Uncertainty evaluation in coordinate metrology: approximate models of CMM behaviour

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CMM uncertainty evaluation

- Uncertainty evaluation a key component of the quality infrastructure: traceability, reproducibility, inter-operability
- CMM used extensively in industry
- CMM uncertainty evaluation not straightforward
- Large number of influence factors: geometrical, environmental
- EUCoM project: practical observation-based methods, model-based methods

Motivation

- Guide to the expression of uncertainty in measurement (GUM, coordinate metrology)
- Model based approach, measurement equation relating the measurand to influence factors
- Principled, probabilistic approach
- Assign distributions (uncertainties) to influence factors
- Calculate sensitivity of measurand with respect to influence factors
- Use the law of propagation of uncertainties to associate an uncertainty with the measurand
- Measurement systems analysis, gauge R&R observational approach: change the influence factors, observe the change in the measurement results

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Implementing a model-based approach

- We need a model that is:
  - plausible, reflecting actual behaviours of CMMs
  - valid for uncertainty evaluation
- We are not using it for error correction, only assessing uncertainty
- We would like to specify the model straightforwardly
  - MPE statement $A + d/B$
- We would like to perform calculations using the model straightforwardly

Basic model for measurement along a single measuring line

- Measurement along a line (call it x)

$$x = x^* + e + \epsilon$$

where

  - x is the recorded position
  - x^* is the 'true' position along the line,
  - e represents a systematic effect, that persists over a measurement cycle, and
  - ϵ a random effect (repeatability effect)

- We only know x, everything else is uncertain.
- Use some information/model about the likely behaviour the measurement system to determine an estimate $\hat{x}$ of x^* and its associated uncertainty, e.g., $\hat{x} = x$, $u(x)$

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Model incorporating a scale effect

- Importance of temperature in affecting scale
- Simple model incorporating a scale effect, measurement of two points along the line

$$x_1 = x_1^*(1 + e) + \epsilon_1, \quad x_2 = x_2^*(1 + e) + \epsilon_2$$

- Distance between x_i^* and x_2^* (x_1, x_2 known, everything else uncertain):

$$\begin{aligned} x_2^* - x_1^* &= x_2 - x_1 + (x_2^* - x_1^*)e + \epsilon_2 - \epsilon_1, \\ &\approx x_2 - x_1 + (x_2 - x_1)e + \epsilon_2 - \epsilon_1 \end{aligned}$$

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Statistical model for influence factors

- Model for $d_{12} = d(x_1^*, x_2^*) = x_2^* - x_1^*$:

$$x_2^* - x_1^* \approx x_2 - x_1 + (x_2 - x_1)e + \epsilon_2 - \epsilon_1$$

- Statistical model: : $e \in \mathcal{N}(0, \sigma_M^2), \quad \epsilon_1, \epsilon_2 \in \mathcal{N}(0, \sigma_S^2)$
- Estimates of influence factors: $\hat{e} = 0, \quad \hat{\epsilon}_1 = \hat{\epsilon}_2 = 0$
- Standard uncertainties associated with influence factors:

$$u(e) = \sigma_M, \quad u(\epsilon_1) = u(\epsilon_2) = \sigma_S$$

- Rules: estimates add linearly, uncertainties add in quadrature:

$$c = a + Kb, \quad \hat{c} = \hat{a} + K\hat{b}, \quad u^2(c) = u^2(a) + K^2u^2(b)$$

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Estimate of the distance

- Model for $d_{12} = d(x_1^*, x_2^*) = x_2^* - x_1^*$:

$$d_{12} = x_2 - x_1 + (x_2 - x_1)e + \epsilon_2 - \epsilon_1$$

- Estimates of influence factors: $\hat{e} = 0, \quad \hat{\epsilon}_1 = \hat{\epsilon}_2 = 0$
- Estimate $\hat{d}_{12}$ of d_{12}:

$$\hat{d}_{12} = x_2 - x_1 + (x_2 - x_1)\hat{e} + \hat{\epsilon}_2 + \hat{\epsilon}_1 = x_2 - x_1$$

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Uncertainty associated with the distance estimate

