions lead to 16 combinations, each associated with a special linear case of the equation. Majority of these 16 cases (except for 4) are then solved, mostly by separate variable method. The results are series of general solutions with special functions and constants. Many well-known solutions, including the potential flow around a sphere, Hill's vortex with and without swirl ^{[2], [3]}, Bogoyavlenskij's solution of Beltrami flow^[4], the Fraenkel-Norbury family of vortex rings ^{[5], [6]}, etc., can be obtained from these general solutions when constants are set to certain values.

Some linear cases of the equation have close relation, so do their solutions and the related flows. For example, Hill's vortex without swirl can be considered as a uniform flow plus an extra velocity. Hill's vortices with swirl can be considered as the Beltrami spherical vortex (the eigenvector field of curl operator in spherical coordinate system, discussed in Section 7) plus a special rotation. These are discussed in detail with the stream functions and velocities.

A new decomposition of vorticity is also derived. This decomposition applies to all flows (that follow B-H equation) and helps to understand the impacts of Bernoulli function and the azimuthal velocity to vorticity and to the flows.

The rest of this paper is organized as follows. In Section 2, we briefly recall the derivation of B-H equation, especially the form in spherical coordinate system. In Section 3, series assumptions for Bernoulli function and azimuthal velocity are listed and briefly discussed. Section 4 is about the vorticity decomposition. Section 5 to Section 10 are devoted to solving different cases of the equation. Section 11 offers a summary table and some further discussion.

2. Bragg-Hawthorne equation in spherical coordinates

For incompressible ideal fluids, the steady motions are described by (steady) Euler equations, which can be written as

$$\begin{cases} \nabla \cdot \mathbf{v} = 0 \\ \mathbf{v} \times (\nabla \times \mathbf{v}) = \nabla H \end{cases} \quad (2.1)$$

$$\mathbf{v} \times (\nabla \times \mathbf{v}) = \nabla H \quad (2.2)$$

where \mathbf{v} is velocity and H is the Bernoulli function. LHS (left-hand side) of (2.2) is also known as the Lamb vector, denoted as

$$\mathbf{L} \equiv \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{v} \times \boldsymbol{\omega} \quad (\boldsymbol{\omega} \equiv \nabla \times \mathbf{v} \text{ is the vorticity}).$$

For (2.2) to stand, $\nabla \times \mathbf{L} = 0$ is required (and is sufficient, assuming the flow is in a simply connected domain). A (non-irrotational) flow meeting this requirement is a generalized Beltrami flow^[7], (or is further a Beltrami flow if $\mathbf{L} = 0$ in the whole domain^[8]).

In axisymmetric case, it is possible to define the Stokes stream function ψ and an independent function C so that

$$v_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, v_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}, v_\varphi = \frac{C}{r \sin \theta} \quad (2.3)$$

in spherical coordinate system (r, θ, φ) , or

$$v_\rho = -\frac{1}{\rho} \frac{\partial \psi}{\partial z}, v_\phi = \frac{c}{\rho}, v_z = \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} \quad (2.4)$$

in cylindrical coordinate system (ρ, ϕ, z) .

With the stream function, (2.1) is automatically satisfied. Taking the spherical coordinate case as example, as all derivatives with respect to azimuthal angle φ vanish, vorticity is

$$\boldsymbol{\omega} = \left(\frac{1}{r^2 \sin \theta} \frac{\partial C}{\partial \theta} \right) \mathbf{e}_r + \left(-\frac{1}{r \sin \theta} \frac{\partial C}{\partial r} \right) \mathbf{e}_\theta + \left(-\frac{1}{r \sin \theta} \frac{\partial^2 \psi}{\partial r^2} + \frac{\cos \theta}{r^3 \sin^2 \theta} \frac{\partial \psi}{\partial \theta} - \frac{1}{r^3 \sin \theta} \frac{\partial^2 \psi}{\partial \theta^2} \right) \mathbf{e}_\varphi \quad (2.5)$$

and the three components of Lamb vector are

$$\left\{ \begin{array}{l} L_r = \left(-\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \right) \left(-\frac{1}{r \sin \theta} \frac{\partial^2 \psi}{\partial r^2} + \frac{\cos \theta}{r^3 \sin^2 \theta} \frac{\partial \psi}{\partial \theta} - \frac{1}{r^3 \sin \theta} \frac{\partial^2 \psi}{\partial \theta^2} \right) - \left(\frac{c}{r \sin \theta} \right) \left(-\frac{1}{r \sin \theta} \frac{\partial C}{\partial r} \right) \quad (2.6) \\ L_\theta = \frac{c}{r \sin \theta} \left(\frac{1}{r^2 \sin \theta} \frac{\partial C}{\partial \theta} \right) - \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \left(-\frac{1}{r \sin \theta} \frac{\partial^2 \psi}{\partial r^2} + \frac{\cos \theta}{r^3 \sin^2 \theta} \frac{\partial \psi}{\partial \theta} - \frac{1}{r^3 \sin \theta} \frac{\partial^2 \psi}{\partial \theta^2} \right) \quad (2.7) \\ L_\varphi = \frac{1}{r^3 \sin^2 \theta} \left(\frac{\partial \psi}{\partial r} \frac{\partial C}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \frac{\partial C}{\partial r} \right). \quad (2.8) \end{array} \right.$$

As the flow is steady, each of H and C is a function of ψ only^[9]. Thus, we have the following ‘‘chain rules’’ for $H(\psi)$ and $C(\psi)$:

$$\frac{\partial H}{\partial r} = \frac{dH}{d\psi} \frac{\partial \psi}{\partial r}, \frac{\partial H}{\partial \theta} = \frac{dH}{d\psi} \frac{\partial \psi}{\partial \theta}, \frac{\partial C}{\partial r} = \frac{dC}{d\psi} \frac{\partial \psi}{\partial r}, \frac{\partial C}{\partial \theta} = \frac{dC}{d\psi} \frac{\partial \psi}{\partial \theta}. \quad (2.9)$$

In this case, the Lamb vector components in (2.6) ~ (2.8) simplify to

$$\left\{ \begin{array}{l} L_r = \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial^2 \psi}{\partial r^2} - \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + C \frac{dC}{d\psi} \right) \frac{\partial \psi}{\partial r} \quad (2.10) \end{array} \right.$$

$$\left\{ \begin{array}{l} L_\theta = \frac{1}{r^3 \sin^2 \theta} \left(\frac{\partial^2 \psi}{\partial r^2} - \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + C \frac{dC}{d\psi} \right) \frac{\partial \psi}{\partial \theta} \quad (2.11) \end{array} \right.$$

$$\left\{ \begin{array}{l} L_\varphi = 0 \quad (2.12) \end{array} \right.$$

Equation (2.2), if rewritten with (2.9) and (2.10) ~ (2.12), becomes an identity in \mathbf{e}_φ direction (both sides vanish) and two identical scalar equations in \mathbf{e}_r and \mathbf{e}_θ direction, both read

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = r^2 \sin^2 \theta \frac{dH}{d\psi} - C \frac{dC}{d\psi} \quad (2.13)$$

This is the Bragg-Hawthorne equation ^[1] (in spherical coordinate system). It is also referred to as Hicks equation ^[10] or Squire-Long equation ^{[11], [12]}. The Grad-Shafranov equation in ideal magnetohydrodynamics also has the same form ^{[13], [14]}.

Changing the variable from θ to $\cos \theta$, we have $\frac{\partial \psi}{\partial \theta} = -\sin \theta \frac{\partial \psi}{\partial (\cos \theta)}$ and $\frac{\partial^2 \psi}{\partial \theta^2} = \sin^2 \theta \frac{\partial^2 \psi}{\partial (\cos \theta)^2} - \cos \theta \frac{\partial \psi}{\partial (\cos \theta)}$. (2.13) becomes

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 \psi}{\partial (\cos \theta)^2} = r^2 \sin^2 \theta \frac{dH}{d\psi} - C \frac{dC}{d\psi} \quad (2.14)$$

Compared to (2.13), LHS of (2.14) only has 2 terms. Considering $\sin^2 \theta = 1 - \cos^2 \theta$, (2.14) has a simple form with respect to the variable $\cos \theta$. This is convenient. A lot of solutions in this paper are based on this form of the equation.

The above derivation of the equation also applies to the case in cylindrical coordinate system (ρ, ϕ, z) . In that, equation (2.13) or (2.14) takes the well-known form as

$$\frac{\partial^2 \psi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = \rho^2 \frac{dH}{d\psi} - C \frac{dC}{d\psi} \quad (2.15)$$

This equation, either in the form of (2.13), (2.14) or (2.15), is the governing equation for ideal incompressible axisymmetric steady flows. For a specific case of the functions $H(\psi)$ and $C(\psi)$, if we can find the stream function ψ satisfying the equation, velocity can be calculated and Euler equation (2.2) is solved.

In general, however, this could be difficult. Mathematically, $H(\psi)$ and $C(\psi)$ can be arbitrary (smooth) functions of ψ . They can make the equation complicated. Nevertheless, when $H(\psi)$ and $C(\psi)$ are in certain forms, the equation can be linear and solvable.

3. Linear cases of Bragg-Hawthorne equation

LHS of the equation, either in (2.13), (2.14) or (2.15), is linear with respect to the stream function. If the two terms on the RHS are also linear, the equation is linear.

Assuming $H_0, \psi_0, \lambda, \lambda_1, \lambda_2, a$ and a_0, a_1, a_2, a_3 are constants, for the first term on RHS to be linear, we have

$$(H_1): H = H_0, \text{ (then } r^2 \sin^2 \theta \frac{dH}{d\psi} = 0, \rho^2 \frac{dH}{d\psi} = 0),$$

$$(H_2): H = H_0 + \lambda\psi, \text{ (then } r^2 \sin^2 \theta \frac{dH}{d\psi} = \lambda r^2 \sin^2 \theta, \rho^2 \frac{dH}{d\psi} = \lambda \rho^2),$$

$$(H_3): H = H_0 + \lambda\psi^2, \text{ (then } r^2 \sin^2 \theta \frac{dH}{d\psi} = 2\lambda r^2 \sin^2 \theta \psi, \rho^2 \frac{dH}{d\psi} = 2\lambda \rho^2 \psi).$$

$$(H_4): H = H_0 + \lambda_1\psi + \lambda_2\psi^2, \text{ (then } r^2 \sin^2 \theta \frac{dH}{d\psi} = 2\lambda_2 r^2 \sin^2 \theta \psi + \lambda_1 r^2 \sin^2 \theta, \\ \rho^2 \frac{dH}{d\psi} = 2\lambda_2 \rho^2 \psi + \lambda_1 \rho^2).$$

For the second term to be linear, we can have

$$(C_1): C = \alpha\psi_0, \text{ (then } C \frac{dC}{d\psi} = 0),$$

$$(C_2): C = a_1\sqrt{\psi_0 + a_2\psi}, \text{ (then } C \frac{dC}{d\psi} = \frac{1}{2}a_1^2 a_2 = \frac{1}{2}a, \text{ re-denoting } a \equiv a_1^2 a_2),$$

$$(C_3): C = \psi_0 + \alpha\psi, \text{ (then } C \frac{dC}{d\psi} = a^2\psi + \alpha\psi_0),$$

$$(C_4): C = a_1\sqrt{\psi_0^2 + a_2\psi + a_3\psi^2}, \text{ (then } C \frac{dC}{d\psi} = a_1^2 a_3 \psi + \frac{1}{2}a_1^2 a_2 = \alpha\psi + a_0, \text{ re-} \\ \text{denoting } a \equiv a_1^2 a_3, a_0 \equiv \frac{1}{2}a_1^2 a_2).$$

Mathematically, some constants in the list can be absolved or combined. But keeping them in the above format will make it convenient for discussion.

Constant ψ_0 in C_1 , C_2 and C_4 does not impact the equation, thus has no impact on the solutions. But when C is interpreted linking to azimuthal velocity, ψ_0 can bring big difference for azimuthal velocity and for the flow. For C_3 , most of the feasible cases require $\psi_0 = 0$. But to maximize generality, ψ_0 is kept here as a constant, and the different impacts of $\psi_0 = 0$ and $\psi_0 \neq 0$ will be discussed specifically.

