Noname manuscript No.
(will be inserted by the editor)

The Riemann Hypothesis

Frank Vega

the date of receipt and acceptance should be inserted later

Abstract For every prime number p_n, we define the sequence $X_n = \frac{\prod_{q \mid N_n} \frac{q}{q-1}}{e^{\gamma} \times \log \log N_n}$, where $N_n = \prod_{k=1}^{n} p_k$ is the primorial number of order n and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The Nicolas theorem states that the Riemann hypothesis is true if and only if the $X_n > 1$ holds for all prime $p_n > 2$. For every prime number p_k, $X_k > 1$ is called the Nicolas inequality. We show if the sequence X_n is strictly decreasing for n big enough, then the Riemann hypothesis should be true. Moreover, we demonstrate that the sequence X_n is indeed strictly decreasing when $n \to \infty$. Notice that, Choie, Planat and Solé in the preprint paper arXiv:1012.3613 have a proof that the Cramér conjecture is false when X_n is strictly decreasing for n big enough. This paper is an extension of their result.

Keywords Riemann hypothesis · Nicolas theorem · Prime numbers · Chebyshev function · Monotonicity

Mathematics Subject Classification (2010) MSC 11M26 · MSC 11A41 · MSC 11A25

1 Introduction

Let $N_n = 2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times p_n$ denotes a primorial. For every prime p_n, we define the sequence

$$X_n = \frac{\prod_{q \mid N_n} \frac{q}{q-1}}{e^{\gamma} \times \log \log N_n}.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, log is the natural logarithm, and $q \mid N_n$ means the prime q divides to N_n. The importance of this property is:

F. Vega
CopSonic, 1471 Route de Saint-Nauphary 82000 Montauban, France
ORCiD: 0000-0001-8210-4126
E-mail: vega.frank@gmail.com

Theorem 1.1 *[6], [7].* $X_n > 1$ *holds for all prime* $p_n > 2$ *if and only if the Riemann hypothesis is true. Moreover, the Riemann hypothesis is false if and only if there are infinitely many prime numbers* q_i *for which* $X_i \leq 1$ *and infinitely many prime numbers* r_j *for which* $X_j > 1$.

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x. We use the following property of the Chebyshev function:

Theorem 1.2 *[3].*

$$\lim_{x \to \infty} \frac{\theta(x)}{x} = 1.$$

We use the Mertens' second theorem which states:

Theorem 1.3 *[5].*

$$\lim_{x \to \infty} (\sum_{q \leq x} \frac{1}{q} - \log \log x - B) = 0,$$

where $B \approx 0.2614972128$ *is the Meissel-Mertens constant.*

We use the following property of the Meissel-Mertens constant:

Theorem 1.4 *[1].*

$$B = \gamma + \log(\prod_{q} \frac{q-1}{q}) + \sum_{q} \frac{1}{q}.$$

Besides, we use the following inequality,

Theorem 1.5 *[4]. For* $0 > x > -1$*:*

$$x > \log(1+x).$$

Choie, Planat and Solé showed that if the sequence X_n is strictly decreasing for n big enough, then the Nicolas inequality is satisfied for a prime big enough [2]. They have confirmed that X_n is strictly decreasing with a numerical computations up to $2 < p_n \leq 104729$ (that is $1 < n \leq 10000$) [2]. In addition, these authors in the same paper arXiv:1012.3613 have shown that the Cramér conjecture is false under the assumption that the sequence X_n is strictly decreasing for n big enough [2]. We make a very similar approach showing the same result: that is, if the sequence X_n is strictly decreasing for n big enough, then the Riemann hypothesis is true. Using the properties of the Chebyshev function, we prove that the sequence X_n is strictly decreasing when $n \to \infty$.