- Model for $d_{12} = d(x_1^*, x_2^*) = x_2^* - x_1^*$:

$$d_{12} = x_2 - x_1 + (x_2 - x_1)e + \epsilon_2 - \epsilon_1$$

- Standard uncertainties associated with influence factors:

$$u(e) = \sigma_M, \quad u(\epsilon_1) = u(\epsilon_2) = \sigma_S$$

- Standard uncertainty $u(d_{12})$ associated with $\hat{d}_{12}$ (sum in quadrature)

$$\begin{aligned} u^2(d_{12}) &= (x_2 - x_1)^2 u^2(e) + u^2(\epsilon_1) + u^2(\epsilon_2) \\ &= (x_2 - x_1)^2 \sigma_M^2 + 2\sigma_S^2, \end{aligned}$$

or

$$u(d_{12}) = \sqrt{2\sigma_S^2 + d_{12}^2 \sigma_M^2} = \sqrt{A^2 + d^2/B^2}$$

- Absolute component, length dependent component

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Models derived from an MPE-type statement

- Length measuring capability:

$$|\hat{d}_{12} - d_{12}| \leq A + d_{12}/B$$

- Basic statistical model: $x_1 = x_1^* + e_1 + \epsilon_1$, $x_2 = x_2^* + e_2 + \epsilon_2$, with

$$e_1, e_2 \in \mathcal{N}(0, \sigma_M^2), \quad \epsilon_1, \epsilon_2 \in \mathcal{N}(0, \sigma_S^2)$$

- MPE implies $|e_2 - e_1 + \epsilon_2 - \epsilon_1| \leq A + d_{12}/B$
- MPE implies $2\sigma_S^2 \leq A^2$
- For $\sigma_S \ll \sigma_M$,

$$e_1 - (A + d_{12}/B) \leq e_2 \leq e_1 + (A + d_{12}/B)$$

- Spatial correlation: for x_2 near x_1, e_2 is similar to e_1

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MPE-type statements and error models

- Length measuring capability:

$$|\hat{d} - d| \leq A + d/B$$

- Error model:

$$x = x^* + e(x^*) + \epsilon \approx x^* + e(x) + \epsilon, \quad \epsilon \in \mathcal{N}(0, \sigma_S^2)$$

- MPE implies $|e(x_2) - e(x_1) + \epsilon_2 - \epsilon_1| \leq A + |x_2 - x_1|/B$
- For $\sigma_S \ll \sigma_M$, $x_1 \neq x_2$,

$$\left| \frac{e(x_2) - e(x_1)}{x_2 - x_1} \right| \leq \frac{A}{|x_2 - x_1|} + \frac{1}{B}$$

- The slope of the error function $e(x)$ cannot be too large.

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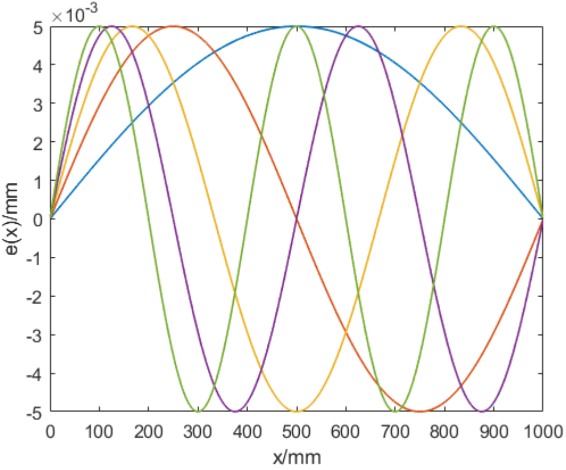
Candidate error functions: Fourier series, polynomials



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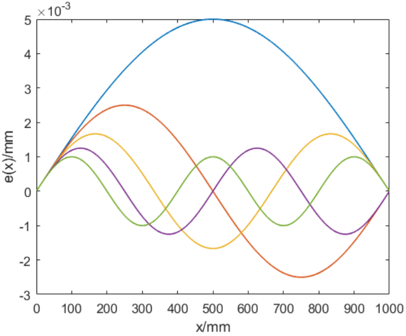
Candidate error functions, slope constraints