The cases in the list are not always independent. For example, H_1 can be considered as a special case of H_2 (when $\lambda = 0$); H_4 is a “combination” of H_2 and H_3 . But the equations and solutions for these cases are so different and are worthy to be discussed separately. For this consideration, they are listed as separated cases.

C_3 and C_4 actually give the same form of equation (with different denotation for the constants). So, they would share the same stream function as mathematical solution of the equation. However, when the same solution is respectively interpreted for C_3 and C_4 , the flows

can be significantly different (on velocity, vorticity and physical feasibility, etc.). For completeness consideration, they are treated as two separated cases in the list.

In a more general view, when the two terms on RHS of the linear equation are expanded, they can become up to four terms, which can be written on RHS of the following “model equation”:

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 \psi}{\partial (\cos \theta)^2} = \Lambda_1 r^2 \sin^2 \theta \psi + \Lambda_0 r^2 \sin^2 \theta - A_1 \psi - A_0 \quad (3.1)$$

or in cylindrical coordinates

$$\frac{\partial^2 \psi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = \Lambda_1 \rho^2 \psi + \Lambda_0 \rho^2 - A_1 \psi - A_0 \quad (3.2)$$

Λ_1 , Λ_0 , A_1 and A_0 here represent certain form of the constants in the list. Different cases of function H and C are associated with presentation/absence of these 4 terms.

Obviously, with $H_1 \sim H_4$ and $C_1 \sim C_4$, there are 16 combinations, each making (2.13), (2.14) or (2.15) a linear equation of ψ (i.e., each is related to a combination of the Λ_1 , Λ_0 , A_1 and A_0 terms in the model equation). For convenience, we will denote these 16 combinations as $H_m C_n$ ($m, n = 1, 2, 3, 4$) in the rest of this paper.

4. Vorticity decomposition and flow properties

Before moving to solutions of particular cases, it is worthy to further investigate impacts of the two functions, H and C to vorticity and to the flow.

Recalling (2.3) and (2.4), we have $v_\varphi = \frac{C}{r \sin \theta}$ (or $v_\phi = \frac{C}{\rho}$ in cylindrical coordinates).

Function C is indeed the linear azimuthal velocity multiplied by distance to the z -axis. $2\pi C$ is often considered as circulation along a circle around z -axis. From another angle of view, C can also be considered as the “azimuthal velocity moment” with respect to z -axis. As mass density is a constant in incompressible fluid, C is also representing angular momentum density of the fluid (with respect to z -axis).

In any of these considerations, an assumption (or a restriction) on C is basically an assumption (restriction) on azimuthal velocity related to the stream function ψ . In other words, C

indicates how the fluid is moving around the symmetric axis. It is intuitive to expect that C has close relation with vorticity of the flow.

Bernoulli function H , on the other hand, is the inverse gradient of Lamb vector, which is just the cross-product of velocity and vorticity. It is also natural to expect that H is closely related to vorticity.

With (2.3), (2.9) and (2.13), vorticity in (2.5) can be rewritten as

$$\boldsymbol{\omega} = \frac{dC}{d\psi} \left(\frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \mathbf{e}_r - \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \mathbf{e}_\theta + \frac{C}{r \sin \theta} \mathbf{e}_\varphi \right) - r \sin \theta \frac{dH}{d\psi} \mathbf{e}_\varphi$$

which is also

$$\boldsymbol{\omega} = \frac{dC}{d\psi} \mathbf{v} - r \sin \theta \frac{dH}{d\psi} \mathbf{e}_\varphi \quad (4.1)$$

Same relation exists as well in cylindrical coordinates, where we have

$$\boldsymbol{\omega} = \frac{dC}{d\psi} \mathbf{v} - \rho \frac{dH}{d\psi} \mathbf{e}_\phi \quad (4.2)$$

Either (4.1) or (4.2) indicates that vorticity $\boldsymbol{\omega}$ can be split into two portions. One is in the velocity direction, proportional to velocity with a scale factor $\frac{dC}{d\psi}$. The second portion is on the azimuthal direction, with magnitude proportional to $\frac{dH}{d\psi}$ (and to the distance to z-axis).

In such a sense, it can be considered that functions H and C construct vorticity, or more specifically, derivatives of H and C compose vorticity in the flow. $\frac{dC}{d\psi}$ decides vorticity in the direction of velocity; $\frac{dH}{d\psi}$ decides vorticity in azimuthal direction. Although these two directions in general are not orthogonal, these two components make up vorticity at each point in the flow.

(4.1) or (4.2) thus can be consider as a decomposition of the vorticity. It applies to all flows that B-H equation stands (i.e., all axisymmetric steady flows of incompressible ideal fluid).

Actually, the relations of H and C to the vorticity components are obvious and are often implied in derivation of equations (see, e.g., Batchelor [9], §7.5). But explicitly putting them in the form of (4.1) or (4.2) straightforwardly shows the impacts of H and C to vorticity and to the flow.

For convenience, we will further denote $\boldsymbol{\omega}_B \equiv \frac{dC}{d\psi} \mathbf{v}$, $\boldsymbol{\omega}_A \equiv -r \sin \theta \frac{dH}{d\psi} \mathbf{e}_\varphi$ (or $\boldsymbol{\omega}_A \equiv -\rho \frac{dH}{d\psi} \mathbf{e}_\varphi$ in cylindrical coordinates), and call $\boldsymbol{\omega}_B$ and $\boldsymbol{\omega}_A$ Beltrami vorticity and azimuthal vorticity, respectively. Total vorticity then is sum of these two vectors, i.e., $\boldsymbol{\omega} = \boldsymbol{\omega}_B + \boldsymbol{\omega}_A$.

As Beltrami vorticity $\boldsymbol{\omega}_B$ is in parallel to velocity, it does not impact Lamb vector. We can just count on the azimuthal vorticity $\boldsymbol{\omega}_A$ when calculating Lamb vector. In other words, regardless of $\boldsymbol{\omega}_B$, we always have $\mathbf{L} = \mathbf{v} \times \boldsymbol{\omega}_A = -r \sin \theta \frac{dH}{d\psi} \mathbf{v} \times \mathbf{e}_\varphi$ (or $\mathbf{L} = -\rho \frac{dH}{d\psi} \mathbf{v} \times \mathbf{e}_\varphi$ in cylindrical coordinates).

This is consistent with the fact that (in ideal incompressible axisymmetric steady flows) Lamb vector is always perpendicular to the azimuthal direction (except for points in the symmetric axis). Following this is also the corollary that Lamb vector is always coplanar with the symmetric axis in such flows.

Applying this decomposition to flows in the list in Section 3, for the H_1C_n cases, as $\frac{dH}{d\psi} = 0$, azimuthal vorticity $\boldsymbol{\omega}_A$ vanishes. Total vorticity contains solely the Beltrami part $\boldsymbol{\omega}_B$. Lamb vector thus vanishes as well. For H_1C_1 , $\boldsymbol{\omega}_B$ also vanishes, so does the total vorticity. It is hence an irrotational/potential axisymmetric flow. For H_1C_2 , $\boldsymbol{\omega} = \boldsymbol{\omega}_B = \frac{a_1 a_2}{2\sqrt{\psi_0 + a_2 \psi}} \mathbf{v}$. Vorticity is in parallel to velocity with a non-constant coefficient. This is a non-linear Beltrami flow (i.e., the velocity field is a non-linear Beltrami field). For H_1C_3 , we have $\boldsymbol{\omega} = \boldsymbol{\omega}_B = \alpha \mathbf{v}$. Vorticity is in parallel to velocity with a constant coefficient. This is linear Beltrami case. H_1C_4 on the other hand, is non-linear Beltrami flow again, as $\boldsymbol{\omega} = \boldsymbol{\omega}_B = \frac{a_1(a_2 + 2a_3\psi)}{2\sqrt{\psi_0^2 + a_2\psi + a_3\psi^2}} \mathbf{v}$.

In the H_2C_n family, azimuthal vorticity $\boldsymbol{\omega}_A = -\lambda r \sin \theta \mathbf{e}_\varphi$ (or $\boldsymbol{\omega}_A = -\lambda \rho \mathbf{e}_\varphi$ in cylindrical coordinates). It is proportional to the distance to z-axis. For H_2C_1 , $\boldsymbol{\omega}_B$ vanishes. Total vorticity is made up solely by $\boldsymbol{\omega}_A$, thus is proportional to the distance to z-axis as well. This is actually the assumption of the Fraenkel-Norbury family of vortex rings ^{[5], [6]} (including the Hill's vortex without swirl). As can be expected, the general solution of H_2C_1 will include Hill's vortex as a special case. H_2C_3 is adding the Beltrami vorticity part $\boldsymbol{\omega}_B = \alpha \mathbf{v}$ to H_2C_1 . This is actually the case of Hill's vortex with swirl, which will be discussed in Section 9. For H_2C_2 and H_2C_4 , both

Beltrami vorticity $\boldsymbol{\omega}_B$ and azimuthal vorticity $\boldsymbol{\omega}_A$ present in the total vorticity, and the flows have more complicated dependency to H and C .

In the H_3C_n and H_4C_n cases, as $\frac{dH}{d\psi}$ explicitly contains ψ , azimuthal vorticity is now directly tied with ψ (in opposite to H_1C_n or H_2C_n where ψ is not explicitly involved in the expression of azimuthal vorticity). The equation is still linear, but this azimuthal vorticity brings an extra term with the unknown function ψ to the equation and makes the solutions complicated. These cases will be further discussed in Section 10.

As an aside, the second term on RHS of the B-H equation, $C \frac{dC}{d\psi}$, has a notable property of symmetric. This can also be analyzed by applying the vorticity decomposition.

Mathematically, function C can change to the opposite sign without impacting the equation. If ψ is a solution of the B-H equation related to $C(\psi)$, ψ is also a solution of the equation related to $-C(\psi)$. Physically, this implies that flows governed by B-H equation are “two-way flows”. The azimuthal velocity $v_\varphi = \frac{C}{r \sin \theta}$ or $v_\phi = \frac{C}{\rho}$ can change to the opposite direction and the flow still satisfies the same equation. Equivalently speaking, for each flow as a solution of B-H equation, there exists its chiral symmetric flow (with opposite azimuthal velocity) to form a pair.

From vorticity decomposition point of view, when $C(\psi)$ changes sign (and ψ remains the same), velocity

$$\boldsymbol{v} = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \boldsymbol{e}_r - \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \boldsymbol{e}_\theta + \frac{C}{r \sin \theta} \boldsymbol{e}_\varphi$$

changes chirality (i.e., from $(v_r, v_\theta, v_\varphi)$ to $(v_r, v_\theta, -v_\varphi)$), but Beltrami vorticity

$$\boldsymbol{\omega}_B = \frac{dC}{d\psi} \left(\frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \boldsymbol{e}_r - \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \boldsymbol{e}_\theta + \frac{C}{r \sin \theta} \boldsymbol{e}_\varphi \right)$$

undergoes a half-round rotation along \boldsymbol{e}_φ direction without changing chirality (i.e., from $(\omega_{Br}, \omega_{B\theta}, \omega_{B\varphi})$ to $(-\omega_{Br}, -\omega_{B\theta}, \omega_{B\varphi})$). At the same time, azimuthal vorticity $\boldsymbol{\omega}_A = -r \sin \theta \frac{dH}{d\psi} \boldsymbol{e}_\varphi$ remains unchanged. Total vorticity $\boldsymbol{\omega} = \boldsymbol{\omega}_B + \boldsymbol{\omega}_A$ thus changes to a new vector that in general is neither rotational symmetric nor chiral symmetric to the original vorticity before C changes sign. Despite the change on vorticity however, Lamb vector remains the same.

5. Axisymmetric potential flow (H₁C₁)

Combination (H₁C₁) is the simplest one in the list. As discussed in Section 4, all flows in this case are axisymmetric potential flow.

