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On completeness in metric spaces and fixed point theorems

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Abstract Complete ultrametric spaces constitute a particular class of the so called, recently, G-complete metric spaces. In this paper we characterize a more general class called weak G-complete metric spaces, by means of nested sequences of closed sets. Then, we also state a general fixed point theorem for a self-mapping of a weak G-complete metric space. As a corollary, every asymptotically regular self-mapping of a weak G-Complete metric space has a fixed point.

Keywords Completeness · fixed point theorem · (non-Archimedean metric) ultrametric.

Mathematics Subject Classification (2000) 54A20 · 54E50 · 54H25

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1 Introduction

A sequence $\{x_n\}$ in a metric space (X, d) satisfying $\lim_n d(x_n, x_{n+1}) = 0$ is called G-Cauchy [14]. This concept is weaker than the classical Cauchy's concept, and it is well known that both concepts agree in an ultrametic space [1]. As usual, a metric (space) has been called G-complete if every G-Cauchy sequence is convergent. A drawback of this concept is that a compact space is not necessarily G-complete [14]. To overcome this inconvenience, in [9] the authors have introduced the concept of weak G-completeness (Definition 2) in such a manner that every compact space is weak G-complete.

Before continuing, it is worth to notice that the corresponding concept of G-completeness in fuzzy setting was introduced by M. Grabiec [6] and it has been extensively used for obtaining fixed point theorems in fuzzy setting [6,7, 12,5,15]. Discussions on this concept can be found in [16,8].

The aim of this paper is, basically, to characterize weak G-complete spaces in a similar way to classical complet metric spaces and to obtain a fixed point theorem. Then, in Theorem 1 we characterize the weak G-completeness by means of nested sequences of non-empty closed sets satisfying that the Hausdorff distance between two consecutive sets of them tends to zero. Consequently, we can state several characterizations of complete ultrametric spaces (Corollary 1), since all the mentioned types of completeness agree in ultrametric spaces. We also observe that in a G-complete metric space it is possible to find, as in the classical case, nested sequences of non-empty closed sets with empty intersection (Example 2).

With respect to the second aim of this paper we introduce a general fixed point theorem for a continuous self-mapping f of a weak G-complete metric space X, under the unique assumption that the classical iterative sequence $\{f^n(x)\}$ to be G-Cauchy, for some $x \in X$ (Theorem 2). As a corollary, every asymptotically regular self-mapping of a weak G-complete metric space has a fixed point. Obviously, the structure on X plays an interesting role in order to obtain fixed point theorems. Indeed, strong structures on X lead to weak contractive conditions on f (compare, for instance, Banach's theorem given for X complete and Edelstein's theorem given for X compact). Then, weak G-completeness concept is strategic to obtain fixed point theorems since this structure is intermediate between completeness and compactness. So, we observe, for instance, that Boyd and Wong's theorem [2] can be stated for weak G-complete spaces by weakening the condition on the gauge function φ which involves f (Remark 2). In Example 5 we compare the usefulness of our Theorem 2 in front of Banach's, Matkowski's and Edelstein's theorems.

We also do some observations about the existence of cluster points for a G-Cauchy sequence. For instance, it is proved that if a G-Cauchy sequence in $\mathbb{R}$ has two distinct cluster points a and b (with $a < b$) then every point of the interval $[a, b]$ is a cluster point of the sequence. As a consequence we give a fixed point theorem for continuous functions defined on closed intervals of $\mathbb{R}$ (Corollary 3). Throughout the paper appropriate examples illustrate the theory.

The structure of the paper is as follows. In Section 3, after the preliminaries, we characterize the weak G-completeness. The short Section 4 is devoted to state the obtained results for ultrametric spaces. In Section 5 we give our fixed point theorems among other considerations.

2 Preliminaries

In the following (X, d) is a metric space. Recall that a sequence $\{x_n\}$ converges to x if $\lim_n d(x, x_n) = 0$ and that $\{x_n\}$ is called Cauchy if $\lim_{n,m} d(x_n, x_m) = 0$. (X, d), or simply X, is called complete if every Cauchy sequence in X is convergent.

Definition 1 [14] A sequence $\{x_n\}$ in X is said to be G-Cauchy if $\lim_n d(x_n, x_{n+1}) = 0$. (X, d), or simply X, is called G-complete if every G-Cauchy sequence in X is convergent.

Definition 2 [8,9] A sequence $\{x_n\}$ in X is called G-convergent if it is G-Cauchy and it has, at least, a cluster point. (X, d), or simply X, is called weak G-complete if every G-Cauchy is G-convergent.