Higher frequency components dampened down



Spatial correlation

- Basic model

$$x = x^* + e + \epsilon, \quad e \in \mathcal{N}(0, \sigma_M^2), \quad \epsilon \in \mathcal{N}(0, \sigma_S^2)$$

- Spatial correlation: if x_1 is 'close' to x_2, then e_1 should be 'close' to e_2
- Implement in terms of a correlation kernel

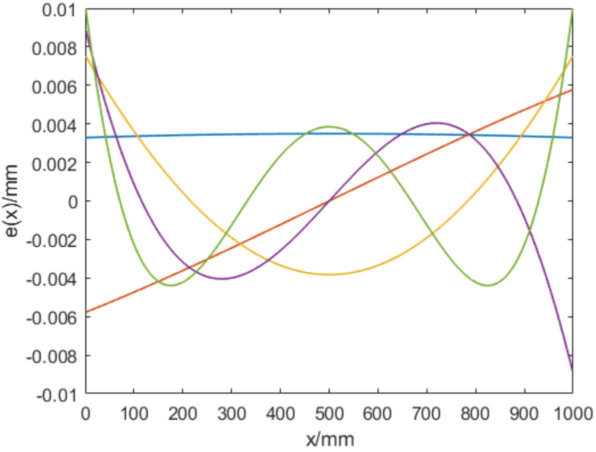
$$\mathrm{cov}(e_1, e_2) = \sigma_M^2 k(|x_2 - x_1|/\lambda)$$

for some function $k(r)$, e.g., $k(r) = e^{-r^2}$.

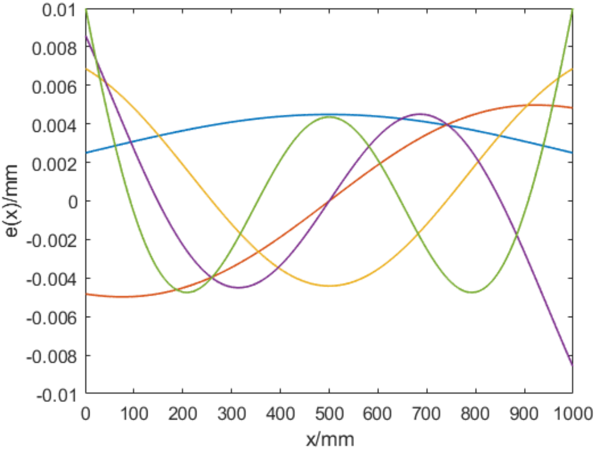
- The length scale parameter λ specifies 'closeness'
- For this case

$$u^2(e_2 - e_1) = 2\sigma_M^2 \left(1 - e^{-(d_{12}/\lambda)^2}\right)$$

Spatially correlated error functions, $\lambda = 2000$ mm



Spatially correlated error functions, $\lambda = 500$ mm



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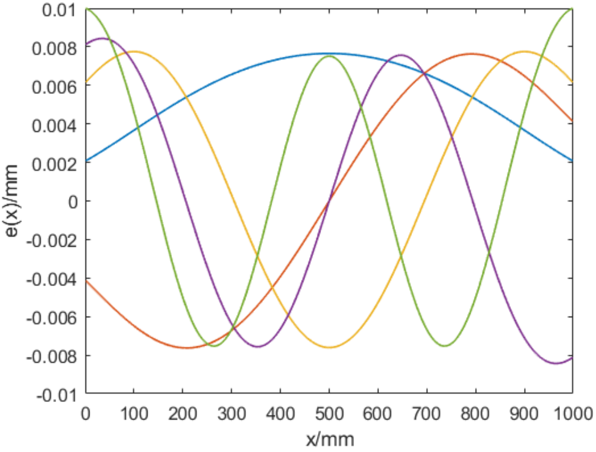
Spatially correlated error functions, $\lambda = 200$ mm



Model involving three length scales, explicit scale effect

- Random effects: very short length scale ($\lambda_S \approx 0$)
- Scale effect: very long length scale ($\lambda_L > L$)
- Spatially correlated systematic effect with medium length scale: $0 < \lambda_M < L$

$$x = x^*(1 + e_L) + e_M + \epsilon,$$

where

$$e_L \in \mathcal{N}(0, \sigma_L^2), \quad e_M \in \mathcal{N}(0, \sigma_M^2), \quad \epsilon \in \mathcal{N}(0, \sigma_S^2)$$