As both terms on RHS of B-H equation vanish, (2.14) becomes

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 \psi}{\partial (\cos \theta)^2} = 0 \quad (5.1)$$

Let $\psi = R(r)\theta(\cos \theta)$ and apply variable separation method. With the separation constant denoted as $n(n + 1)$, the two separated questions are

$$r^2 \ddot{R} - n(n + 1)R = 0 \quad (5.2)$$

$$\sin^2 \theta \ddot{\theta} + n(n + 1)\theta = 0 \quad (5.3)$$

Solution of (5.2) is $R = k_1 r^{n+1} + k_2 r^{-n}$.

k_1, k_2, n here, as well as $k_3, k_4, k_5, k_6, k_0, k$ and K_n in expressions later in this paper are independent constants. Mathematically, they can be arbitrary real numbers, or theoretically even be imaginary numbers in some cases. However, to have the stream function and velocity physically meaningful, restrictions may apply to them.

In this paper, we will consider a flow (from a solution of the B-H equation) with no singular velocity (i.e., no infinite nor discontinuous velocity) in a domain as “physically feasible”. When the domain is not the whole space, the boundary conditions can be provided by solid boundary, or by other flows outside the domain. In the latter case, the flows inside and outside the domain should have zero normal velocity and continuous tangential velocity at the boundary. Equivalently, this requires inside and outside flows to have constant (normally zero) stream function and continuous stream function gradient on the boundary.

For (5.3), θ is function of $\cos \theta$, and derivative $\ddot{\theta}$ is with respect to $\cos \theta$. If we replace the unknown function $\theta(\cos \theta)$ with $\sin \theta T(\cos \theta)$ and apply the identities $\frac{d(\sin \theta)}{d(\cos \theta)} = -\frac{\cos \theta}{\sin \theta}$, $\frac{d^2(\sin \theta)}{d(\cos \theta)^2} = -\frac{1}{\sin^3 \theta}$, (5.3) becomes $\sin^2 \theta \ddot{T} - 2 \cos \theta \dot{T} + \left[n(n + 1) - \frac{1}{\sin^2 \theta} \right] T = 0$. This is an associated Legendre equation of $T(\cos \theta)$, thus is solved by linear combination of the associated Legendre function of the first and second kind, $P_n^1(\cos \theta)$ and $Q_n^1(\cos \theta)$. Solution of (5.3) hence can be written as

$$\theta(\cos \theta) = \sin \theta [k_3 P_n^1(\cos \theta) + k_4 Q_n^1(\cos \theta)]. \quad (5.4)$$

Combining it with solution of (5.2) gives the general solution of (5.1) as

$$\psi = (k_1 r^{n+1} + k_2 r^{-n}) \sin \theta [k_3 P_n^1(\cos \theta) + k_4 Q_n^1(\cos \theta)]. \quad (5.5)$$

With special settings for the constants, stream function ψ can be in simple forms. For example, when $n = 1$, $k_2 = 0$, $k_3 = -1$, $k_4 = 0$, (5.5) gives $\psi = k_1 r^2 \sin^2 \theta$. This is a uniform flow with velocity $U = 2k_1$ in z -direction.

If k_1 and k_2 are both non-zero, the radial portion in (5.5), $R(r) = k_1 r^{n+1} + k_2 r^{-n}$ has a zero at $r = r_0 = \left(-\frac{k_2}{k_1}\right)^{\frac{1}{2n+1}}$. In the case that such r_0 is a positive real number, stream function ψ is zero at surface of the sphere $r = r_0$. The flow outside the sphere is a potential flow around that sphere (while the flow inside the sphere is singular at $r = 0$, thus is not physically feasible.)

In this case, if we further have $n = 1$, $k_4 = 0$, we have $\psi = \frac{U}{2} r^2 \left(1 - \frac{a^3}{r^3}\right) \sin^2 \theta$ (where $U = -2k_1 k_3$, $a = r_0 = \left(-\frac{k_2}{k_1}\right)^{1/3}$). This is the well-known potential flow past the sphere $r = a$.

When $n > 1$, the associated Legendre functions in (5.5) can give more complicated potential flows around the sphere. Some examples can be found in Section 7 (see equation (7.8b) and Fig. 5).

As a property of the C_1 flows (i.e., flows in any $H_m C_1$ case), as long as function C is a constant, its value does not impact the equation, thus will not impact the solution. Azimuthal velocity $v_\varphi = \frac{C}{r \sin \theta}$ hence is independent to the solution. In principle, velocity (v_r, v_θ) in the meridian plane derived from solution of an $H_m C_1$ equation can be with any azimuthal velocity $v_\varphi = \frac{C}{r \sin \theta}$ as long as C is a constant. When $C \neq 0$, this v_φ introduces a circular movement with uniform angular momentum density with respect to z -axis. This is an ‘‘irrotational rotation’’ (as it does not impact the vorticity). In this sense, constant C brings a rotation to $H_m C_1$ flows without impacting vorticity nor impacting velocity in the meridian plane.

Such a rotation has singular velocity at z -axis, though. It is physically feasible only for special cases (e.g., in a domain that is not overlapping z -axis. An example can be found in Section 8, which is the swirled case of Fraenkel-Norbury solutions.)

Case H₁C₁ can also be solved in cylindrical coordinates. Equation (2.15) for H₁C₁ is

$$\frac{\partial^2 \psi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = 0 \quad (5.6)$$

Let $\psi(\rho, z) = P(\rho)Z(z)$, (5.6) can be separated to two equations as $\rho \ddot{P} - \dot{P} + k\rho P = 0$ and $\ddot{Z} - kZ = 0$ (k is the separation constant). The latter one is solved by $Z(z) = k_3 e^{\sqrt{k}z} + k_4 e^{-\sqrt{k}z}$. The former one can be converted to a Bessel equation of $F(x)$ (of order 1) by further replacing $P(\rho)$ by $xF(x)$ (with $x = \sqrt{k}\rho$), thus is solved by linear combination of the Bessel function of the first and second kind, J_1 and Y_1 . Overall, the general solution of (5.6) is

$$\psi = [k_1 \rho J_1(\sqrt{k}\rho) + k_2 \rho Y_1(\sqrt{k}\rho)] (k_3 e^{\sqrt{k}z} + k_4 e^{-\sqrt{k}z}) \quad (5.7)$$

When $k > 0$, the section related to z is unbounded on one side (or both sides, depending on k_3 and k_4) of z when $z \rightarrow \pm\infty$, thus the feasible flow only can exist in a domain excluding upper or lower “end” of z -axis. When $k < 0$, the flow is periodic (and bounded) along the whole z -axis.

In passing, $\psi = k_1 \rho^2 + k_2 z + k_3$ is also a solution of (5.6). This is not from variable separation method, thus is not included in (5.7). When $k_2 = 0$ and $k_3 = 0$, it converts to $\psi = k_1 \rho^2$, which is equivalent to the special form of (5.5), i.e., the uniform flow $\psi = k_1 r^2 \sin^2 \theta$.

6. Axisymmetric non-linear Beltrami flow (H₁C₂)

The solution of H₁C₁ can be “extended” to solve H₁C₂. Taking the spherical coordinates case first, equation (2.14) for H₁C₂ is

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 \psi}{\partial (\cos \theta)^2} = -\frac{a}{2} \quad (6.1)$$

This can be considered as the homogeneous equation (5.1) plus a constant inhomogeneous term on the RHS. It thus can be solved by adding a particular solution, $\psi^* = -\frac{a}{4} r^2$ to general solution of (5.1). This gives

$$\psi = (k_1 r^{n+1} + k_2 r^{-n}) \sin \theta [k_3 P_n^1(\cos \theta) + k_4 Q_n^1(\cos \theta)] - \frac{a}{4} r^2 \quad (6.2)$$

Compared to (5.5), the particular solution ψ^* brings new features for the flow of (6.2). It actually turns the potential flow in (5.5) into a non-linear Beltrami flow.

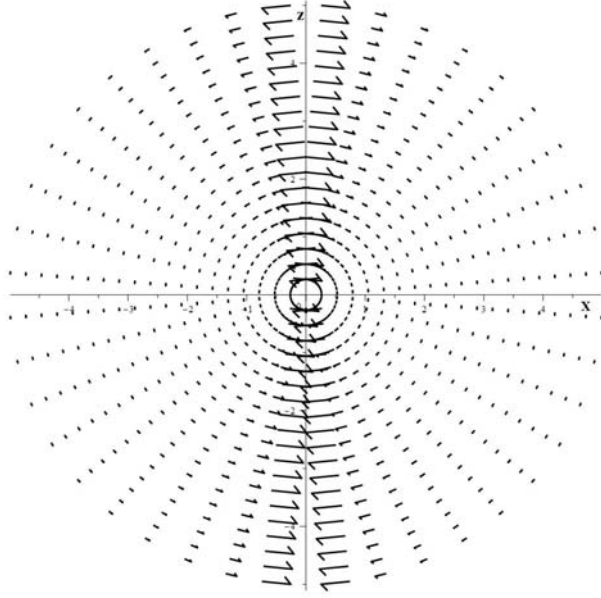


Fig. 1. v_θ^* in x-z plane brought by ψ^* for H_1C_2 flow

If we calculate the velocity from (6.2) and compare it to velocity from (5.5), ψ^* does not impact v_r , but it brings an addition $v_\theta^* = \frac{a}{2 \sin \theta}$ to the polar velocity v_θ . In the meridian plane, this v_θ^* represents a “rotation” around the origin point, with different magnitudes at different polar angles (and is singular on z-axis). Fig. 1 shows this v_θ^* in x-z plane (when $a = 1$).

The impact of ψ^* to azimuthal velocity is more complicated. As $v_\phi = \frac{a_1 \sqrt{\psi_0 + a_2 \psi}}{r \sin \theta}$ is a non-linear function of ψ , the impact of ψ^* to v_ϕ is not simply a linear addition.

Theoretically, (6.2) can be specified to various of flows depending on the constants. However, most (if not all) of them may inherit the singularity from v_θ^* thus need to be within a domain excluding z-axis. Besides, for v_ϕ to be real, it is required that $\psi_0 + a_2 \psi \geq 0$. This may add more restrictions to the flow.

As an example of the feasible H_1C_2 cases, when $k_1 = \frac{a}{2}$ and $n = 1$, $k_2 = 0$, $k_3 = -1$, $k_4 = 0$, stream function (6.2) becomes $\psi = -\frac{a}{4} r^2 \cos(2\theta)$. This is a combination of the uniform flow of (5.5) with $U = a$ in z-direction (i.e., $\psi = \frac{a}{2} r^2 \sin^2 \theta$) and the rotation of v_θ^* shown on Fig. 1 (that is brought by the particular solution $\psi^* = -\frac{a}{4} r^2$).

Velocity in this case is $v_r = a \cos \theta$, $v_\theta = \frac{a \cos(2\theta)}{2 \sin \theta}$, $v_\varphi = \frac{a_1 \sqrt{\psi_0 - \frac{a_1^2 a_2^2}{4} r^2 \cos(2\theta)}}{r \sin \theta}$ (as defined in Section 3, $a = a_1^2 a_2$, ψ_0 is a constant). A real v_φ requires $\psi_0 - \frac{a_1^2 a_2^2}{4} r^2 \cos(2\theta) \geq 0$. When $\psi_0 < 0$, this requirement is satisfied outside of the revolution-solid defined by hyperboloid $r^2 \cos(2\theta) = \frac{4\psi_0}{a_1^2 a_2^2}$. This is the domain the flow exists (as shown in Fig. 2). Velocity has no singularity in this domain. On surface of the hyperboloid, ψ is a constant, v_φ is zero and velocity is tangential to the surface.

The requirement of $\psi_0 < 0$ is critical to ensure feasibility of the flow. If $\psi_0 > 0$, the hyperboloid has two sheets. The domain (in which $\psi_0 - \frac{a_1^2 a_2^2}{4} r^2 \cos(2\theta) \geq 0$) intersects z-axis. In that z-axis segment, v_θ and v_φ are singular and the flow is not feasible.

Similar to the spherical coordinate case, H_1C_2 in cylindrical coordinates,

$$\frac{\partial^2 \psi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = -\frac{a}{2} \quad (6.3)$$

can be solved by adding a particular solution $\psi^* = -\frac{a}{4} z^2$ or $\psi^* = \frac{a}{8} \rho^2 (1 - 2 \ln \rho)$ or their combination $\psi^* = k_0 \frac{a}{8} \rho^2 (1 - 2 \ln \rho) - (1 - k_0) \frac{a}{4} z^2$ to (5.7).

