THE RIEMANN HYPOTHESIS

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Abstract. Robin criterion states that the Riemann Hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. This is known as the Robin inequality. We obtain a contradiction just assuming the smallest counterexample of the Robin inequality exists for some $n > 5040$. In this way, we prove that the Robin inequality is true for all $n > 5040$. Consequently, the Riemann Hypothesis is also true.

1. Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of n [Cho+07]:

$$\sum_{d|n} d$$

where $d \mid n$ means the integer d divides to n. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

The constant γ is the Euler-Mascheroni constant, and log is the natural logarithm. The importance of this property is:

Theorem 1.1. [RH] *If* $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *if and only if the Riemann Hypothesis is true [Rob84].*

We demonstrate that there is a contradiction just assuming the existence of the smallest number $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold. By contraposition, we show that $\mathsf{Robins}(n)$ holds for all $n > 5040$ and thus, the Riemann Hypothesis is true.

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2. A Basic Case

We can easily prove that $\mathsf{Robins}(n)$ holds for certain kind of numbers:

Lemma 2.1. [less-than-7] $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when* $q \leq 5$, *where* q *is the largest prime divisor of* n.

Proof. Let $n > 5040$ and let all its prime divisors be $q_1 < \cdots < q_m \leq 5$, then we need to prove

$$f(n) < e^{\gamma} \times \log \log n$$

that is true when

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq e^{\gamma} \times \log \log n$$

is also true. Certainly, for $n \geq 2$ [Cho+07]:

$$f(n) < \prod_{q \mid n} \frac{q}{q-1}.$$

For $q_1 < \cdots < q_m \leq 5$,

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^{\gamma} \times \log \log(5040) \approx 3.81.$$

However, we note that for $n > 5040$

$$e^{\gamma} \times \log \log(5040) < e^{\gamma} \times \log \log n$$

and therefore, the proof is complete when $q_1 < \cdots < q_m \leq 5$. □

3. Some Useful Inequalities

The following lemma is a very helpful inequality:

Lemma 3.1. [1-ineq] *For* $x > 0$, *we have*

$$\frac{x}{1-x} \leq \frac{1}{y + y^2 + \frac{y^3}{2}}$$

where $y = 1 - x$.

Proof. For $k > -1$, we know $1 + k \leq e^k$ [Koz20]. Therefore,

$$\frac{x}{1-x} \leq \frac{e^{x-1}}{1-x} = \frac{1}{(1-x) \times e^{1-x}} = \frac{1}{y \times e^{y}}.$$

However, for every real number $y \in \mathbb{R}$ [Koz20]:

$$y \times e^{y} \geq y + y^2 + \frac{y^3}{2}$$

and this can be transformed into

$$\frac{1}{y \times e^y} \leq \frac{1}{y + y^2 + \frac{y^3}{2}}.$$

Consequently, we show

$$\frac{x}{1-x} \leq \frac{1}{y + y^2 + \frac{y^3}{2}}.$$

□

This is another inequality that we use:

Lemma 3.2. [2-ineq] *For* $x \geq 2$,

$$\frac{x}{x-1} \geq e^{\frac{1}{x}}.$$

Proof. If we apply the logarithm to the both sides of the inequality, then we obtain that

$$\log \frac{x}{x-1} \geq \frac{1}{x}.$$

We know that

$$\log \frac{x}{x-1} = \log(1 + \frac{1}{x-1}).$$

For $x > -1$ [Koz20]:

$$\frac{x}{x+1} \leq \log(1+x).$$

We use this property to show that:

$$\log(1 + \frac{1}{x-1}) \geq \frac{\frac{1}{x-1}}{1 + \frac{1}{x-1}} = \frac{1}{(x-1) \times (1 + \frac{1}{x-1})} = \frac{1}{x}.$$

Therefore, the proof is complete. □

Here, it is another practical inequality:

Lemma 3.3. [property] *Suppose that* $n > 5040$ *and let* $n = r \times q$, *where* q *denotes the largest prime factor of* n *and* $r > 1$ *is a natural number. We have that*

$$f(n) \leq (1 + \frac{1}{q}) \times f(r).$$

Proof. Suppose that n is the form of $m \times q^k$ where m and q are coprimes such that m and k are natural numbers. We have that

$$f(n) = f(m \times q^k) = f(m) \times f(q^k)$$

since f is multiplicative and m and q are coprimes [Voj20]. However, we know that

$$f(q^k) \leq f(q^{k-1}) \times f(q)$$

because of we notice that $f(a \times b) \leq f(a) \times f(b)$ when $a, b \geq 2$ [Voj20]. In this way, we obtain that

$$f(q^{k-1}) \times f(q) = f(q^{k-1}) \times (1 + \frac{1}{q})$$

according to the value of $f(q) = (1+\frac{1}{q})$ [Voj20]. In addition, we analyze that

$$f(m) \times f(q^{k-1}) = f(m \times q^{k-1}) = f(r)$$

because f is multiplicative and m and q are coprimes [Voj20]. Finally, we obtain that

$$f(n) = f(m) \times f(q^k) \leq f(m) \times f(q^{k-1}) \times f(q) = f(r) \times (1 + \frac{1}{q})$$

and as a consequence, the proof is done. □

4. Proof of Main Theorem

Theorem 4.1. [main] $\mathsf{Robins}(n)$ *holds for all* $n > 5040$.

Proof. Suppose that n is the smallest integer exceeding 5040 that does not satisfy the Robin inequality. Let $n = r \times q$, where q denotes the largest prime factor of n. We prove that $\mathsf{Robins}(n)$ holds for all $n > 5040$ when $q \leq 5$ according to the lemma 2.1 [less-than-7]. As result, this implies that $q > 5$ for this possible counterexample. Recall that $p_1, p_2, \ldots$ denote the consecutive primes. An integer of the form $\prod_{i=1}^{s} p_i^{e_i}$ with $e_1 \geq e_2 \geq \cdots \geq e_s \geq 0$ we will call an Hardy-Ramanujan integer [Cho+07]. A natural number n is called superabundant precisely when, for all $m < n$

$$f(m) < f(n).$$

If n is superabundant, then n is an Hardy-Ramanujan integer [AE44]. Moreover, the smallest counterexample of Robin inequality greater than 5040 must be a superabundant number [AF09]. Consequently, it is necessary that $r \geq 2 \times 3 \times 5 = 30$. In this way, the following inequality

$$f(n) \geq e^{\gamma} \times \log \log n$$

should be true. We know that

$$(1 + \frac{1}{q}) \times f(r) \geq f(q \times r) \geq f(n) \geq e^{\gamma} \times \log \log n$$

due to the lemma 3.3 [property]. Besides, this shows that

$$(1 + \frac{1}{q}) \times e^{\gamma} \times \log \log r > e^{\gamma} \times \log \log n$$

should be also true, because of $f(r) < e^{\gamma} \times \log\log r$. Certainly, if n is the smallest counterexample exceeding 5040 of the Robin inequality, then $\mathsf{Robins}(r)$ holds [Cho+07]. That is the same as

$$(1 + \frac{1}{q}) \times \log\log r > \log\log n.$$

We have that

$$(1 + \frac{1}{q}) \times \log\log r > \log(\log r + \log q)$$

where we notice that

$$\log(a + c) = \log\left(a \times (1 + \frac{c}{a})\right) = \log a + \log(1 + \frac{c}{a})$$

for $a \geq 1$ and $c \geq 1$. This follows as

$$(1 + \frac{1}{q}) \times \log\log r > \log\log r + \log(1 + \frac{\log q}{\log r})$$

since $\log r \geq 1$ and $\log q \geq 1$ for $q > 5$ and $r \geq 30$. This is equal to

$$(1 + q) \times \log\log r > q \times \log\log r + q \times \log(1 + \frac{\log q}{\log r})$$