- Uncertainty associated with a distance $d_{12} = |x_2 - x_1|$:

$$u^2(d_{12}) = 2\sigma_S^2 + d_{12}^2\sigma_L^2 + 2\sigma_M^2 \left(1 - e^{-(d_{12}/\lambda_M)^2}\right)$$

- $u(d_{12})$ can be compared with $A + d_{12}/B$

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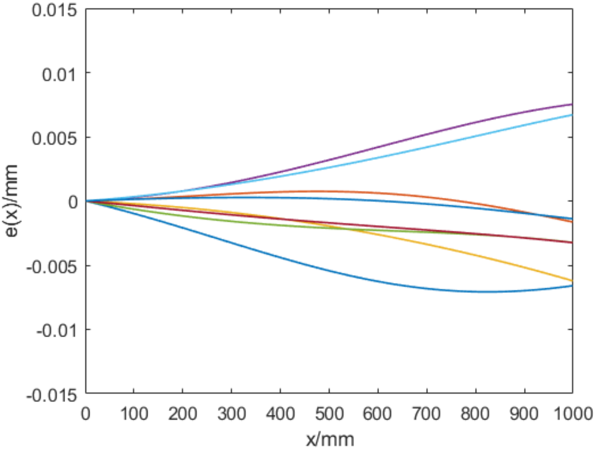
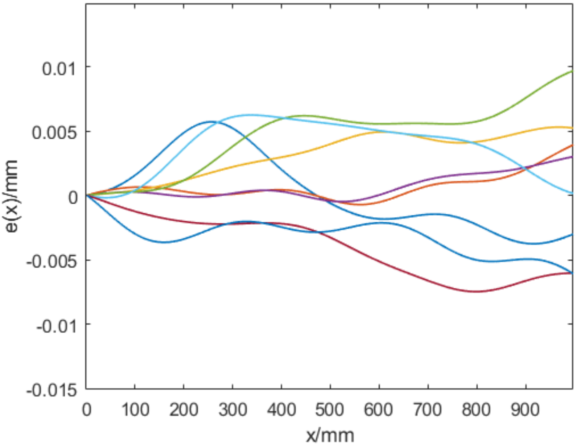
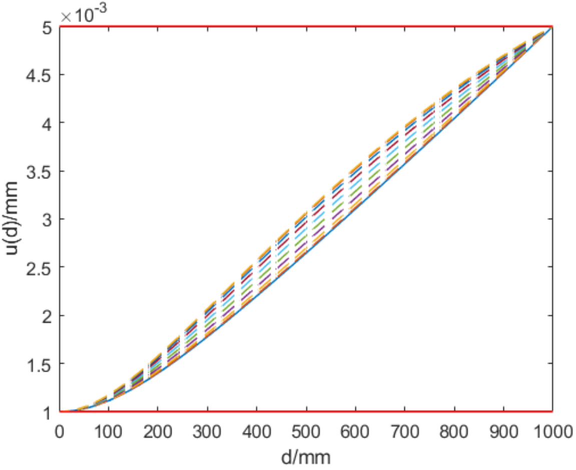
Plausible error functions, scale effect, $\lambda_M = 1000$ mm



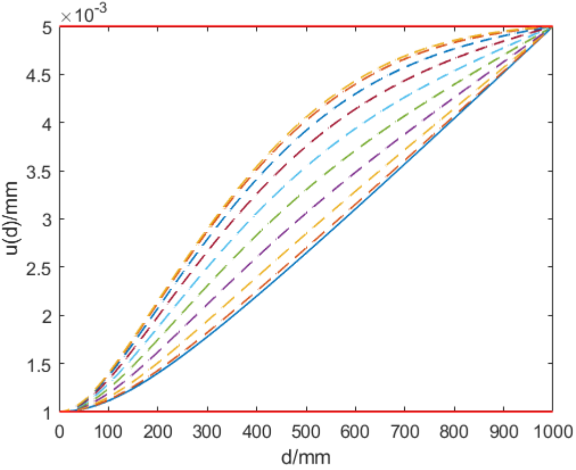
Plausible error functions, scale effect, $\lambda_M = 200$ mm