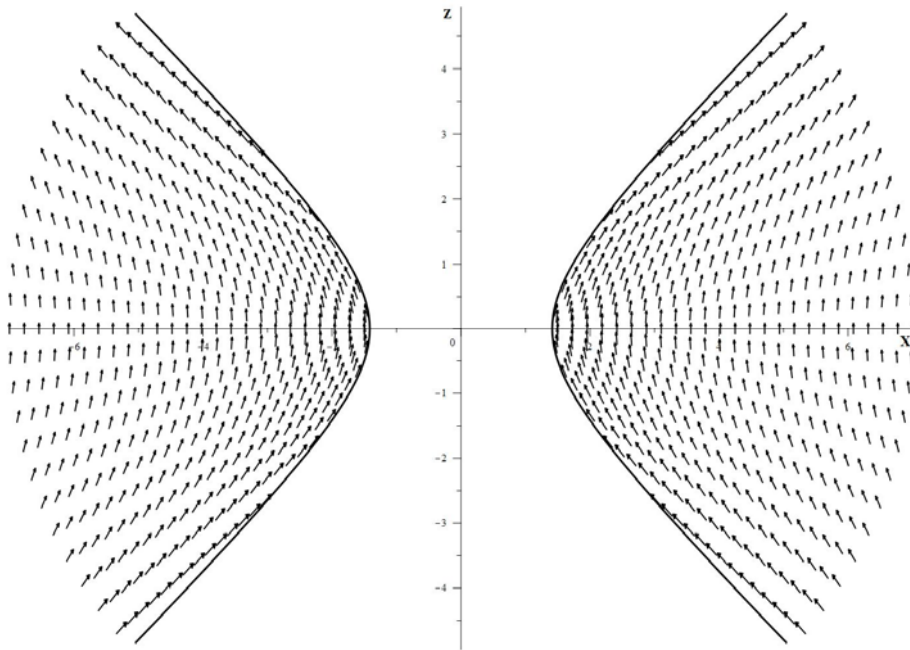


Fig. 2. An H_1C_2 flow outside a hyperboloid

7. Beltrami spherical vortex (H_1C_3)

As discussed in Section 4, for H_1C_3 , as $\boldsymbol{\omega} = a\boldsymbol{v}$, velocity field is a linear Beltrami field. Velocity in this case is eigenvector of the curl operator (and a is the eigenvalue).

In [15] or [16], a solution of velocity field for this case is given by directly solving $\boldsymbol{\omega} = \boldsymbol{v}$ with variable separation method (i.e., without employing stream function). As flows in this case have some significant properties and are important for further discussion, here we will re-solve it in stream function form.

General case of C_3 is $C = \alpha\psi + \psi_0$. For convenience, consider $\psi_0 = 0$ first. Equation (2.14) in this case is

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 \psi}{\partial (\cos \theta)^2} = -a^2 \psi \quad (7.1)$$

Let $\psi = R(r)\Theta(\cos \theta)$, (7.1) can be separated to the following 2 equations (with $n(n+1)$ being the separation constant):

$$r^2 \ddot{R} + [a^2 r^2 - n(n+1)]R = 0 \quad (7.2)$$

$$\sin^2 \theta \ddot{\Theta} + n(n+1)\Theta = 0 \quad (7.3)$$

(7.2) can be converted to a spherical Bessel equation of $F(x)$ by replacing $R(r)$ by $xF(x)$ (with $x = ar$). Thus, it is solved by linear combination of the spherical Bessel function $J_{n+1/2}(ar)$ and $Y_{n+1/2}(ar)$. (7.3) is the same as (5.3), and hence is solved by (5.4). Combining these gives the general solution of (7.1) as

$$\psi = r^{1/2} [k_1 J_{n+1/2}(ar) + k_2 Y_{n+1/2}(ar)] \sin \theta [k_3 P_n^1(\cos \theta) + k_4 Q_n^1(\cos \theta)] \quad (7.4)$$

If we set $a = 1$, $k_2 = 0$, and $k_3 = 1$, $k_4 = 0$, consider the case when n is a positive integer, and re-denote constant k_1 as $-K_n$, we have the special case of (7.4) as

$$\psi = -K_n r^{1/2} J_{n+1/2}(r) \sin \theta P_n^1(\cos \theta) \quad (7.5)$$

Submitting this into (2.3), the velocity can be found the same as in [15] and [16]:

$$\begin{cases} v_r = K_n n(n+1) r^{-3/2} J_{n+1/2}(r) P_n(\cos \theta) & (7.6a) \end{cases}$$

$$\begin{cases} v_\theta = K_n r^{-1/2} [J_{n-1/2}(r) - n J_{n+1/2}(r)/r] P_n^1(\cos \theta) & (7.6b) \end{cases}$$

$$\begin{cases} v_\varphi = -K_n r^{-1/2} J_{n+1/2}(r) P_n^1(\cos \theta) & (7.6c) \end{cases}$$

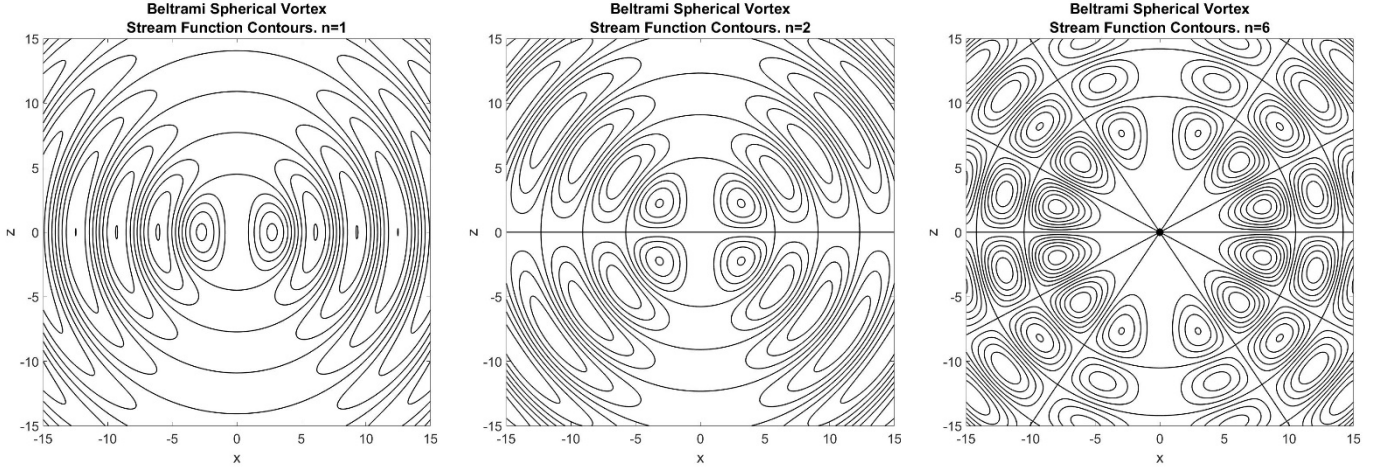


Fig. 3. Stream function contours of some Beltrami spherical vortices (in x-z plane)

This is a family of multi-layer spherical vortices, indexed by n (and will be referred to as Beltrami spherical vortices in this paper). As described in [16], the field is split by zeros of $J_{n+1/2}(r)$ (i.e., zeros of v_r and v_ϕ) into homocentric spherical layers. Inside each layer, there are n count of vortex rings, separated by the surfaces $P_n^1(\cos \theta) = 0$ (i.e., $v_\theta = 0$, $v_\phi = 0$). Fig. 3 shows contours of stream function when n is 1, 2 and 6. Examples of velocity field (when $n = 1, 2, 3$) can be found in Section 15 of [16].

As shown by the contours, the zero surfaces of $J_{n+1/2}(r)$ and $P_n^1(\cos \theta)$ (i.e., zero surfaces of ψ) split the whole field into axisymmetric and coaxial cells that each contains one vortex ring. On the zero surfaces, velocity is only on the tangential direction. Hence, the fluid inside each cell is contained within that cell all the time.

From (7.5) and (7.6), stream function and velocity have no singularity in the whole space. Thus, such a Beltrami vortex theoretically can steadily exist by itself in the whole space. In other words, it does not have to be moving in a potential flow (like Hill's vortex) or to be surrounded by other flow outside a restricted domain to be physically feasible.

Nevertheless, it is also possible to assemble a Beltrami vortex with potential flow outside a sphere, similar to the case of Hill's vortex.

For the first order vortices in the family (i.e., $n = 1$), at the spherical interfaces between layers, where $J_{n+1/2}(r) = 0$ thus $\psi = 0$, both v_r and v_ϕ vanish, and the polar velocity v_θ is

proportional to $P_1^1(\cos \theta) = -\sin \theta$. This exactly matches the case of irrotational flow past a sphere. Thus, we can have this Beltrami vortex inside the sphere and match it with a potential flow outside.

Specifically, in this case, the stream function can be written as

$$\left\{ \begin{array}{l} \psi = K_1 \left(\frac{\sin r}{r} - \cos r \right) \sin^2 \theta, \quad (r \leq D, D \text{ is one of the solutions of } r = \tan r) \quad (7.7a) \\ \psi = \frac{1}{3} K_1 \cos D \left(r^2 - \frac{D^3}{r} \right) \sin^2 \theta, \quad (r \geq D). \quad (7.7b) \end{array} \right.$$

Obviously, in this case the inside and outside flow both have vanished stream function on the surface of sphere $r = D$, and gradient of stream function is continuous on that surface.

Depending on which solution of $r = \tan r$ parameter D is set to, the vortex inside the sphere can have single or multiple layers. Fig. 4 shows 2 examples of such vortices with 1 layer ($D \approx 4.49$) and 3 layers ($D \approx 10.90$), respectively.

When $n > 1$, v_r and v_ϕ still vanish on the interfaces between spherical layers, but the tangential velocity v_θ at these interfaces has more complicated dependency to θ (rather than just being proportional to $\sin \theta$ when $n = 1$). Even in this case, theoretically it is still possible to match the vortex inside a sphere by a (high order) irrotational flow outside.

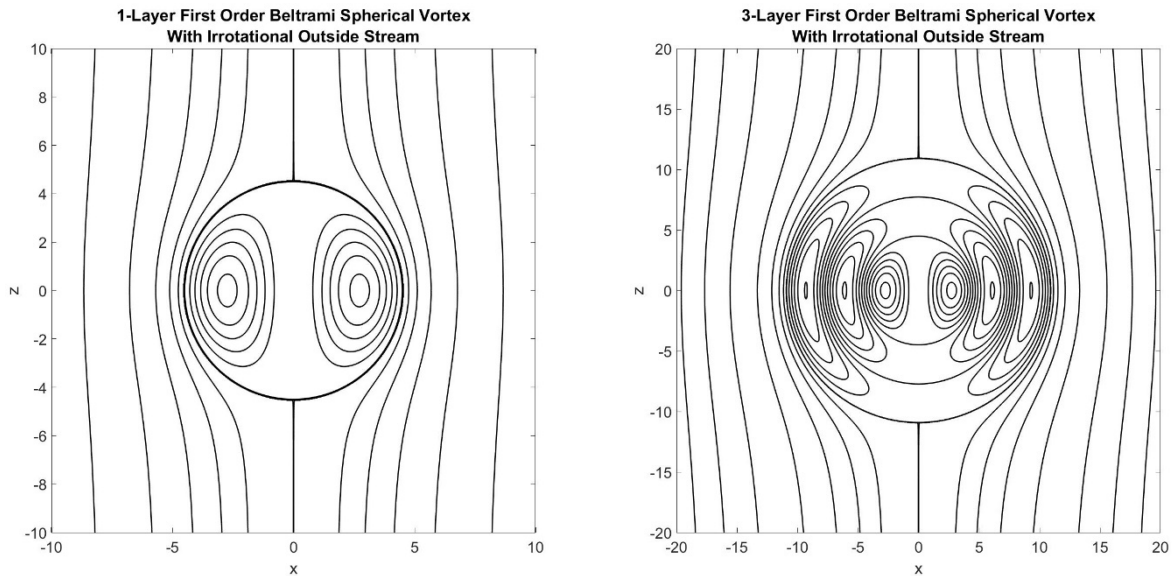


Fig. 4. 1-layer and 3-layer first order Beltrami spherical vortices

Note that for H_1C_1 , stream function (5.5) has the same polar angle section as in (7.4). With constants properly selected, a high order (i.e., $n > 1$) axisymmetric potential flow of (5.5) outside a sphere can match the same order of Beltrami vortex of (7.4) inside that sphere with zero stream function and continuous stream function gradient at the interface. This is an extension of the “assembled flow” of (7.7).