Recall that every Cauchy sequence with a cluster point is convergent and that every sequence in a compact space has a cluster point. Now, it is clear that a Cauchy sequence is G-Cauchy and also that a convergent sequence is G-convergent. So, the following diagram of implications summarizes the relationship among compactness and the distinct concepts of completeness.

$$\begin{array}{ccccc} \text{compact} & \longrightarrow & \text{weak G-complete} & \longrightarrow & \text{complete} \\ & & \uparrow & & \\ & & \text{G-complete} & & \end{array}$$

Our basic reference for general topology is [10].

3 Characterizing weak G-completeness

First, notice that if A, B are two subsets of X with $A \subset B$ then the Hausdorff distance between A and B is given by $d_H(B, A) = \sup\{d(b, A) : b \in B\}$, where $d(b, A) = \inf\{d(b, a) : a \in A\}$. This distance could be infinite but this observation has not any interest in our next context.

Definition 3 Let $\{F_n\}$ be a nested sequence ($F_{n+1} \subset F_n$, $n = 1, 2, \ldots$) of non-empty subsets of X. We will say that $\{F_n\}$ has Hausdorff diameter (H-diameter, for simplicity) zero if $\lim_n d_H(F_n, F_{n+1}) = 0$.

Recall that the diameter of a subset A of (X,d) is $diam(A) = \sup\{d(x,y) : x,y \in A\}$.

Theorem 1 *(X,d) is weak G-complete if and only if every nested sequence of non-empty closed sets with H-diameter zero has a non-empty intersection.*

Proof Suppose that (X,d) is weak G-complete. Let $\{F_n\}$ be a nested sequence of non-empty closed sets which has H-diameter zero. We will prove that $\bigcap_n F_n \neq \emptyset$.

Put $\delta_n = d_H(F_n, F_{n+1})$ for all $n \in \mathbb{N}$. Then, by hypothesis $\lim_n \delta_n = 0$ and, without lose of generality, we can suppose that δ_n is finite for each $n \in \mathbb{N}$.

Take $x_1 \in F_1$. Since $d_H(F_1, F_2) = \delta_1$ we can find $x_2 \in F_2$ such that $d(x_1, x_2) < \delta_1 + 1$. For x_2, analogously, we can find $x_3 \in F_3$ such that $d(x_2, x_3) < \delta_2 + \frac{1}{2}$. In this way we construct, by induction on n, a sequence $\{x_n\}$ where $x_n \in F_n$, for all $n \in \mathbb{N}$, and such that $d(x_n, x_{n+1}) < \delta_n + \frac{1}{n}$. Consequently, the sequence $\{x_n\}$ is G-Cauchy, since $\lim_n d(x_n, x_{n+1}) \leq \lim_n(\delta_n + \frac{1}{n}) = 0$.

Hence, $\{x_n\}$ has a cluster point x, since X is weak G-complete. Suppose that the subsequence $\{x_{n_p}\}$ of $\{x_n\}$ converges to x.

For all $p \in \mathbb{N}$ we have, by construction, that $\{x_{n_p}, x_{n_p+1}, x_{n_p+2}, \dots\} \subset F_{n_p}$. In particular, $\{x_{n_p}, x_{n_{p+1}}, x_{n_{p+2}}, \dots\} \subset F_{n_p}$. So, $x \in F_{n_p}$ for all $p \in \mathbb{N}$, since F_{n_p} is closed. Then $x \in \bigcap_{p=1}^{\infty} F_{n_p} \neq \emptyset$.

Conversely, let $\{x_n\}$ be a G-Cauchy sequence in (X,d), and suppose that each nested sequence of closed (non-empty) subsets of X with H-diameter zero has a non-empty intersection.

Put $A_n = \{x_n, x_{n+1}, x_{n+2}, \dots\}$, and let $F_n = \overline{A_n}$ for all $n \in \mathbb{N}$ ($\overline{A_n}$ denotes the closure of A_n). Then $\{F_n\}$ is a nested sequence of closed subsets of X. We will show that $\{F_n\}$ has H-diameter zero.

Let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+1}) < \frac{\varepsilon}{2}$ for all $n \geq n_0$, since $\{x_n\}$ is G-Cauchy, and consequently $d_H(A_n, A_{n+1}) = d(x_n, A_{n+1}) < \frac{\varepsilon}{2}$ for all $n \geq n_0$.