2 On Sequence X_n

Theorem 2.1

$$\lim_{n\to\infty} X_n = 1.$$

Proof By the theorem 1.3,

$$\lim_{n\to\infty}(\sum_{q\leq p_n}\frac{1}{q} - \log\log p_n - B) = 0,$$

and by the theorem 1.4,

$$B = \gamma + \log(\prod_q \frac{q-1}{q}) + \sum_q \frac{1}{q}.$$

Putting all this together yields the result,

$$\lim_{n\to\infty}(\sum_{q\leq p_n}\frac{1}{q} - \log\log p_n - \gamma - \log(\prod_{q\leq p_n}\frac{q-1}{q}) - \sum_{q\leq p_n}\frac{1}{q}) = 0,$$

that is equivalent to

$$\lim_{n\to\infty}(\log(\prod_{q\leq p_n}\frac{q}{q-1}) - \gamma - \log\log p_n) = 0.$$

We use that theorem 1.2:

$$\lim_{n\to\infty}(\log(\prod_{q|N_n}\frac{q}{q-1}) - \gamma - \log\log\log N_n) = 0.$$

Finally, we can apply the exponentiation to show:

$$\lim_{n\to\infty}(\frac{\prod_{q|N_n}\frac{q}{q-1}}{e^{\gamma}\times\log\log N_n}) = 1.$$

Theorem 2.2 *If X_n is strictly decreasing for n big enough, then the Riemann hypothesis is true.*

Proof Suppose that $N_n > 2$ is the smallest primorial number such that the Nicolas inequality is false under the assumption that X_i is strictly decreasing (that is $X_i > X_{i+1}$). In this way, we have

$$X_n \leq 1$$

and thus

$$X_{n+1} < X_n \leq 1.$$

This implies

$$\limsup_{n\to\infty} X_n < 1$$

which is a contradiction with the theorem 2.1. By contraposition, the Nicolas inequality could be satisfied for all prime p_n big enough. Consequently, there would be no infinitely many prime numbers for which the Nicolas inequality is unsatisfied. Using the theorem 1.1, we can conclude that the Riemann hypothesis is true when X_n is strictly decreasing for n big enough.

Theorem 2.3 *The inequality $X_n > X_{n+1}$ is equivalent to*

$$\frac{\log\theta(p_{n+1})}{\log\theta(p_n)} > \frac{p_{n+1}}{p_{n+1}-1}.$$

Proof The inequality $X_n > X_{n+1}$ can be written as

$$\frac{\prod_{q|N_n}\frac{q}{q-1}}{e^{\gamma}\times\log\log N_n} > \frac{\prod_{q|N_{n+1}}\frac{q}{q-1}}{e^{\gamma}\times\log\log N_{n+1}}$$

which is the same as

$$\frac{\prod_{q|N_n}\frac{q}{q-1}}{\log\log N_n} > \frac{\prod_{q|N_{n+1}}\frac{q}{q-1}}{\log\log N_{n+1}}.$$

However, we know that

$$\prod_{q|N_{n+1}}\frac{q}{q-1} = \frac{p_{n+1}}{p_{n+1}-1}\times\prod_{q|N_n}\frac{q}{q-1}.$$

In this way, we have that

$$\frac{1}{\log\log N_n} > \frac{\frac{p_{n+1}}{p_{n+1}-1}}{\log\log N_{n+1}}$$

which is equivalent to

$$\frac{\log\log N_{n+1}}{\log\log N_n} > \frac{p_{n+1}}{p_{n+1}-1}$$

that is equal to

$$\frac{\log\theta(p_{n+1})}{\log\theta(p_n)} > \frac{p_{n+1}}{p_{n+1}-1}.$$

3 Main Theorem

Theorem 3.1 *When $n\to\infty$:*

$$\frac{\log\theta(p_{n+1})}{\log\theta(p_n)} > \frac{p_{n+1}}{p_{n+1}-1}.$$

Proof We know that

$$\begin{aligned}
\log\theta(p_n) &= \log\log N_n\\
&= \log\log\frac{N_{n+1}}{p_{n+1}}\\
&= \log\left(\log N_{n+1} - \log(p_{n+1})\right)\\
&= \log\left(\log N_{n+1}\times(1-\frac{\log(p_{n+1})}{\log N_{n+1}})\right)\\
&= \log\log N_{n+1} + \log(1-\frac{\log(p_{n+1})}{\log N_{n+1}})\\
&= \log\theta(p_{n+1}) + \log(1-\frac{\log(p_{n+1})}{\theta(p_{n+1})}).
\end{aligned}$$