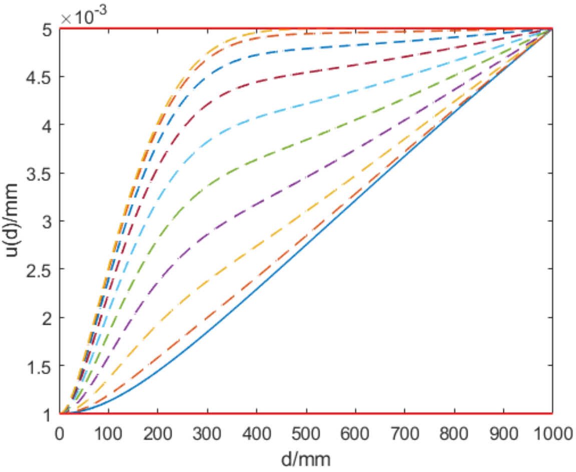
Uncertainties for distances, scale effect, $\lambda_M = 1000$ mm



Uncertainties for distances, scale effect, $\lambda_M = 500$ mm



Uncertainties for distances, scale effect, $\lambda_M = 200$ mm



Model for three dimensions

- Three length scales: short (random effects), long (scale and squareness), medium (spatially correlated effects)
- Scale and squareness model in three dimensions

$$\begin{aligned} x &= x^* + (e_L + e_{L,xx})x + e_{L,xy}y + e_{L,xz}z \\ y &= y^* + (e_L + e_{L,yy})y + e_{L,yz}z \\ z &= z^* + (e_L + e_{L,zz})z \end{aligned}$$

where

$$e_L \in \mathcal{N}(0, \sigma_L^2), \quad e_{L,xx}, e_{L,yy}, e_{L,zz} \in \mathcal{N}(0, \sigma_{L,a}^2),$$

and

$$e_{L,xy}, e_{L,xz}, e_{L,yz} \in \mathcal{N}(0, \sigma_Q^2)$$

- e_L represents a global scale effect
- $e_{L,xx}$ etc., represent additional scale effects for each axis
- e_{xy} etc., represent squareness effects, x-axis to y-axis, etc.

Scale and squareness effects: uncertainties in distances

- Scale and squareness model in three dimensions

$$
\begin{aligned}
x &= x^* + (e_L + e_{L,xx})x + e_{L,xy}y + e_{L,xz}z \\
y &= y^* + (e_L + e_{L,yy})y + e_{L,yz}z \\
z &= z^* + (e_L + e_{L,zz})z
\end{aligned}
$$

- If $d_{12} = \|\boldsymbol{x}_2 - \boldsymbol{x}_1\|$, then

$$u^2(d_{12}) = \sigma_L^2 d_{12}^2 + \sigma_{L,a}^2 D_{L,a}^2 + \sigma_Q^2 D_Q^2,$$

where

$$D_{L,a}^2 = \frac{1}{d_{12}^2}\left[x_{12}^4 + y_{12}^4 + z_{12}^4\right],$$

and

$$D_Q^2 = \frac{1}{d_{12}^2}\left[x_{12}^2 y_{12}^2 + x_{12}^2 z_{12}^2 + y_{12}^2 z_{12}^2\right].$$

with $x_{12} = x_2 - x_1$, etc.

- Non isotropic behaviour: measurement along an axis different from measurement along diagonals

Scale and squareness effects: uncertainties in distances II

- If $d_{12} = \|\boldsymbol{x}_2 - \boldsymbol{x}_1\|$ and $d_{34} = \|\boldsymbol{x}_4 - \boldsymbol{x}_3|$, then

$$u^2(d_{12} - d_{34}) = \sigma_L^2(d_{12} - d_{34})^2 + \sigma_{L,a}^2 D_{L,a}^2 + \sigma_Q^2 D_Q^2,$$

where

$$D_{L,a}^2 = \left(\frac{x_{12}^2}{d_{12}} - \frac{x_{34}^2}{d_{34}}\right)^2 + \left(\frac{y_{12}^2}{d_{12}} - \frac{y_{34}^2}{d_{34}}\right)^2 + \left(\frac{z_{12}^2}{d_{12}} - \frac{z_{34}^2}{d_{34}}\right)^2$$

and $D_Q^2 =$

$$\left(\frac{x_{12}y_{12}}{d_{12}} - \frac{x_{34}y_{34}}{d_{34}}\right)^2 + \left(\frac{x_{12}z_{12}}{d_{12}} - \frac{x_{34}z_{34}}{d_{34}}\right)^2 + \left(\frac{y_{12}z_{12}}{d_{12}} - \frac{y_{34}z_{34}}{d_{34}}\right)^2$$