Let $x \in F_n = \overline{A_n}$. Then, for the open ball $B(x, \frac{\varepsilon}{2})$, centered at x, we have that $B(x, \frac{\varepsilon}{2}) \cap A_n \neq \emptyset$. Take $a \in B(x, \frac{\varepsilon}{2}) \cap A_n$. We have that

$$
\begin{aligned}
d(x, F_{n+1}) &\leq d(x,a) + d(a, F_{n+1}) \leq d(x,a) + d(a, A_{n+1}) \leq \\
&\leq d(x,a) + d_H(A_n, A_{n+1}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for all } n \geq n_0
\end{aligned}
$$

Consequently, $d_H(F_n, F_{n+1}) \leq \varepsilon$ for all $n \geq n_0$. So, $\{F_n\}$ has H-diameter zero, and, by hypothesis, $\bigcap_n F_n \neq \emptyset$.

Now, $\bigcap_n F_n = \bigcap_n \overline{A_n}$ and this last set is the set of cluster points of $\{x_n\}$ [10], and so $\{x_n\}$ has, at least, a cluster point. Hence, $\{x_n\}$ is G-convergent.

By means of the previous theorem we obtain the following result.

Proposition 1 *Let d be the Euclidean metric on $\mathbb{R}^n$. Then $(\mathbb{R}^n, d)$ is not weak G-complete.*

Proof Let $\{a_m\}$ be the harmonic series in $\mathbb{R}$, i.e., $a_m = \sum_{i=1}^{m} \frac{1}{i}$, $m = 1, 2, \ldots$ Consider the open balls in $\mathbb{R}^n$ $B(\mathbf{0}, a_m)$ centered at the origin $\mathbf{0}$ with radius a_m for $m = 1, 2, \ldots$

Let $F_m = \mathbb{R}^n - B(\mathbf{0}, a_m)$ for $m = 1, 2, \ldots$ Obviously, $\{F_m\}$ is a nested sequence of closed sets of $\mathbb{R}^n$.

Now, $\delta_m = d_H(F_m, F_{m+1}) = \frac{1}{m} - \frac{1}{m+1} = \frac{1}{m(m+1)}$. Then $\lim_m d_H(F_m, F_{m+1}) = 0$, i.e., $\{F_m\}$ has H-diameter zero, and $\bigcap_m F_m = \emptyset$. Hence $(\mathbb{R}^n, d)$ is not weak G-complete.

The following example is appropriate to illustrate the diagram of Section 2

Example 1 (A weak G-complete non-compact non-G-complete space)
Let $c, d \in \mathbb{R}$ and $k \in \mathbb{N}$ such that $c < d < k$ and consider $A = [c, d]$ and $B = \{k, k+1, k+2, \ldots\}$. Take $X = A \cup B$ and suppose X endowed with the usual topology of $\mathbb{R}$, restricted to X. Obviously, X is non-compact.

Clearly, a G-Cauchy sequence in X cannot take frequently values in B. So, every G-Cauchy sequence is eventually in A and then it has a cluster point in A, since A is compact. Consequently, X is weak G-complete.

Now, X is not G-complete since A is not G-complete. Indeed, the sequence $\{x_n\}$, where $x_n = c + (d - c)|\sin \sqrt{n}|$, is G-Cauchy in A and obviously $\{x_n\}$ is not convergent.

It is well known that a nested sequence of closed sets in a complete metric space can have empty intersection. Is this fact possible in a G-complete metric space? The answer is affirmative as proves the next example.

Example 2 [13] (Sierpinski's metric space)
Let $X = \{1, 2, 3, \ldots\}$. The function $d(i, j) = 1 + \frac{1}{i+j}$ for $i \neq j$ and $d(i, i) = 0$ for all $i, j \in X$, is a metric on X [13].

Clearly $B(i, \frac{1}{2}) = \{i\}$, so each point of X is isolated and the topology on X generated by d is the discrete topology.

Clearly (X, d) is G-complete since all G-Cauchy sequences are eventually constant.

Let $S_n = \{j \in X : d(j, n) \leq 1 + \frac{1}{2n}\}$. Obviously, $S_n = \{n, n+1, n+2, \ldots\}$. So $\{S_n\}$ is a nested sequence of non-empty closed sets and $\bigcap_n S_n = \emptyset$.

Notice that in a G-complete metric space the intersection of a nested sequence of closed sets with H-diameter zero is not necessarily a unique point, as shows the next example.