In this case, the stream function is

$$\begin{cases} \psi = K_n r^{1/2} J_{n+1/2}(r) \sin \theta P_n^1(\cos \theta), & (r \leq D, D \text{ is one of the zeros of } J_{n+1/2}(r)) & (7.8a) \\ \psi = K_n \frac{J_{n-1/2}(D)}{2n+1} (D^{-n+1/2} r^{n+1} - D^{n+3/2} r^{-n}) \sin \theta P_n^1(\cos \theta), & (r \geq D). & (7.8b) \end{cases}$$

When $n = 1$, (7.8) simplifies to (7.7) (with some constants re-denoted). Two examples of (7.8) are shown in Fig. 5. Note that when $n > 1$ the outside flow has unbounded velocity in far field. Such a vortex is more of a theoretical model, as physical feasibility is limited due to the outside flow.

(7.5) is only a special case of the general solution of (7.1). Setting constants in (7.4) to other values will give other kinds of flows.

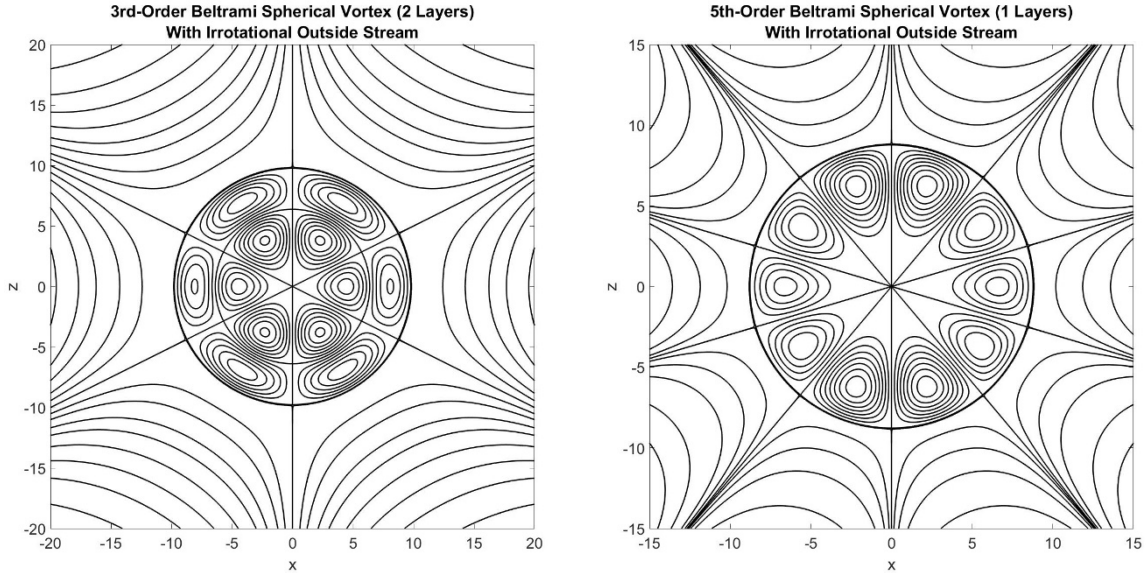


Fig. 5. High-order Beltrami spherical vortices surrounded by potential flow

When $\psi_0 \neq 0$ in $C = \alpha\psi + \psi_0$, the equation for H_1C_3 is $\frac{\partial^2\psi}{\partial r^2} + \frac{\sin^2\theta}{r^2} \frac{\partial^2\psi}{\partial(\cos\theta)^2} = -a^2\psi + \alpha\psi_0$. This is equation (7.1) plus a constant inhomogeneous term $\alpha\psi_0$ on the RHS. It is solved by adding the particular solution $\psi^* = \frac{\psi_0}{\alpha}$ to (7.4). (Another way to solve this equation is to replace ψ by $\psi - \frac{\psi_0}{\alpha}$ so to convert it to the same form as (7.1). This gives the same result.)

In this case, the particular solution $\psi^* = -\frac{\psi_0}{\alpha}$ brings to the flow an additional azimuthal velocity, which is singular at z-axis. Such an H_1C_3 flow with $\psi_0 \neq 0$ is feasible only when it is inside a domain not overlapping z-axis.

In cylindrical coordinates, when $\psi_0 = 0$, equation (2.15) for H_1C_3 , is

$$\frac{\partial^2\psi}{\partial\rho^2} - \frac{1}{\rho} \frac{\partial\psi}{\partial\rho} + \frac{\partial^2\psi}{\partial z^2} = -a^2\psi \quad (7.9)$$

This can be solved the same way as that for (5.6). The solution is in the same form as (5.7), with constant k in the $P(\rho)$ or in $Z(z)$ section of (5.7) biased by a^2 . That is,

$$\psi = [k_1\rho J_1(\sqrt{k}\rho) + k_2\rho Y_1(\sqrt{k}\rho)] \left(k_3 e^{\sqrt{k-a^2}z} + k_4 e^{-\sqrt{k-a^2}z} \right) \quad (7.10)$$

Same as in the spherical coordinates case, adding $\psi^* = -\frac{\psi_0}{\alpha}$ to (7.10) gives the solution to the $\psi_0 \neq 0$ case for H_1C_3 in cylindrical coordinates.

Mathematically, H_1C_4 and H_1C_3 have the same form of equation, thus they share the same solutions. They also have the same v_r and v_θ (or same v_ρ and v_z in cylindrical coordinates), but as these 2 cases have different function C , azimuthal velocity v_φ and vorticity are different. As discussed in Section 4, the flow in H_1C_4 is a non-linear Beltrami flow (rather than a linear Beltrami flow for H_1C_3).

8. Hill's vortex and extension (H_2C_1)

For Hill's vortex ^[2], or a bigger group, the Fraenkel-Norbury family of vortex rings ^{[5], [6], [8]}, the flow in the "core region" is featured by (a) swirl free and (b) vorticity is proportional to distance to the symmetry axis. In terms of B-H equation, feature (a) requires $v_\varphi = 0$, thus $C = 0$. Considering the vorticity decomposition, vorticity by (4.1) becomes $\boldsymbol{\omega} = -r \sin\theta \frac{dH}{d\psi} \mathbf{e}_\varphi$ (or $\boldsymbol{\omega} =$

$-\rho \frac{dH}{d\psi} \mathbf{e}_\phi$ in cylindrical coordinates). In this case, feature (b) translates to $\frac{dH}{d\psi} = \text{constant}$. Flows meeting (a) and (b) hence can be described by $H = H_0 + \lambda\psi$ and $C = 0$. Obviously, this is a subset of H_2C_1 .

As discussed in Section 5, for any H_mC_1 case, value of constant C is independent to the equation. Mathematically, a solution in such case is compatible with arbitrary constant C . A non-zero C brings non-zero v_ϕ which is an irrotational rotation around the z-axis (and is singular in the z-axis, thus is feasible only in a region not intersecting z-axis). Theoretically speaking (i.e., regardless of the singularity), feature (b) does not have to be paired with feature (a) in a flow.

For any flow of H_2C_1 , (b) is always valid. In this sense, the inner flow in Fraenkel-Norbury family can be considered as the swirl-free subset of H_2C_1 (while Hill's vortex is a special case in the Fraenkel-Norbury family).

Equation (2.14) for H_2C_1 is

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 \psi}{\partial (\cos \theta)^2} = \lambda r^2 \sin^2 \theta \quad (8.1)$$

This is equation (5.1) plus an inhomogeneous term on the RHS. A particular solution for (8.1) is $\psi^* = \frac{\lambda}{10} r^4 \sin^2 \theta + \psi_0$. Adding it to the solution of (5.1) gives the solution of (8.1) as

$$\psi = (k_1 r^{n+1} + k_2 r^{-n}) \sin \theta [k_3 P_n^1(\cos \theta) + k_4 Q_n^1(\cos \theta)] + \frac{\lambda}{10} r^4 \sin^2 \theta + \psi_0 \quad (8.2)$$

Similar to the case of H_1C_2 , as per (2.3) velocity in radial and polar directions are linear with respect to the stream function ψ , the particular solution ψ^* in (8.2) is linearly adding an extra velocity \mathbf{v}^* in meridian plane to the flow of (5.5). This extra velocity can be calculated from ψ^* as $v_r^* = \frac{\lambda}{5} r^2 \cos \theta$, $v_\theta^* = -\frac{2\lambda}{5} r^2 \sin \theta$. Fig. 6 shows such a \mathbf{v}^* when $\lambda = 1$. It is this extra velocity that turns the potential flows of (5.5) into H_2C_1 flows whose vorticity is proportional to distance to the symmetry axis.

By itself \mathbf{v}^* is unbounded in far field. As vorticity is proportional to distance to z-axis, vorticity is also unbounded when $\rho = r \sin \theta$ approaches infinity. To avoid such infinite velocity and vorticity, a physically feasible H_2C_1 flow in general should be inside a bounded region, with proper boundary conditions provided by another flow outside (or by a solid boundary). This is the case of the Fraenkel-Norbury vortex solution.

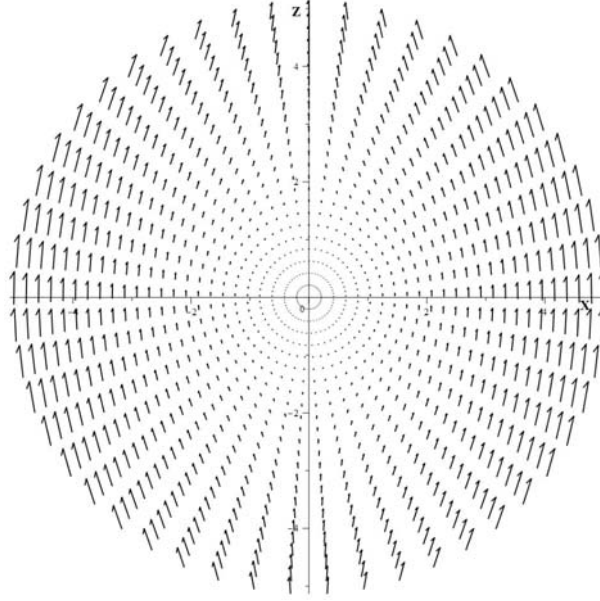


Fig. 6. Extra velocity in x-z plane brought by ψ^* for H_2C_1 flow

To better discuss, we can consider a more general situation that an H_2C_1 flow (i.e., solution (8.2)) exists inside an axisymmetric domain, \mathbb{A} , that (i) on the boundary $\partial\mathbb{A}$ the stream function ψ vanishes, and (ii) outside \mathbb{A} (i.e., in domain $R^3 - \mathbb{A}$) exists an axisymmetric flow whose ψ and $\nabla\psi$ are continuous with that of the inside flow at $\partial\mathbb{A}$, and (iii) the inside flow has no singularity in \mathbb{A} , the outside flow has no singularity in $R^3 - \mathbb{A}$ (including $r = \infty$).

Obviously, such an “assembled flow” is a candidate for Fraenkel-Norbury vortex ring solution. In other words, we can consider a flow meeting (i)/(ii)/(iii) as an extended or generalized Fraenkel-Norbury vortex solution. In this case, the inner flow does not have to be swirl-free, and the outside flow does not have to be vorticity-free. These make it more general than the original Fraenkel-Norbury solutions.

In general, such a flow could be complicated. But when $n = 1$, (8.2) can be in simple form and we have a chance to study the inside flow explicitly.