Scale and squareness effects: III

- Two diameters of length d along the x- and y-axes
- For this case, $d = d_{12} = d_{34}$, $x_{12} = d$, $y_{34} = d$ and

$$u^2(d_{12} - d_{34}) = 2d^2\sigma_{L,a}^2.$$

- Two diameters of length d along the $x = y$ and $x = -y$
- For this case, $d = d_{12} = d_{12}$, $x_{12} = y_{12} = x_{34} = -y_{34} = d/\sqrt{2}$ and

$$u^2(d_{12} - d_{34}) = \sigma_Q^2 d^2$$

- Likely impact on evaluated form error depends on $\sigma_{L,a}d$ and $\sigma_Q d$.

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Model in 3D with three length scales

- Length scales: random, scale and squareness, spatially correlated effects:

$$\boldsymbol{x} = \boldsymbol{x}^* + \boldsymbol{e}_L + \boldsymbol{e}_M + \epsilon,$$

where $\boldsymbol{e}_L$ is a scale and squareness effect, $\boldsymbol{e}_M$ is a spatially correlated effect with length scale λ_M

$$u^2(d_{12}) = 2\sigma_S^2 + \sigma_L^2 d_{12}^2 + \sigma_Q^2 c_{12}^2 + 2\sigma_M^2 \left(1 - e^{-d_{12}^2/\lambda_M^2}\right).$$

- $u^2(d_{12})$ can be compared with an $A + d/B$ statement.
- Example: two diameters of length d, contribution from spatially correlated effects

$$u_M^2(d_{12} - d_{34}) \leq 4\sigma_M \left(1 - e^{-d^2/\lambda_M^2}\right),$$

maximised when the diameters are a 90 degrees to each other.

Example: quantifying Abbe effects

- Measurement of a length standard against a calibrated standard
- Two measuring lines parallel to the x-axis, one offset by L_0 mm from the other.
- Four measurement points $\boldsymbol{x}_1 = (0.0, 0.0, 0.0)$, $\boldsymbol{x}_2 = (500.0, 0.0, 0.0)$, $\boldsymbol{x}_3 = (0.0, L_0, 0.0)$ and $\boldsymbol{x}_4 = (500.0, L_0, 0.0)$
- Scale and squareness effects make no contribution to $u(d_{12} - d_{34})$
- Contribution from random effects and spatially correlated effects:

$$4\sigma_S^2 \leq u^2(d_{12} - d_{34}) \approx 4\sigma_S^2 + 4\sigma_M^2 \left(1 - e^{-L_0^2/\lambda_M^2}\right) \leq 4(\sigma_S^2 + \sigma_M^2)$$

Uncertainties associated with point clouds, derived features

- So far looked at uncertainties associated with distances ($A + d/B$ statements)
- Statistical model:

$$\boldsymbol{x}_i = \boldsymbol{x}_i^* + \boldsymbol{e}_{L,i} + \boldsymbol{e}_{M,i} + \epsilon_i, \quad i = 1, 2, \ldots, m$$

- Point cloud variance $3m \times 3m$ variance matrix V_X
- Diagonal elements are $u^2(\boldsymbol{x}_i)$
- Derived parameters $\boldsymbol{a} = (a_1, \ldots, a_n)^\top = \boldsymbol{a}(\boldsymbol{x}_{1:m})$ depending on data $X = \boldsymbol{x}_{1:m}$
- $n \times 3m$ sensitivity matrix G_{AX} evaluating change in a_j due to a change in each coordinate in $\boldsymbol{x}_{1:m}$
- Variance V_A associated with $\boldsymbol{a}$ is given by matrix multiplication

$$V_A = G_{AX} V_X G_{AX}^\top$$

- The ith diagonal element of V_A is $u^2(a_j)$.