Example 3 (A compact G-complete metric space)
Let $X = A \cup B$ where $A = \{0\} \cup \{\frac{1}{n} : n = 1, 2, 3, \ldots\}$, and $B = \{2\} \cup \{2 + \frac{1}{n} : n = 1, 2, 3, \ldots\}$endowed with the usual metric d of $\mathbb{R}$, restricted to X.

Notice that $\{\frac{1}{i}\}$ and $\{2+\frac{1}{i}\}$ are open for each $i = 1, 2, \dots$ and that local bases at 0 and 2 are $\{0\} \cup \{\frac{1}{i}, \frac{1}{i+1}, \frac{1}{i+2}, \dots\}$, and $\{2\} \cup \{2+\frac{1}{i}, 2+\frac{1}{i+1}, 2+\frac{1}{i+2}, \dots\}$, respectively, where $i = 1, 2, \dots$

Obviously (X, d) is compact (and consequently, weak G-complete). We will prove that (X, d) is G-complete.

Let $\{a_n\}$ be a G-Cauchy sequence in X. If $\{a_n\}$ is eventually constant then $\{a_n\}$ is convergent. Suppose that $\{a_n\}$ is not eventually constant and, without lose of generality, that $a_n \neq a_{n+1}$ for all $n = 1, 2, \dots$

Clearly, $\{a_n\}$ cannot be frequently in A and B simultaneously since $d_H(A, B) = 1$ and $\{a_n\}$ is G-Cauchy. Consequently, the sequence is eventually in A or B. Suppose that $\{a_n\}$ is eventually on A. We claim that $\{a_n\}$ converges to 0. To prove it, we suppose the contrary. If $\{a_n\}$ is not convergent to 0 then for some $k_0 \in \mathbb{N}$ we have that $\{a_n\}$ is frequently in $[\frac{1}{k_0-1}, 1]$. Choose $\varepsilon > 0$ such that $0 < \varepsilon < \frac{1}{k_0} - \frac{1}{k_0+1}$. Since $\{a_n\}$ is G-Cauchy, given $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that $|a_n - a_{n+1}| < \varepsilon$ for all $n \geq n_0$. Now, for each $i \in \mathbb{N}$ then exists a_{n_i} with $n_i > n_0$ such that $a_{n_i} \in [\frac{1}{k_0-1}, 1]$ and then $|a_{n_i} - a_{n_i+1}| \geq \frac{1}{k_0} - \frac{1}{k_0+1} > \varepsilon$, a contradiction.

If $\{a_n\}$ is eventually in B then with a similar argument we can prove that $\{a_n\}$ converges to 2. So, (X, d) is G-complete.

Now, consider the nested sequence of closed sets $\{F_n\}$ given by $F_n = A \cup \{2+\frac{1}{n}, 2+\frac{1}{n+1}, 2+\frac{1}{n+2}, \dots\}$ for $n = 1, 2, 3, \dots$ We have that $d_H(F_n, F_{n+1}) = d(2+\frac{1}{n}, 2+\frac{1}{n+1}) = \frac{1}{n(n+1)}$. Then $\lim_n d_H(F_n, F_{n+1}) = 0$ and $\bigcap_n F_n = A \cup \{2\}$.

Then, at the light of Example 3 it arises the following open question.

Question 1 If every nested sequence of non-empty closed sets of X with H-diameter zero has an intersection constituted by a unique element then, is (X, d) G-complete?

4 Only for ultrametrics

Recall that a metric space (X, d) is called an ultrametric space if $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for all $x, y, z \in X$. The following is a well-known result.

Proposition 2 *[1] Let (X, d) be an ultrametric space. Then, a sequence $\{x_n\}$ is Cauchy if and only if it is G-Cauchy.*

The following corollary is a consequence of the above concepts and results.

Corollary 1 *Let (X, d) be an ultrametric space. They are equivalent:*

(i) *(X, d) is G-complete.*
(ii) *(X, d) is weak G-complete.*
(iii) *(X, d) is complete.*
(iv) *Every nested sequence $\{F_n\}$ of non-empty closed subsets of X with $\lim_n \mathrm{diam}(F_n) = 0$ has non-empty intersection.*

(v) *Every nested sequence* $\{F_n\}$ *of non-empty closed subsets of* X *with H-diameter zero has non-empty intersection.*

Proof By Proposition 2 and general topology results, we have that (i), (iii) and (iv) are equivalent. Also, by Theorem 1 we have that (ii) and (v) are equivalent.