In this way, we have that

$$\frac{\log\theta(p_{n+1})}{\log\theta(p_n)} = \frac{\log\theta(p_{n+1})}{\log\theta(p_{n+1}) + \log(1 - \frac{\log(p_{n+1})}{\theta(p_{n+1})})}.$$

We use the theorem 1.5 to show that

$$-\frac{\log(p_{n+1})}{\theta(p_{n+1})} > \log(1 - \frac{\log(p_{n+1})}{\theta(p_{n+1})})$$

since $0 > -\frac{\log(p_{n+1})}{\theta(p_{n+1})} > -1$ for $p_{n+1} > 2$. Hence, we would have that

$$\frac{\log\theta(p_{n+1})}{\log\theta(p_n)} > \frac{\log\theta(p_{n+1})}{\log\theta(p_{n+1}) - \frac{\log(p_{n+1})}{\theta(p_{n+1})}}.$$

Then, it is enough to prove that

$$\frac{\log\theta(p_{n+1})}{\log\theta(p_{n+1}) - \frac{\log(p_{n+1})}{\theta(p_{n+1})}} \geq \frac{p_{n+1}}{p_{n+1} - 1}.$$

However, due to the theorem 1.2, we know that

$$\lim_{n\to\infty} \frac{\theta(p_{n+1})}{p_{n+1}} = 1.$$

If we replace the value of $\theta(p_{n+1})$ by p_{n+1} in the following expression:

$$\frac{\log\theta(p_{n+1})}{\log\theta(p_{n+1}) - \frac{\log(p_{n+1})}{\theta(p_{n+1})}}$$

then, we obtain that

$$\begin{aligned}
\frac{\log p_{n+1}}{\log p_{n+1} - \frac{\log(p_{n+1})}{p_{n+1}}} &= \frac{\log p_{n+1}}{\log p_{n+1}} \times \frac{1}{1 - \frac{1}{p_{n+1}}} \\
&= \frac{1}{1 - \frac{1}{p_{n+1}}} \\
&= \frac{p_{n+1}}{p_{n+1}} \times \frac{1}{1 - \frac{1}{p_{n+1}}} \\
&= \frac{p_{n+1}}{p_{n+1} - 1}.
\end{aligned}$$

Therefore, the proof is complete.

References

1. Choie, Y., Lichiardopol, N., Moree, P., Solé, P.: On Robin's criterion for the Riemann hypothesis. Journal de Théorie des Nombres de Bordeaux **19**(2), 357–372 (2007). DOI 10.5802/jtnb.591
2. Choie, Y., Planat, M., Solé, P.: On Nicolas criterion for the Riemann hypothesis. arXiv preprint arXiv:1012.3613 (2010). URL `https://arxiv.org/pdf/1012.3613.pdf`
3. Grönwall, T.H.: Some asymptotic expressions in the theory of numbers. Transactions of the American Mathematical Society **14**(1), 113–122 (1913). DOI 10.2307/1988773
4. Kozma, L.: Useful Inequalities. `http://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf` (2021). Accessed on 2021-07-12
5. Mertens, F.: Ein Beitrag zur analytischen Zahlentheorie. J. reine angew. Math. **1874**(78), 46–62 (1874). DOI 10.1515/crll.1874.78.46. URL `https://doi.org/10.1515/crll.1874.78.46`
6. Nicolas, J.L.: Petites valeurs de la fonction d'Euler et hypothese de Riemann. Séminaire de Théorie des nombres DPP, Paris **82**, 207–218 (1981)
7. Nicolas, J.L.: Petites valeurs de la fonction d'Euler. Journal of number theory **17**(3), 375–388 (1983). DOI 10.1016/0022-314X(83)90055-0