Derived feature calculations

- Example: uncertainties associated with distances:
- 6×6 variance matrix associated with $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, $V_{12} =$

$$\begin{bmatrix} u^2(x_1) & u(x_1,y_1) & u(x_1,z_1) & u(x_1,x_2) & u(x_1,y_2) & u(x_1,z_2) \\ & u^2(y_1) & u(y_1,z_1) & u(y_1,x_2) & u(y_1,y_2) & u(y_1,z_2) \\ & & u^2(z_1) & u(z_1,x_2) & u(z_1,y_2) & u(z_1,z_2) \\ & & & u^2(x_2) & u(x_2,y_2) & u(x_2,z_2) \\ & & & & u^2(y_2) & u(y_2,z_2) \\ & & & & & u^2(z_2) \end{bmatrix}$$

- Uncertainty in the distance $d_{12} = \|\boldsymbol{x}_2 - \boldsymbol{x}_1\|$ is given by

$$u^2(d_{12}) = \begin{bmatrix} \boldsymbol{n}_{12} \\ -\boldsymbol{n}_{12} \end{bmatrix}^\top V_{12} \begin{bmatrix} \boldsymbol{n}_{12} \\ -\boldsymbol{n}_{12} \end{bmatrix}, \quad \boldsymbol{n}_{12} = \frac{\boldsymbol{x}_2 - \boldsymbol{x}_1}{d_{12}}$$

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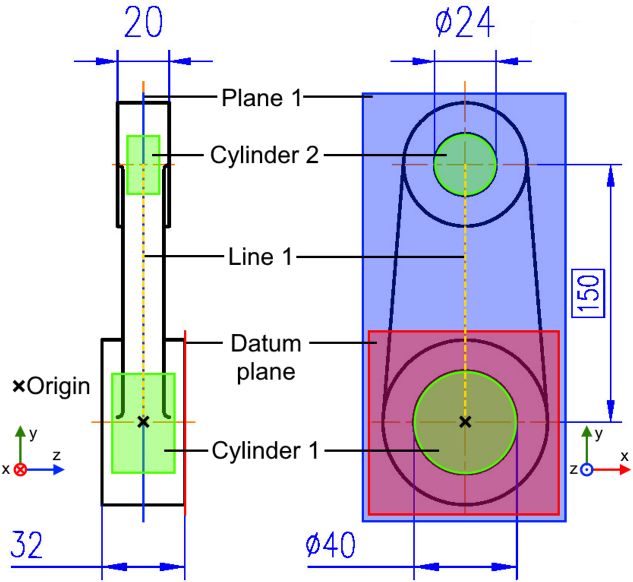
Example: connecting rod



Key characteristics and model

- Key characteristics
  - Radii of the two circles, nominally $r_{0,1} = 20$ mm, $r_{0,2} = 12$ mm
  - Distance D between the two circle centres, nominally $D = 120$ mm
  - Form error associated with the two circles, contribution associated with CMM measurement
- Model involving three length scales: scale (long), random effects, (short), spatially correlated effects (medium)

$$\boldsymbol{x} = \boldsymbol{x}^* + \boldsymbol{e}_L + \boldsymbol{e}_M + \epsilon$$

- Statistical parameters σ_S, σ_L (scale), σ_M and λ_M

Measurement strategy, calculations

- Align artefact with a CMM axis, e.g., x-axis
- Measurement strategy: m points uniformly spaced around each circle, special case $m = 4$
- Determine Gaussian (least squares) associated circle to measured data to determine estimates $\hat{\boldsymbol{a}}_1$ and $\hat{\boldsymbol{a}}_2$ of $\boldsymbol{a}_1 = (x_{0,1}, y_{0,2}, r_{0,1})$ and $\boldsymbol{a}_2$
- Estimate $\hat{D}$ of D given by $\hat{D} = \hat{x}_{0,2} - \hat{x}_{0,1}$
- Derived features (centres, radii) only depend on uncertainty contributions orthogonal (normal) to the profile/surface

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Uncertainty contribution associated with random effects (short length scales)