Obviously (i) implies (ii). We prove that (ii) implies (i). Indeed, let $\{x_n\}$ be a G- Cauchy sequence in X. By definition $\{x_n\}$ has a cluster point. Now, by Proposition 2 $\{x_n\}$ is Cauchy and then $\{x_n\}$ is convergent in X.

Remark 1 Notice that every compact ultrametric space is G-complete.

Example 4 (A non-compact G-complete ultrametric space)
Consider the ultrametric d on $[0,1]$ given by $d(x,y) = \max\{1-x, 1-y\}$ if $x \neq y$, and $d(x,x) = 0$ for all $x, y \in [0,1]$. Then, each $\{x\}$ is open for $x \neq 1$. The open balls of radius $r > 0$ centered at 1 are $B(1,r) =]1-r, 1]$.

Then, a sequence $\{x_n\}$ is G-Cauchy if and only if $\{x_n\}$ is eventually constant or $\{x_n\}$ is a sequence converging to 1 with respect to the usual topology of $\mathbb{R}$. Consequently, every G-Cauchy sequence is convergent in $([0,1], d)$ and so $([0,1], d)$ is G-complete.

Now, $\tau(d)$ is not compact. Indeed, for instance $[0,1] = \big\{\{x\} : x \leq \frac{1}{2}\big\} \cup]\frac{1}{2}, 1]$, and clearly this open cover has not any finite subcover.

5 Fixed point theorems

Next, under the assumption that (X,d) is weak G-complete we will state our fixed point theorem. The goal of our theorem is that we only need that the iterative sequence $\{x_n\}$ to be G-Cauchy, instead of Cauchy which is mostly demanded when X is complete.

Theorem 2 *Let* (X,d) *be a weak G-complete metric space and let* $f : X \to X$ *be a continuous mapping.*

(i) *Suppose there exists* $x \in X$ *such that the iterative sequence* $\{x_n\}$ *is G-Cauchy, where* $x_1 = f(x)$, $x_n = f(x_{n-1})$ *for* $n = 2, 3, \ldots$ *Then* f *has a fixed point (more precisely, the cluster points of* $\{x_n\}$ *are fixed points for* f*).*
(ii) *If in addition* $d(f(x), f(y)) < d(x,y)$ *for* $x \neq y$, $x, y \in X$, *then the fixed point is unique.*

Proof (i) Suppose that the sequence $\{x_n\}$ is G-Cauchy, where $x_1 = f(x)$ for some $x \in X$ and $x_n = f(x_{n-1})$ for $n = 2, 3, \ldots$ Then $\{x_n\}$ has, at least, a cluster point, say, c. Thus, there exists a subsequence $\{x_{p_n}\}_n$ of $\{x_n\}_n$ such that $\{x_{p_n}\}$ converges to c. We will prove that $\{x_{p_n-1}\}_n$ also converges to c.

Indeed, let $\varepsilon > 0$. We can find $n_1 \in \mathbb{N}$ such that $d(x_{p_n-1}, x_{p_n}) < \frac{\varepsilon}{2}$ for all $n \geq n_1$ since $\{x_n\}$ is G-Cauchy. Also, we can find $n_2 \in \mathbb{N}$ such that $d(x_{p_n}, c) < \frac{\varepsilon}{2}$ for all $n \geq n_2$, since $\{x_{p_n}\}_n$ converges to c.

Take $n_0 = \max\{n_1, n_2\}$. Then for all $n \geq n_0$ we have that $d(x_{p_n-1}, c) \leq d(x_{p_n-1}, x_{p_n}) + d(x_{p_n}, c) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ and hence $\{x_{p_n-1}\}_n$ converges to c.

Now we have that $0 = \lim_n d(x_{p_n}, c) = \lim_n d\left(f^{p_n}(x), c\right) =$
$\lim_n d\left(f(f^{p_n-1}(x)), c\right) = d\left(f(\lim_n f^{p_n-1}(x)), c\right) = d\left(f(\lim_n x_{p_n-1}), c\right) =$
$d(f(c), c)$ by continuity of f and by the previous paragraph. Hence $f(c) = c$ and so c is a fixed point of f.

(ii) If $d(f(x), f(y)) < d(x, y)$ for $x, y \in X$ and $x \neq y$, and we suppose that y_0 is another fixed point of f, with $y_0 \neq c$, then we have $d(c, y_0) = d(f(c), f(y_0)) < d(c, y_0)$, a contradiction.