and thus,

$$\log\log r > q \times \log(1 + \frac{\log q}{\log r}).$$

This implies that

$$\begin{aligned}
\frac{\log\log r}{\log(1 + \frac{\log q}{\log r})} &= \\
\frac{\log\log r}{\log \frac{\log r + \log q}{\log r}} &= \\
\frac{\log\log r}{\log \frac{\log n}{\log r}} &= \\
\frac{\log\log r}{\log\log n - \log\log r} &= \\
\frac{\log\log r}{\log\log n \times (1 - \frac{\log\log r}{\log\log n})} &= \\
\frac{\frac{\log\log r}{\log\log n}}{(1 - \frac{\log\log r}{\log\log n})} &> q
\end{aligned}$$

should be true. If we assume that $y = 1 - \frac{\log\log r}{\log\log n}$, then we analyze that

$$\frac{1}{y + y^2 + \frac{y^3}{2}} \geq \frac{\frac{\log\log r}{\log\log n}}{(1 - \frac{\log\log r}{\log\log n})}$$

because of lemma 3.1 [1-ineq]. As result, we have that

$$\frac{1}{y + y^2 + \frac{y^3}{2}} > q$$

and therefore,

$$\frac{1}{1 + y + \frac{y^2}{2}} > q \times y.$$

Since we have

$$1 + y + \frac{y^2}{2} > 1$$

then

$$\frac{1}{1 + y + \frac{y^2}{2}} < 1.$$

Consequently, we obtain that

$$1 > q \times y$$

which is the same as

$$e > e^{q \times y}.$$

For $y > 0$, we have that $1 + y \leq e^y$ [Koz20] and therefore,

$$e > e^{q \times y} \geq (1 + y)^q = (2 - \frac{\log\log r}{\log\log n})^q$$

that is

$$\sqrt[q]{e} > (2 - \frac{\log\log r}{\log\log n})$$

and finally,

$$1 > (2 - \frac{\log\log r}{\log\log n}) \times \frac{1}{e^{\frac{1}{q}}}.$$

According to the lemma 3.2 [2-ineq], we know that

$$\frac{q}{q-1} \geq e^{\frac{1}{q}}$$

which is equivalent to

$$\frac{q-1}{q} \leq \frac{1}{e^{\frac{1}{q}}}.$$

In this way, we obtain that

$$(2 - \frac{\log\log r}{\log\log n}) \times \frac{1}{e^{\frac{1}{q}}} \geq (2 - \frac{\log\log r}{\log\log n}) \times \frac{q-1}{q}$$

and thus,

$$1 > (2 - \frac{\log\log r}{\log\log n}) \times \frac{q-1}{q}.$$

This the same as

$$\frac{\log\log r}{\log\log n} \times \frac{q-1}{q} > 2 \times \frac{q-1}{q}$$

which is equal to

$$\frac{\log\log r}{\log\log n} \times \frac{q-1}{q} + \frac{2}{q} > 2.$$

We know that

$$\frac{q-1}{q} > \frac{\log\log r}{\log\log n} \times \frac{q-1}{q}$$

since we can assure that $a > c$ and $b > c$ when $c = a \times b$ such that $0 < a < 1$ and $0 < b < 1$. In fact, we note that $0 < \frac{\log\log r}{\log\log n} < 1$ and $0 < \frac{q-1}{q} < 1$. Consequently, we would have that

$$\frac{q-1}{q} + \frac{2}{q} > 2.$$

However, this is contradiction because of

$$\frac{q-1}{q} < 1$$

and

$$\frac{2}{q} < 1$$

for $q > 5$. Indeed, if we sum the previous inequalities, then we can see that

$$\frac{q-1}{q} + \frac{2}{q} < 1 + 1 = 2.$$

Hence, we obtain a contradiction when $n > 5040$ is the possible smallest number such that $\mathsf{Robins}(n)$ does not hold. By contraposition, we have that $\mathsf{Robins}(n)$ holds for all $n > 5040$. □

Theorem 4.2. [conclusion] *The Riemann Hypothesis is true.*

Proof. This is a direct consequence of theorems 1.1 [RH] and 4.1 [main]. □

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