To avoid singularity, we can further set $k_2 = 0$, $k_3 = 1$ and $k_4 = 0$. With that (8.2) becomes

$$\psi = \left(\frac{\lambda}{10}r^4 - k_1r^2\right) \sin^2\theta + \psi_0 \quad (8.3)$$

This is a combination of a uniform flow with $U = -2k_1$ in z-direction (i.e., $\psi = -k_1r^2 \sin^2\theta$ as a special case of (5.5)) and the extra velocity \mathbf{v}^* shown in Fig. 6. The polar

angle factor, $\sin^2 \theta$, is the same as in the axisymmetric potential flow (5.5) when also with $k_4 = 0$ and $n = 1$ (i.e., the potential flow past a sphere).

Boundary of the region, $\partial\mathbb{A}$, is defined by $\psi = 0$. That is $r^2 \left(k_1 - \frac{\lambda}{10} r^2 \right) \sin^2 \theta = \psi_0$. This definition is equivalent to the definition by Fraenkel and Norbury in [5] and [6], which appears as $\psi = k$ (k is a constant). As ψ_0 is a free constant in the solution, ψ_0 can be set to zero then ψ is a constant on $\partial\mathbb{A}$. In this sense, ψ_0 here is equivalent to $-k$ in [5] or [6].

By this definition, $\partial\mathbb{A}$ is significantly impacted by ψ_0 . Without losing generality, assume $\lambda > 0, k_1 > 0$. Fig. 7 shows some cross sections of $\partial\mathbb{A}$ with different values of ψ_0 (when $\lambda = 1, k_1 = 1$).

When $\psi_0 = 0$, $\partial\mathbb{A}$ is surface of the sphere $r = \sqrt{\frac{10k_1}{\lambda}}$. When $\psi_0 > 0$, $\partial\mathbb{A}$ becomes a “donut shape” inside the sphere. When $\psi_0 = \frac{5k_1^2}{2\lambda}$, $\partial\mathbb{A}$ reaches the extreme case of a “thin core circle” in x-y plane, defined by $r = \sqrt{\frac{5k_1}{\lambda}}, \theta = \frac{\pi}{2}$. The case $\psi_0 < 0$ is not feasible as $\partial\mathbb{A}$ (shown by the dot-line in Fig. 7) then encloses the whole z-axis and velocity is unbounded.

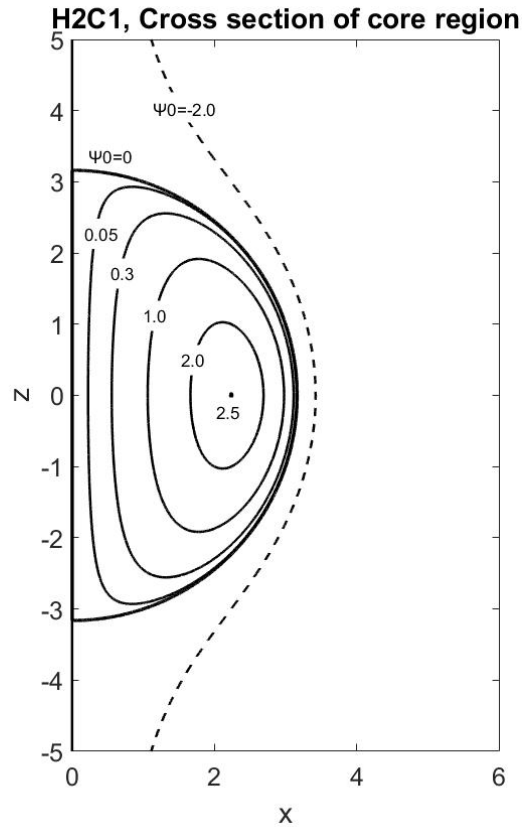


Fig. 7. Cross sections of core region of H₂C₁ flow

When $0 < \psi_0 < \frac{5k_1^2}{2\lambda}$, as the core region is avoiding z-axis, the flow inside the core region can have any irrotational rotation (i.e., with any non-zero C and azimuthal velocity $v_\varphi = \frac{C}{r \sin \theta}$). This is a swirled case of the (generalized) Fraenkel-Norbury solutions.

The case of $\psi_0 = 0$ is actually Hill's vortex. With $\psi_0 = 0$, if we further denote $\lambda = A$ and $k_1 = \frac{A}{10} \alpha^2$, (8.3) becomes $\psi = \frac{A}{10} r^2 \sin^2 \theta (r^2 - \alpha^2)$. This is the well-known stream function of Hill's vortex (in spherical coordinates).

In this case, as a segment of z-axis is involved, there should be no irrotational rotation in the core region (so to avoid singular velocity). In other words, Hill's vortex (as the solution of H_2C_1 with spherical core region) is always swirl-free. In comparison, the case of Hill's vortex with swirl, found by Moffat ^[3], is actually a solution of H_2C_3 , which will be discussed in next section. In that case, vorticity is no longer proportional to distance to z-axis, thus it does not belong to the Fraenkel-Norbury family of vortex rings.

When $\psi_0 > 0$, the outside flow with ψ and $\nabla\psi$ matching the inside flow at the donut-shape boundary could be complicated. But when $\psi_0 = 0$, as $\partial\mathbb{A}$ is a sphere surface with polar angle factor $\sin^2 \theta$ for stream function, a potential flow past that sphere matches well the inside flow. That forms the well-known case of Hill's vortex surrounded by uniform potential flow.

Recall that the first order Beltrami vortex (7.7a) also has polar angle factor $\sin^2 \theta$ on the layer interfaces. It is also possible to match Hill's vortex on $\partial\mathbb{A}$ with a "hollow" Beltrami vortex outside. In that case, the stream function can be written as

$$\begin{cases} \psi = K_1 \left(\frac{\sin r}{r} - \cos r \right) \sin^2 \theta, & (r \geq D, D \text{ is one of the solutions of } r = \tan r), & (8.4a) \\ \psi = -K_1 \frac{\cos D}{2D^2} r^2 \sin^2 \theta (D^2 - r^2) & (r \leq D). & (8.4b) \end{cases}$$

Fig. 8 shows an example of such assembled vortices (with $K_1 = 1$ and $D \approx 4.49$).

The stream function contours in Fig. 8 appear to be similar to that of the first order Beltrami spherical vortex (Fig. 3). But certain properties are quite different inside and outside the interface (shown by the dark line in Fig. 8). Vorticity inside is only in azimuthal direction, with magnitude proportional to distance to the z-axis; vorticity outside is in parallel to velocity at each point, with magnitude proportional to magnitude of velocity. On the interface, outside flow has

vorticity (and velocity) tangent to the sphere surface (i.e., in the azimuthal direction only). This matches vorticity of inside flow and provides a continuous vorticity at the interface. Another obvious difference between the inside and the outside flow is, one is swirl-free and the other one is with swirl.

Switching to cylindrical coordinates, equation for H_2C_1 ,

$$\frac{\partial^2 \psi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = \lambda \rho^2 \quad (8.5)$$

is the inhomogeneous case of equation (5.6). Adding a particular solution, $\psi^* = \psi_0 +$

$\frac{\lambda}{2(4k_5+k_6)} \rho^2 (k_5 \rho^2 + k_6 z^2)$ to (5.7) gives the solution of (8.5) as

$$\psi = [k_1 \rho J_1(\sqrt{k} \rho) + k_2 \rho Y_1(\sqrt{k} \rho)] (k_3 e^{\sqrt{k} z} + k_4 e^{-\sqrt{k} z}) \psi_0 + \frac{\lambda}{2(4k_5+k_6)} \rho^2 (k_5 \rho^2 + k_6 z^2) \quad (8.6)$$

When $\psi_0 = 0$ and $k_5 = 1$, $k_6 = 1$, the particular solution becomes $\psi^* = \frac{\lambda}{10} \rho^2 (\rho^2 + z^2)$.

If we consider the special solution of (5.6), $\psi = k_1 \rho^2$ (see the note in Section 5 following (5.7)),

and add to it with particular solution $\psi^* = \frac{\lambda}{10} \rho^2 (\rho^2 + z^2)$, we can also get the Hill's vortex

solution $\psi = \frac{A}{10} \rho^2 (\alpha^2 - \rho^2 - z^2)$ by re-denoting $\lambda = -A$ and $k_1 = \frac{A}{10} \alpha^2$.

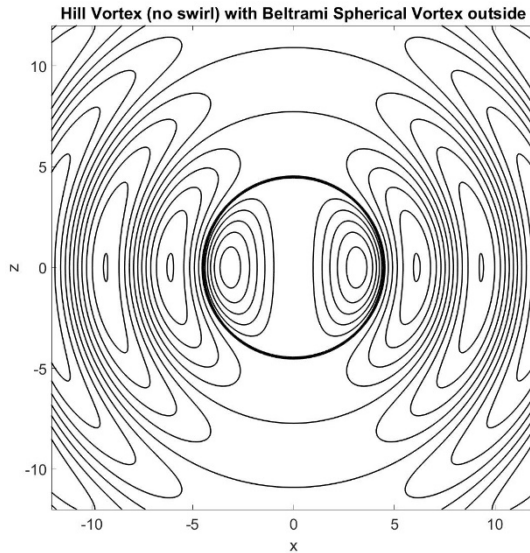


Fig. 8. Hill's vortex (without swirl) with hollow Beltrami spherical vortex

For H_2C_2 , another inhomogeneous term, $-\frac{a}{2}$, is added to RHS of the H_2C_1 equation (8.1) or (8.5). In spherical coordinates that leads to

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 \psi}{\partial (\cos \theta)^2} = \lambda r^2 \sin^2 \theta - \frac{a}{2}, \quad (8.7)$$

and in cylindrical coordinates, that is

$$\frac{\partial^2 \psi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = \lambda \rho^2 - \frac{a}{2}. \quad (8.8)$$

These are still inhomogeneous equation of (5.1) and (5.6), respectively. The particular solutions for (8.7) can be found by combining the particular solution of (6.1) and the particular solution of (8.1). That gives

$$\psi^* = \frac{\lambda}{10} r^4 \sin^2 \theta - \frac{a}{4} r^2 + \psi_0. \quad (8.9)$$

Similarly, particular solution for (8.8) can be found by combining the particular solution of (6.3) and the particular solution of (8.5). That yields

$$\psi^* = \frac{\lambda}{2(4k_5 + k_6)} \rho^2 (k_5 \rho^2 + k_6 z^2) + k_0 \frac{a}{8} \rho^2 (1 - 2 \ln \rho) - (1 - k_0) \frac{a}{4} z^2 + \psi_0 \quad (8.10)$$

Adding (8.9) and (8.10) respectively to (5.5) and (5.7) gives the general solutions of (8.7) and (8.8).

For H_2C_2 , vorticity is neither proportional to distance to z-axis, nor in parallel and proportional to velocity. According to decomposition (4.1) and (4.2), vorticity in this case has non-zero Beltrami vorticity part and non-zero azimuthal vorticity part. Thus, it is in a combined direction of azimuthal direction and direction of the velocity.

Compared to the H_1C_2 or H_2C_1 case, the restriction in H_2C_2 for velocity and vorticity to be bounded and for $C = a_1 \sqrt{\psi_0 + a_2 \psi}$ to be real is tighter. A feasible flow in this case would be in a more restricted region compared to that of H_1C_2 or H_2C_1 flow.

9. Hill's vortex with swirl (H_2C_3)

The Hill's vortex with swirl, discovered by Moffatt^[3] and Hicks^[10], is a solution of the B-H equation with $C = a\psi$, $H = H_0 + \lambda\psi$. This is the case of H_2C_3 (with $\psi_0 = 0$ in C_3).