Corollary 2 *Let (X, d) be a weak G-complete metric space. Every asymptotically regular self-mapping of X [3] has a fixed point.*

Remark 2 (Explanatory notes about completeness and contractivity)
Let f be a self-mapping of (X, d). Many contractive conditions on f have been given, in the literature, in order to assert the existence of fixed points for f. These contractive conditions are related with the structure of X. The stronger the structure on X, the weaker the contractive condition on f.

So, Banach's fixed point theorem asserts that a self-contraction f of a complete metric space X has a unique fixed point in X (f is a Banach contraction of X if there exists $k \in [0, 1[$ such that $d(f(x), f(y)) \leq k\, d(x, y)$ for all $x, y \in X$). The same conclusion was obtained by Edelstein [4] on a compact space X with the weaker contractive condition $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$.

Then, the weak G-complete structure is appropriate in order to relax contractive conditions for f, given on a complete metric space. For instance, denote by Ψ the set of all non-decreasing functions $\varphi : [0, \infty[\to [0, \infty[$ such that $\lim_n \varphi^{n)}(t) = 0$ for all $t > 0$. A mapping $f : (X, d) \mapsto (X, d)$ is called φ-contractive if there exists $\varphi \in \Psi$ such that $d(f(x), f(y)) \leq \varphi(d(x, y))$ for all $x, y \in X$. Clearly a Banach contractive mapping is of φ-contractive type. Boyd and Wong [2] proved that a φ-contractive mapping on a complete metric space has a unique fixed point, under the assumption that φ be upper semicontinuous from the right. This result can be obtained from our Theorem 2 without demanding any continuity condition on φ, but in a weak G-complete metric space. We do not incorporate this result here, because Matkowski ([11]Theorem 1.2) has improved it proving such a result in a complete metric space. Notice, following our argument, that every φ-contractive mapping satisfies the Edelstein's contractive condition, since $\varphi(t) < t$ for all $t > 0$, whenever $\varphi \in \Psi$.

Obviously, all the mentioned theorems can be applied on a compact space X if f has the appropriated properties. Now, if f does not satisfy the Edelstein's contractive condition on X, obviously f is not φ-contractive neither a Banach contraction. The following example test the goodness of the mentioned theorems.

Example 5 Consider the self-mapping f of $]0,+\infty[$ given by $f(x)=\ln(1+\frac{1}{x})$. We suppose $]0,+\infty[$ endowed with the usual metric of $\mathbb{R}$.

At the moment, none of the above mentioned results can be applied since $]0,+\infty[$ is not complete.

Consider the iterative sequence $\{x_n\}$ where $x_0=1$ and $x_n=f(x_{n-1})$ for $n=1,2,3,\dots$ One can check that x_3 satisfies $x_1<x_3<x_2$ and, consequently, $f(I)\subset I$, where $I=[x_1,x_2]$, since f is decreasing. In particular, x_n is in I for $n\geq 1$. Also, $\lim_n |x_{n+1}-x_n|=0$ (it is immediate using equation (1) below). Hence $\{x_n\}$ is G-Cauchy and by Theorem 2 f has a unique fixed point in I.

Due to the form of f it is difficult to find a compact in which applying Edelstein's theorem. It is easier to consider the gauge function $\varphi\in\Psi$ where $\varphi(t)=\ln(1+t)$ for all $t\geq 0$, and look for, supposing $x<y$, where the next equation is satisfied

$$|f(x)-f(y)|=\ln\frac{y(x+1)}{(y+1)x}\leq \ln\left(1+(y-x)\right)<y-x. \tag{1}$$

After an easy computation it is obtained that $a=\frac{-1+\sqrt{5}}{2}$ is the least positive number where (1) is satisfied whenever $x,y\geq a$ (curiously, a is the inverse of the golden proportion). Now, if we consider the compact $J=[a,f(a)]$ it is immediate, as above, to check that $f(J)\subset J$. Then the theorems of Edelstein and Matkowski prove the existence and uniqueness of a fixed point for f in J.

On the other hand, Banach's theorem cannot be applied to f defined on J. Indeed, suppose that for $x,y\in J$ there exists $k\in[0,1[$ such that $|f(x)-f(y)|\leq k\,|x-y|$.

Take $x=a$, $y\in J$. We have that $|f(a)-f(y)|=\ln\frac{(a+1)y}{(y+1)a}\leq k\,(y-a)$ and so $\frac{\ln\frac{(a+1)y}{(y+1)a}}{y-a}\leq k$ for all $y\geq a$. Consequently $\lim_{y\to a}\frac{\ln\frac{(a+1)y}{(y+1)a}}{y-a}\leq k<1$. Now, it is easy to prove that the last limit is 1, a contradiction.