- For a uniform distribution of$m$ points around a circle $(x_i, y_i) = (r_0 \cos\theta_i, r_0, \sin\theta_i,$

$$u^2(x_0) = u^2(y_0) = \frac{2\sigma_S^2}{m}, \quad u^2(r_0) = \frac{\sigma_S^2}{m}$$

- Uncertainty associated with the distance D between centres

$$u^2(D) = u^2(x_{0,1}) + u^2(x_{0,2}) = \frac{4}{m}\sigma_S^2$$

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Uncertainty contribution associated with scale effects (long length scales)

- Circle radius

$$u(r_0) = \sigma_L r_0$$

- Uncertainty associated with the distance D between centres (measurement along an axis)

$$u(D) = \sigma_L D$$

Uncertainty contribution from spatially correlated effects

- Special case $m = 4$, measurement points $(\pm r_0, 0)$ and $(0, \pm r_0)$

$$\hat{x}_0 = \frac{x_1 + x_2}{2}, \quad \hat{y}_0 = \frac{y_3 + y_4}{2}$$

- Uncertainties associated with circle parameters

$$u^2(x_0) = u^2(y_0) = \frac{\sigma_M^2}{2}(1 + e^{-4r_0^2/\lambda^2}), \quad u^2(r_0) = \frac{\sigma_M^2}{4}(1 - e^{-4r_0^2/\lambda^2})$$

- Uncertainty associated with the estimate of D:

$$u^2(D) \approx \frac{\sigma_M^2}{2}\left(2 + e^{-4r_{0,1}^2/\lambda^2} + e^{-4r_{0,2}^2/\lambda^2} - 4e^{-D^2/\lambda^2}\right)$$

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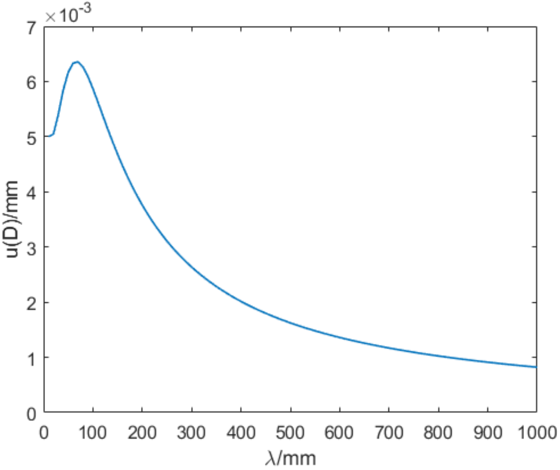
Example: uncertainty contribution from spatially correlated effects, $\sigma_M = 0.005$ mm



Estimating statistical parameters

- Model based on a small number of statistical parameters σ_S^2, σ_L^2, σ_Q^2, σ_M^2 and λ_M
- In general: $u(d) \leq (A + d/B)/k$, $2 \leq k \leq 3$
- σ_S relates directly to repeatability and A in $A + d/B$, $2\sigma_S^2 \leq A^2/k^2$
- σ_L relates to scale effects and B: $\sigma_L d \leq (A + d/B)/k$
- σ_Q relates to scale effects
- σ_M relates to kinematic errors, form errors
- λ_M: not too large $\lambda_M < L$, not too small, $\lambda_M \approx 100$ mm (can use more than one λ)
- Estimating the balance between scale errors and spatially correlated errors, try a few

$$\sigma_L^2 d^2 + 2\sigma_M^2(1 - e^{-d^2/\lambda^2}) \leq (A + d/B)^2/k^2$$

Summary

- Length measuring capability statement $|\hat{d} - d| \leq A + d/B$
- Constraint on behaviour over short and longer length scales
- Statistical interpretation in terms of random effects associated short, medium and long length scales
- Long length scale behaviour described in terms of scale and squareness errors
- Behaviour characterised by a small number of statistical parameters σ_S^2, σ_L^2, σ_Q^2, σ_M^2 and λ_M
- Straightforward uncertainty calculus based on the GUM and the law of propagation of uncertainty
- Formulæ for uncertainties associated with distances, other cases
- Implementation in spreadsheets straightforward in principle: all calculations are direct

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