In spherical coordinates, the equation for H₂C₃ is

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 \psi}{\partial (\cos \theta)^2} = \lambda r^2 \sin^2 \theta - a^2 \psi + \alpha \psi_0 \quad (9.1)$$

Obviously, this is the inhomogeneous case of (7.1), i.e., the equation of H₁C₃. A particular solution of (9.1) is $\psi^* = \frac{\lambda}{a^2} r^2 \sin^2 \theta + \frac{\psi_0}{\alpha}$. Adding it to (7.4) brings the solution of (9.1) as

$$\psi = r^{\frac{1}{2}} [k_1 J_{n+1/2}(ar) + k_2 Y_{n+1/2}(ar)] \sin \theta [k_3 P_n^1(\cos \theta) + k_4 Q_n^1(\cos \theta)] + \frac{\lambda}{a^2} r^2 \sin^2 \theta + \frac{\psi_0}{\alpha} \quad (9.2)$$

The same procedure can be employed to solve the H₂C₃ equation in cylindrical coordinates,

$$\frac{\partial^2 \psi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = \lambda \rho^2 - a^2 \psi + \alpha \psi_0. \quad (9.3)$$

With general solution (7.10) and the particular solution $\psi^* = \frac{\lambda}{a^2} \rho^2 + \frac{\psi_0}{\alpha}$, solution of (9.3) is

$$\psi = [k_1 \rho J_1(\sqrt{a^2 + k} \rho) + k_2 \rho Y_1(\sqrt{a^2 + k} \rho)] (k_3 e^{\sqrt{k} z} + k_4 e^{-\sqrt{k} z}) + \frac{\lambda}{a^2} \rho^2 + \frac{\psi_0}{\alpha} \quad (9.4)$$

Alternatively, there is a more straightforward approach to solve (9.1). It can be described by the following statement (as a theorem): if ψ^* is solution of (5.1), and $\tilde{\psi}$ is solution of (7.1), then $\psi = \tilde{\psi} + \psi^*$ is solution of the following equation

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 \psi}{\partial (\cos \theta)^2} = a^2 \psi^* - a^2 \psi \quad (9.5)$$

The proof is straightforward. As ψ^* is solution of equation (5.1), $\psi = \psi^*$ is a particular solution of (9.5). Thus $\psi = \tilde{\psi} + \psi^*$ solves (9.5).

This theorem is valid for the cylindrical coordinates case as well. In that case ψ^* is solution of (5.6), $\tilde{\psi}$ is solution of (7.9) and $\psi = \tilde{\psi} + \psi^*$ solves the corresponding equation of (9.5) in cylindrical coordinates, which is

$$\frac{\partial^2 \psi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = a^2 \psi^* - a^2 \psi \quad (9.6)$$

Solution of (5.1) and (5.6) can be in many different forms. Each of them will bring an equation in form of (9.5) or (9.6) (not necessarily a B-H equation). Theoretically, this theorem enables an approach to solve these equations.

Equation (9.1) happens to be an example of (9.5). When we take $\psi^* = \frac{\lambda}{a^2} r^2 \sin^2 \theta + \frac{\psi_0}{\alpha}$ as solution of (5.1) and apply the theorem, we obtain (9.2). Similarly, (9.3) as an example of (9.6) when we take $\psi^* = \frac{\lambda}{a^2} \rho^2 + \frac{\psi_0}{\alpha}$ as solution of (5.6). Applying the theorem leads to (9.4).

As discussed in Section 7, solution (7.4) represents a family of Beltrami spherical vortices. ψ^* in (9.2) is adding to them an extra velocity \mathbf{v}^* , which by (2.3) can be calculated as

$$v_r^* = \frac{2\lambda}{a^2} \cos \theta, v_\theta^* = -\frac{2\lambda}{a^2} \sin \theta, v_\varphi^* = \frac{\lambda}{a^2} r \sin \theta + \frac{\psi_0}{\alpha} \frac{1}{r \sin \theta} \quad (9.7)$$

In a meridian plane, v_r^* and v_θ^* form a constant velocity $U = \frac{2\lambda}{a^2}$ in z direction. v_φ^* has two terms. The first term $\frac{\lambda}{a^2} r \sin \theta$ is related to a rigid rotation around z -axis; the second term is the irrotational rotation discussed previously, which has singularity on the z -axis. As the flows discussed in this case are mainly the vortices centered at $r = 0$, to avoid the singularity, we will assume $\psi_0 = 0$ (i.e., without the irrotational rotation) hereafter in this section.

In this case, the extra velocity \mathbf{v}^* represents a special spiral movement: uniform velocity in z -direction and a rigid rotation around z -axis. Adding this to the linear Beltrami flows of (7.4) yields the H₂C₃ flows in (9.2).

With $\psi_0 = 0$, when $n = 1$, $k_1 = A\alpha^{3/2}$, $k_2 = 0$, $k_3 = -1$, $k_4 = 0$, (9.2) is exactly the inner flow of Hill's vortex with swirl, presented by Moffatt ^[3] as

$$\psi = r^2 \sin^2 \theta \left[\frac{\lambda}{a^2} + A \left(\frac{\alpha}{r} \right)^{3/2} J_{3/2}(ar) \right] \quad (9.8)$$

Setting constants to other values finds other forms of solutions (9.2).

As nature of rigid rotations, azimuthal velocity v_φ^* is unbounded in far field. As a result of that, a feasible H₂C₃ flow should be inside a bounded region with proper condition on the boundary. When $n = 1$ (and $k_4 = 0$), (9.2) has a polar angle factor $\sin^2 \theta$. This is the same as the irrotational flow past a sphere. Thus, that irrotational flow can provide the boundary conditions.

Taking (9.8) as the special case of (9.2) and matching it with the irrotational flow outside, with some constants re-denoted, the stream function can be written as

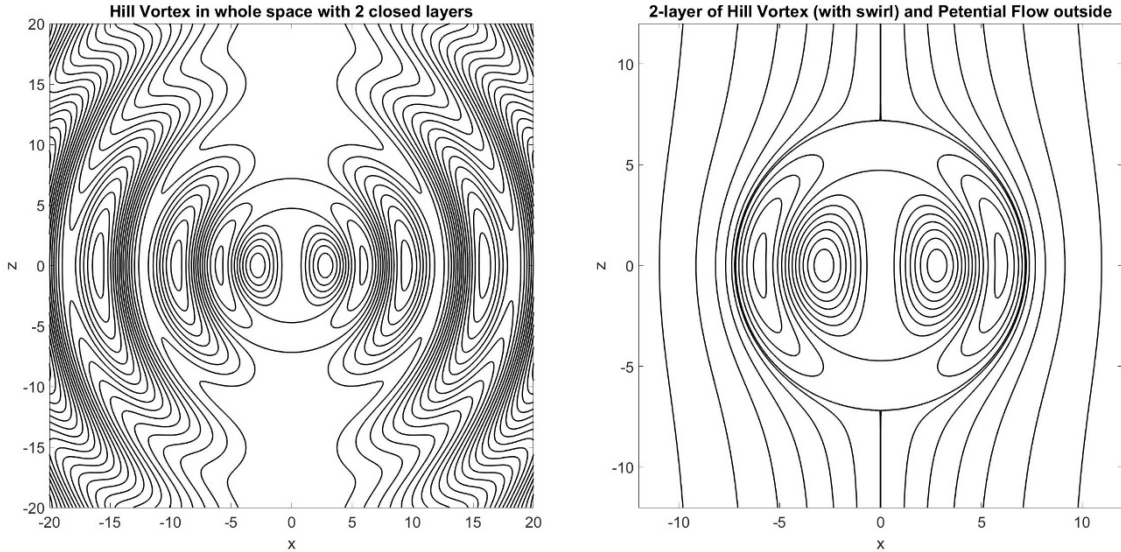
$$\left\{ \begin{array}{l} \psi = A \left[\frac{\sin(ar)}{ar} - \cos(ar) + \frac{\lambda}{Aa^2} r^2 \right] \sin^2 \theta, \quad (r \leq D), \end{array} \right. \quad (9.9a)$$

(D is a positive solution of $\frac{\sin(ar)}{ar} - \cos(ar) + \frac{\lambda}{Aa^2} r^2 = 0$ as an equation of r),

$$\left\{ \begin{array}{l} \psi = \left[\frac{\lambda}{a^2} + A \frac{a}{3D} \sin(aD) \right] \left(r^2 - \frac{D^3}{r} \right) \sin^2 \theta, \quad (r \geq D). \end{array} \right. \quad (9.9b)$$

Note that $\frac{\sin(ar)}{ar} - \cos(ar) + \frac{\lambda}{Aa^2} r^2 = 0$ can have multiple positive solutions (depending on the value of $\frac{\lambda}{Aa^2}$). Each solution represents a spherical interface of a closed layer. Fig. 9a shows the flow of (9.9a) in the whole domain when $A = 1$, $a = 1$ and $\lambda = 0.01$. Two closed layers exist near the center in this case. Outside of the second layer, the flow is no longer contained in close layers, and velocity increases (unboundedly) as r increases.

The flow inside any closed interface can be matched by a potential flow outside. In other words, it does not have to be the first interface from the center. In the case D in (9.9a) is set to solution other than the first one from the center, the inner vortex can have multiple layers. Fig. 9b shows the central section of Fig. 9a with 2 layers ($D \approx 7.18$) surrounded by potential flow (9.9b).



(a) Hill's Vortex with swirl in whole space (b) 2-Layer Hill vortex with swirl in potential flow

Fig. 9. Multi-layer of Hill's vortex with swirl

Similar to Hill's vortex without swirl in (8.4), the inner vortex with a polar angle factor $\sin^2\theta$ can also be matched by a hollow Beltrami Spherical vertex (similar as in Fig. 8). In this case, the inner flow is still the same as (9.9a), and the outer flow is from (7.4) with dedicated constants to have stream function and its gradient matching the inner flow at the interface:

$$\left\{ \begin{array}{l} \psi = A \left[\frac{\sin(ar)}{ar} - \cos(ar) + \frac{\lambda}{Aa^2} r^2 \right] \sin^2\theta, \quad (r \leq D), \quad (9.10a) \\ (D \text{ is a positive solution of } \frac{\sin(ar)}{ar} - \cos(ar) + \frac{\lambda}{Aa^2} r^2 = 0 \text{ as an equation of } r), \\ \psi = K \left[\frac{\sin(br)}{(br)} - \cos(br) \right] \sin^2\theta, \quad (r \geq D), \quad (9.10b) \\ (K = \frac{1}{b \sin(bD)} \left[\frac{3\lambda D}{a^2} + aA \sin(aD) \right], b \text{ is a solution of } \frac{\sin(bD)}{(bD)} - \cos(bD) = 0). \end{array} \right.$$

An example of such vortices is shown in Fig. 10. (with $A = 1$, $a = 1$ and $\lambda = 0.01$, $D \approx 7.1798$, $b \approx 1.0760$).

Similar as for H_1C_4/H_1C_3 , the H_2C_4 case has the same equation as H_2C_3 , thus share with H_2C_3 the solution (9.2) (or (9.4) in in cylindrical coordinates). Feasibility and properties of the H_2C_4 flow, however, are quite different than the H_2C_3 flow even when they are from the same stream function.

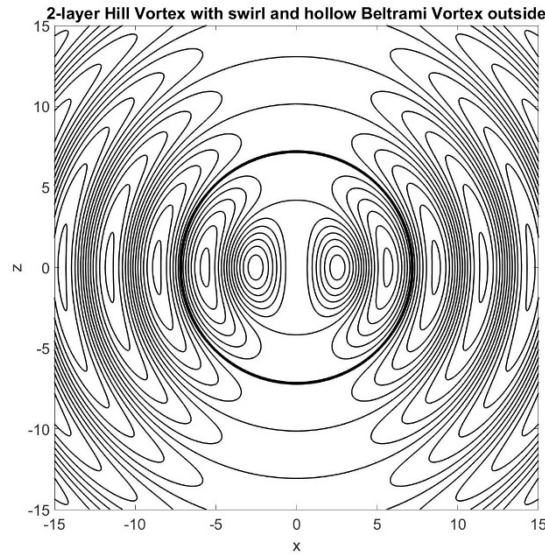


Fig. 10. Hill's vortex (with swirl) with hollow Beltrami vortex

10. Z-periodic axisymmetric flows (H_3C_n and H_4C_n)

Some physical background of the H_3C_3 case can be found in, e.g., [17] and [18].

For these two families (i.e., H_3C_n and H_4C_n), as H is quadratic function of ψ , (homogeneous part of) the first term on RHS explicitly contains ψ . Moreover, in the spherical coordinates case, two coordinate variables, r and θ , are explicitly coupled with ψ in this term (i.e., $2\lambda^2 r^2 \sin^2 \theta \psi$). This makes it difficult for variables to be separated. In cylindrical coordinates, this term (i.e., $2\lambda^2 \rho^2 \psi$) involves only one coordinate variable ρ . This allows variable separation method to work on the equation.