Remark 3 It is worth to notice that, in an artificial way, one can construct a self-mapping g, modifying f in Example 5 on a week G-complete space, in which the theorems of Edelstein and Matkowski cannot be applied.

Indeed, consider $X=J\cup\{b,1,2,3,\dots\}$ with $0<b<a$. Let $g:X\to X$ defined by $g(x)=\begin{cases} f(x) & x\in[a,f(a)]\\ f(a) & x\in\{1,2,3,\dots\}\\ 2 & x=b\end{cases}$

Notice that g is well defined. The iterative sequence $\{y_n\}$ given by $y_0=1$ and $y_n=g(y_{n-1})$ for $n=1,2,3,\dots$ is in J for $n\geq 1$, and it agrees with the above sequence $\{x_n\}$. Then, it is easy to prove that $\{y_n\}$ is G-Cauchy in X, which is a weak G-complete metric space (by simple comparison with Example 1), and then by (i) of Theorem 2 g has a fixed point in X.

Now, X is not compact. On the other hand $|g(b)-g(a)|=2-f(a)>1>a-b$. Then Edelstein's contractive condition is not satisfied and, consequently, there is not any gauge function $\varphi\in\Psi$ satisfying on X the Matkowski's contractive condition for g.

Hence, Eldestein's and Matkowski's theorems cannot be applied on g.

Remark 4 It is well known that every Cauchy sequence in a metric space with a cluster point converges to it. The case of G-Cauchy sequences is different. A G-Cauchy sequence $\{x_n\}$ can have many cluster points and even if $\{x_n\}$ has only a cluster point it could not be convergent to it ([9], Example 3.7).

In the case of $\mathbb{R}^n$ endowed with the usual Euclidean metric the situation is nice. Indeed, every G-Cauchy sequence with a unique cluster point converges to it, since $\mathbb{R}^n$ is locally compact ([8], Proposition 3.9).

The study of the particular case of $\mathbb{R}$ is completed in the following proposition.

Proposition 3 *Let $\{a_n\}$ be a G-Cauchy sequence in $\mathbb{R}$ provided with the usual metric. If a and b are two distinct cluster points of $\{a_n\}$, and suppose $a < b$, then each point of the interval $[a, b]$ is a cluster point of $\{a_n\}$.*

Proof Suppose that a and b, with $a < b$, are two cluster points of $\{a_n\}$.

Let $\{a_{p_n}\}$ and $\{a_{q_n}\}$ be two subsequences of $\{a_n\}$ converging to a and b, respectively. Let $x \in]a, b[$. We will prove that x is a cluster point of $\{a_n\}$.

Let $\varepsilon > 0$. We choose ε such that $\varepsilon < \frac{1}{2}\min\{x - a,\, b - x\}$ to avoid trivial discussions. We will show that the sequence $\{a_n\}$ is frequently in $]x-\varepsilon, x+\varepsilon[$. We can find $n_0 \in \mathbb{N}$ such that the following three conditions are simultaneously satisfied:

$$a_{p_n} \in]a - \varepsilon, a + \varepsilon[\,,\ a_{q_n} \in]b - \varepsilon, b + \varepsilon[\,,\ |a_n - a_{n+1}| < \varepsilon \text{ for all } n \geq n_0.$$

We choose $a_{q_m} \in]b - \varepsilon, b + \varepsilon[$ such that $m \geq n_0$. Let a_{p_l} with $p_l > q_m$ and such that a_{p_l} is the first element after a_{q_m} satisfying $a_{p_l} \in]a - \varepsilon, a + \varepsilon[$. We have that $a + \varepsilon < x - \varepsilon < x + \varepsilon < b - \varepsilon$, hence we get $i_m \in \mathbb{N}$ such that $q_m < i_m < p_l$ and such that $a_{i_m} \in [x-\varepsilon, x+\varepsilon[$, since $\{a_n\}$ is G-Cauchy, where $i_m > m$.

Now choose $a_{q_{m'}}$ with $a_{q_{m'}} > p_l$ such that $a_{q_{m'}}$ is the first element after a_{p_l} satisfying $a_{q_{m'}} \in]b-\varepsilon, b+\varepsilon[$. Now choose $a_{p_{l'}}$ as we did earlier considering $a_{q_{m'}}$. Again we will get $p_l < q_{m'} < i_{m'} < p_{l'}$ such that $a_{i_{m'}} \in [x - \varepsilon, x + \varepsilon]$, thus we get frequent elements of the sequence in $[x - \varepsilon, x + \varepsilon]$.

Example 6 Consider the sequence $\{a_n\}$ where $a_n = \sin(\sqrt{n}\frac{\pi}{2})$. This sequence takes values in $[-1, 1]$, and one can check that $\{a_n\}$ is G-Cauchy. We have that $\sin(\sqrt{(4m+1)^2}\frac{\pi}{2}) = 1$ for $m = 0, 1, 2, \ldots$ and $\sin(\sqrt{(4m+3)^2}\frac{\pi}{2}) = -1$ for $m = 0, 1, 2, \ldots$ Then -1 and 1 are cluster points of $\{a_n\}$ and consequently all points of $[-1, 1]$ are cluster points of $\{a_n\}$.

The following is a Corollarly of Theorem 2, using Proposition 3.

Corollary 3 *Let $f : [a, b] \to [a, b]$ be a continuous mapping on the finite interval $[a, b]$ of $\mathbb{R}$ endowed with the usual metric of $\mathbb{R}$. If the iterative sequence $x_1 = f(x), x_n = f(x_{n-1})$, $n = 2, 3, \ldots$, is G-Cauchy for some $x \in [a, b]$, then $\{x_n\}$ is convergent to a fixed point of f.*

Proof Obviously $\{x_n\}$ has a cluster point since $[a,b]$ is compact. We will see that $\{x_n\}$ has only a cluster point. Suppose the contrary and let c_1 and c_2 be two cluster points of $\{x_n\}$ with $c_1 < c_2$. By Proposition 3 every point of $[c_1, c_2]$ is a cluster point of $\{x_n\}$. Then by (i) of Theorem 2 each point of $[c_1, c_2]$ is a fixed point for f and therefore f is the identity function on $[c_1, c_2]$. Let x_p be the first element of $\{x_n\}$ which is in $[c_1, c_2]$. Then $x_{p+1} = f(x_p) = x_p$ and clearly $x_{p+i} = x_p$ for all $i \geq 1$ and consequently x_p is the only point of $[c_1, c_2]$ which is cluster point of $\{x_n\}$, a contradiction. Then, $\{x_n\}$ has only a cluster point and by (i) of Theorem 2 and Remark 4 $\{x_n\}$ converges to a fixed point of f.

Remark 5 The last corollary shows the most commonly method, (roughly) used in computation, to prove the existence of a fixed point for a real function.

Example 7 Suppose $\mathbb{R}$ endowed with the usual metric and let k be an integer with $k \geq 2$. Denote by $[x]$ the floor function and let f be the real function given by

$$f(x) = \begin{cases} 0 & x \leq 0 \\ [x] + \sqrt{x - [x]} & 0 < x < k \\ k & x \geq k \end{cases}$$

It is an easy exercise to verify that f is continuous on $\mathbb{R}$. If we denote $K = [0, k]$ then it is obvious that f has not any fixed point out of K. So, consider the restriction g of f to K.

Let $x \in K$ with $x < k$. We construct the iterative sequence $\{x_n\}$ as follows: $x_1 = g(x) = [x] + \sqrt{x - [x]}$ and $x_n = g(x_{n-1})$ for $n \geq 2$. Observing that $\left[[x] + \sqrt{x - [x]}\right] = [x]$, it is easy to verify that

$$x_n = [x] + \sqrt[2^n]{x - [x]} \tag{2}$$

Then, $x_{n+1} - x_n = \sqrt[2^{n+1}]{x - [x]} - \sqrt[2^n]{x - [x]}$.

Consequently, $\lim_n (x_{n+1} - x_n) = 0$ and thus $\{x_n\}$ is G-Cauchy in the compact K. Then, by Corollary 3, the sequence $\{x_n\}$ converges to a fixed point of f.

Attending to Equation (2) we have that $\lim_n x_n = x$ if $[x] = x$, i.e. if $x \in \mathbb{Z}$ and, in other case, $\lim_n x_n = [x] + 1$. So, all integers in K are fixed points for g and they can be obtained as limit of iterative sequences. (It is easy to observe that they are the only fixed points for g).

Now, g does not fulfill the Edelstein's contractive condition. Indeed, take $c \in \mathbb{Z}$ with $0 \leq c < k$ and let $s = c + \frac{1}{2}$. Then we have $g(s) - g(c) = \sqrt{\frac{1}{2}} > \frac{1}{2} = s - c$. Consequently, neither Banach's theorem nor Matkowski's theorem can be applied on g.

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