In the rest of this section, we will concentrate on possible solutions in cylindrical coordinates. The approach to solve the equation in spherical coordinates is yet to be explored.

For H_3C_3 with $\psi_0 = 0$ in C_3 , equation (2.15) is

$$\frac{\partial^2 \psi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = 2\lambda \rho^2 \psi - a^2 \psi \quad (10.1)$$

This is the basic homogeneous equation for the two families. We will start the discussion with this equation.

Let $\psi = P(\rho)Z(z)$ and denote the separation constant by k , (10.1) could be separated to two equations: $\rho \ddot{P} - \dot{P} - (2\lambda \rho^2 + k - a^2)\rho P = 0$, $\ddot{Z} + kZ = 0$. The former one is solved by

$$P(\rho) = \rho^2 e^{-\sqrt{\frac{\lambda}{2}}\rho^2} \left[k_1 M\left(\frac{k-a^2}{4\sqrt{2\lambda}} + 1, 2, \sqrt{2\lambda}\rho^2\right) + k_2 U\left(\frac{k-a^2}{4\sqrt{2\lambda}} + 1, 2, \sqrt{2\lambda}\rho^2\right) \right] \quad (10.2)$$

where $M(a, b, x)$ and $U(a, b, x)$ are Kummer's confluent hypergeometric functions of the first and second kind, respectively ^[19].

General solution of (10.1) thus is

$$\psi = \rho^2 e^{-\sqrt{\frac{\lambda}{2}}\rho^2} \left[k_1 M\left(\frac{k-a^2}{4\sqrt{2\lambda}} + 1, 2, \sqrt{2\lambda}\rho^2\right) + k_2 U\left(\frac{k-a^2}{4\sqrt{2\lambda}} + 1, 2, \sqrt{2\lambda}\rho^2\right) \right] \left(k_3 e^{\sqrt{k}z} + k_4 e^{-\sqrt{k}z} \right) \quad (10.3)$$

Similar to (5.7), in (10.3) the section related to z is periodic along z -axis when $k < 0$ (and is unbounded on one side of z -axis when $z \rightarrow \pm\infty$ if $k > 0$). For this reason, most of the feasible flows in these 2 families are periodic along z -axis.

H₃C₄ with $\psi_0 = 0$ has the same form of equation as (10.1), thus is also solved by (10.3) (with some constants re-denoted).

When $a = 0$, (10.1) reduces to equation of H₃C₁ as

$$\frac{\partial^2 \psi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = 2\lambda \rho^2 \psi \quad (10.4)$$

and (10.3) reduces to solution of (10.4) as

$$\psi = \rho^2 e^{-\sqrt{\frac{\lambda}{2}} \rho^2} \left[k_1 M\left(\frac{k}{4\sqrt{2\lambda}} + 1, 2, \sqrt{2\lambda} \rho^2\right) + k_2 U\left(\frac{k}{4\sqrt{2\lambda}} + 1, 2, \sqrt{2\lambda} \rho^2\right) \right] \left(k_3 e^{\sqrt{k} z} + k_4 e^{-\sqrt{k} z} \right) \quad (10.5)$$

Equation (10.1) can also be solved by a method developed by Bogoyavlenskij^[17]. It is a variable separation method with series of variable replacements. The solution is a special case of (10.3) when variables are properly replaced and constants are properly set.

For H₃C₂, a constant inhomogeneous term is added to RHS of (10.4). Equation (2.15) thus is

$$\frac{\partial^2 \psi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = 2\lambda \rho^2 \psi - \frac{a}{2} \quad (10.6)$$

Mathematically, a particular solution of (10.6) can be found by solving $\psi^*(\rho)$ as a function of ρ in the following equation

$$\frac{d^2 \psi^*(\rho)}{d\rho^2} - \frac{1}{\rho} \frac{d\psi^*(\rho)}{d\rho} = 2\lambda \rho^2 \psi^*(\rho) - \frac{a}{2}$$

The solution can be written with hyperbolic functions and hyperbolic integrals as

$$\psi^* = [k_1 \cosh(\sigma) + k_2 \sinh(\sigma)] + \frac{a}{4\sqrt{2\lambda}} [\text{chi}(\sigma) \sinh(\sigma) - \text{shi}(\sigma) \cosh(\sigma)], \quad (\sigma = \sqrt{\frac{\lambda}{2}} \rho^2) \quad (10.7)$$

Adding (10.7) to (10.5) gives the solution for (10.6).

For H₄C₁ and H₄C₂, a non-constant inhomogeneous term, $\lambda_1 \rho^2$ appears on the RHS as well. For H₄C₁ the equation is

$$\frac{\partial^2 \psi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = 2\lambda_2 \rho^2 \psi + \lambda_1 \rho^2 \quad (10.8)$$

and for H₄C₂ the equation is

$$\frac{\partial^2 \psi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = 2\lambda_2 \rho^2 \psi + \lambda_1 \rho^2 - \frac{a}{2} \quad (10.9)$$

By replacing the unknown function $\psi(\rho, z)$ with $\Phi(\rho, z) - \frac{\lambda_1}{2\lambda_2}$, the inhomogeneous term $\lambda_1 \rho^2$ will disappear, and equations (10.8) and (10.9) convert to equation of $\Phi(\rho, z)$ in the same form as (10.4) and (10.6), respectively, thus they can be solved the same way as for (10.4) and (10.6).

Actually, the cases discussed so far in this section (H₃C₁, H₃C₂, H₄C₁, H₄C₂, H₃C₃ with $\psi_0 = 0$, H₃C₄ with $a_0 = 0$) are the only cases in H₃C_n and H₄C_n families with known solution in this paper. For the rest of cases in these two families, i.e., H₃C₃ with $\psi_0 \neq 0$, H₃C₄ with $a_0 \neq 0$, H₄C₃, and H₄C₄, due to difficulty on finding the pedicular solutions, the equations are still unsolved in this paper.

Considering these 4 cases with the model equation (3.2) in Section 3, on the RHS, both of the homogeneous terms, Λ_1 and A_1 , exist, together with one or both of the inhomogeneous terms, Λ_0 and A_0 . Solution of the homogeneous equation with Λ_1 and A_1 (i.e., equation (10.1)) has been obtained as (10.2). So, the question is to find the pedicular solution in these cases.

Actually, by replacement $\psi(\rho, z) \rightarrow \Phi(\rho, z) - \frac{\Lambda_0}{\Lambda_1}$, the $\Lambda_0 \rho^2$ term will disappear; by replacement $\psi(\rho, z) \rightarrow \Phi(\rho, z) - \frac{A_0}{A_1}$, the A_0 term will disappear. By these, one of the two inhomogeneous terms (but not both) can be eliminated. If we have an approach to find the pedicular solution for the equation with both Λ_1 and A_1 terms and either of Λ_0 and A_0 terms, adding that pedicular solution to (10.3) will solve that equation (and hence solve all the four cases). However, the approach to find such a pedicular solution is still lacking in this paper.

11. Summary and Discussion

As discussed in Section 3, any equation for an H_mC_n case can be written in (3.1) or (3.2) form. With that, when Λ_0 and A_0 terms both vanish, the equation is homogeneous (and is solved by variable separation method). When at least one of Λ_0 and A_0 terms is non-zero, the equation is inhomogeneous and is to be solved by adding a pedicular solution to solution of the related homogeneous equation. Basically, this is the approach taken in this paper.

For convenience, a summary is given in Table 1 for the combinations of these 4 terms and the related forms of function H and C , as well as the known solution of each case.

In the table, presentation/absence of the four terms are indicated by “1”/”0” on the “RHS terms” column. ψ^* is representing the particular solution for each inhomogeneous equation. As ψ_0 in C_3 and C_4 has big impacts on the equation and solution for the H_1 , H_3 and H_4 families, cases of $\psi_0 = 0$ and $\psi_0 \neq 0$ are listed separately. With that, there are totally 20 cases in the table, of which 4 (in the last 4 rows) are still unsolved.

By nature of the variable separation method, solutions can be different when variables are separated in different ways (e.g., in different coordinate systems). Thus, other ways to separate variables may find other solutions. The solutions obtained so far in this paper are not expected to cover all possible solutions.

| H/C function ($H_m C_n$) | RHS terms | | | | spherical coordinates | | cylindrical coordinates | | | |
|----------------------------------|--------------|-------------|-------|-------|------------------------|--------------------------|-------------------------|---------------------------|-----------------------|---|
| | Λ_1 | Λ_0 | A_1 | A_0 | Equation | Solution | Equation | Solution | | |
| $H_1 C_1$ | 0 | 0 | 0 | 0 | (5.1) | (5.5) | (5.6) | (5.7) | | |
| $H_1 C_2$ | 0 | 0 | 0 | 1 | (6.1) | (5.5) + $\psi^* = (6.2)$ | (6.3) | (5.7) + ψ^* | | |
| $H_1 C_3, \psi_0 = 0$ | 0 | 0 | 1 | 0 | (7.1) | (7.4) | (7.9) | (7.10) | | |
| $H_1 C_3, \psi_0 \neq 0$ | 0 | 0 | 1 | 1 | (7.1) + $\alpha\psi_0$ | (7.4) + ψ^* | (7.9) | (7.10) + ψ^* | | |
| $H_1 C_4, \alpha_0 = 0$ | 0 | 0 | 1 | 0 | (7.1) | (7.4) | (7.9) | (7.10) | | |
| $H_1 C_4, \alpha_0 \neq 0$ | 0 | 0 | 1 | 1 | (7.1) + $\alpha\psi_0$ | (7.4) + ψ^* | (7.9) | (7.10) + ψ^* | | |
| $H_2 C_1$ | 0 | 1 | 0 | 0 | (8.1) | (5.5) + $\psi^* = (8.2)$ | (8.5) | (5.7) + $\psi^* = (8.6)$ | | |
| $H_2 C_2$ | 0 | 1 | 0 | 1 | (8.7) | (5.5) + (8.9) | (8.8) | (5.7) + (8.10) | | |
| $H_2 C_3$ | 0 | 1 | 1 | 1 | (9.1) | (7.4) + $\psi^* = (9.2)$ | (9.3) | (7.10) + $\psi^* = (9.4)$ | | |
| $H_2 C_4$ | 0 | 1 | 1 | 1 | (9.1) | (7.4) + $\psi^* = (9.2)$ | (9.3) | (7.10) + $\psi^* = (9.4)$ | | |
| $H_3 C_1$ | 1 | 0 | 0 | 0 | (yet to be solved) | | (10.4) | (10.5) | | |
| $H_3 C_2$ | 1 | 0 | 0 | 1 | | | (10.6) | (10.5) + (10.7) | | |
| $H_3 C_3, \psi_0 = 0$ | 1 | 0 | 1 | 0 | | | (10.1) | (10.3) | | |
| $H_3 C_4, \alpha_0 = 0$ | 1 | 0 | 1 | 0 | | | (10.1) | (10.3) | | |
| $H_4 C_1$ | 1 | 1 | 0 | 0 | | | (10.8) | (10.5) | | |
| $H_4 C_2$ | 1 | 1 | 0 | 1 | | | (10.9) | (10.5) + (10.7) | | |
| $H_3 C_3, \psi_0 \neq 0$ | 1 | 1 | 1 | 1 | | | (yet to be solved) | | (10.10) or (10.11) | (10.5) + $\psi^* (?)$ (ψ^* is yet to be found) |
| $H_3 C_4, \alpha_0 \neq 0$ | 1 | 1 | 1 | 1 | | | | | | |
| $H_4 C_3$ | 1 | 1 | 1 | 1 | | | | | | |
| $H_4 C_4$ | 1 | 1 | 1 | 1 | | | | | | |

Table 1. Summary of equation forms and solutions

As the major attention in this paper is on general mathematical solutions, only a few obvious specific flows were discussed as examples. By setting constants to other values, there could be more specific flows of interest. However, physical feasibility (and stability) of them is to be carefully investigated, especially for those “assembled vortices”.

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