

A study on the relationship between relaxed metrics and indistinguishability operators

Pilar Fuster-Parra · Javier Martín ·
Juan-José Miñana · Oscar Valero.

Received: date / Accepted: date

Abstract In 1982, E. Trillas introduced the notion of indistinguishability operator with the main aim of fuzzifying the crisp notion of equivalence relation. In the study of such a class of operators, an outstanding property must be stressed. Concretely, there exists a duality relationship between indistinguishability operators and metrics. The aforesaid relationship was deeply studied by several authors that introduced a few techniques to generate metrics from indistinguishability operators and vice-versa. In the last years a new generalization of the metric notion has been introduced in the literature with the purpose of developing mathematical tools for quantitative models in Computer Science and Artificial Intelligence. The aforesaid generalized metrics are known as relaxed metrics. The main purpose of the present paper is to explore the possibility of making explicit a duality relationship between indistinguishability operators and relaxed metrics in such a way that the aforementioned classical techniques to generate both concepts, one from the other, can be extended to the new framework.

Keywords Additive generator · pseudo-inverse · continuous Archimedean t-norm · relaxed indistinguishability operator · relaxed (pseudo-)metric

This work was partially supported by the Spanish Ministry of Economy and Competitiveness under Grants DPI2017-86372-C3-3-R, TIN2016-81731-REDT (LODISCO II) and AEI/FEDER, UE funds, by Programa Operatiu FEDER 2014-2020 de les Illes Balears, by project ref. PROCOE/4/2017 (Direcció General d'Innovació i Recerca, Govern de les Illes Balears), and by project ROBINS. The latter has received research funding from the EU H2020 framework under GA 779776. This publication reflects only the authors views and the European Union is not liable for any use that may be made of the information contained therein.

Pilar Fuster-Parra, Javier Martín, Juan-José Miñana, Oscar Valero
Universitat de les Illes Balears, Cra. de Valldemossa km 7.5, Palma (Illes Balears), Spain
Tel.: +34-971 172901
E-mail: {pilar.fuster, javier.martin, jj.minana, o.valero}@uib.es

1 Introduction

Throughout this paper we will assume that the reader is familiar with the basics of triangular norms (see [15] for a deeper treatment of the topic). In [22], E. Trillas introduced the notion of T -indistinguishability operator with the aim of fuzzifying the classical (crisp) notion of equivalence relation. Let us recall that, according to [22] (see also [15, 19]), given a t -norm $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$, a T -indistinguishability operator on a nonempty set X is a fuzzy relation $E : X \times X \rightarrow [0, 1]$ satisfying for all $x, y, z \in X$ the following conditions

- (i) $E(x, x) = 1$ (Reflexivity),
- (ii) $E(x, y) = E(y, x)$ (Symmetry),
- (iii) $T(E(x, y), E(y, z)) \leq E(x, z)$ (T -Transitivity).

A T -indistinguishability operator E is said to separate points provided that $E(x, y) = 1 \Rightarrow x = y$ for all $x, y \in X$.

In the literature the relationship between metrics and T -indistinguishability operators has been studied in depth for several authors [3, 8, 12, 15, 18, 19, 23]. Let us recall a few facts about metric spaces in order to explicitly state the aforesaid relationship. Following [5], a pseudo-metric on a nonempty set X is a function $d : X \times X \rightarrow [0, \infty]$ such that, for all $x, y, z \in X$, the following properties hold:

- (i) $d(x, x) = 0$,
- (ii) $d(x, y) = d(y, x)$,
- (iii) $d(x, z) \leq d(x, y) + d(y, z)$.

A pseudo-metric d on X is called pseudo-ultrametric if it satisfies, in addition, for all $x, y, z \in X$ the following inequality:

- (iv) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

Of course, a pseudo-metric (pseudo-ultrametric) d on X is called a metric (ultrametric) provided that it satisfies, in addition, the following axiom for all $x, y \in X$:

- (i') $d(x, y) = 0 \Rightarrow x = y$.

Regarding the relationship between (pseudo-)metrics and indistinguishability operators, the next results make it explicit. The first one introduces a technique that allows to construct (pseudo-)metrics from indistinguishability operators.

Theorem 1 *Let X be a nonempty set and let T^* be a t -norm with additive generator $f_{T^*} : [0, 1] \rightarrow [0, \infty]$. Let $d_E : X \times X \rightarrow [0, \infty]$ be the function defined by*

$$d_E(x, y) = f_{T^*}(E(x, y))$$

for all $x, y \in X$. If T is a t -norm, then the following assertions are equivalent:

- 1) $T^* \leq T$ (i.e., $T^*(x, y) \leq T(x, y)$ for all $x, y \in [0, 1]$).
- 2) For any T -indistinguishability operator E on X the function d_E is a pseudo-metric on X .
- 3) For any T -indistinguishability operator E on X that separates points the function d_E is a metric on X .

The second one introduces a method to revert the technique given by Theorem 1, i.e., to construct now indistinguishability operators from (pseudo-)metrics.

Theorem 2 Let T^* be a continuous Archimedean t -norm with additive generator $f_{T^*} : [0, 1] \rightarrow [0, \infty]$. Let $E_d : X \times X \rightarrow [0, 1]$ be the fuzzy binary relation defined by

$$E_d(x, y) = f_{T^*}^{(-1)}(d(x, y))$$

for all $x, y \in X$. If d is a pseudo-metric on X , then E_d is a T^* -indistinguishability operator. Moreover, the T^* -indistinguishability operator E_d separates points if and only if d is a metric on X .

In the last years a few generalizations of the metric notion have been introduced in the literature with the purpose of developing suitable mathematical tools for quantitative models in Computer Science and Artificial Intelligence. Concretely, the notion of dislocated metric, dislocated ultrametric, weak partial (pseudo-)metric and partial (pseudo-)metric have been studied and applied to Logic Programming in [10,11], Domain Theory in [9,20,21], Denotational Semantics in [16,17] and Asymptotic Complexity of Programs in [1], respectively. Each of the preceding generalized metric notions can be retrieved as a particular case of a new notion, called relaxed metric, which has been introduced recently in [5]. Let us recall, according to [5], the notion of relaxed metric.

Definition 1 A relaxed pseudo-metric on a nonempty set X is a function $d : X \times X \rightarrow [0, \infty]$ which satisfies for all x, y, z the following:

- (i) $d(x, y) = d(y, x)$,
- (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

We will say that a relaxed pseudo-metric d on a nonempty set satisfies the small self-distances (SSD for short) property in the spirit of [9] whenever $d(x, x) \leq d(x, y)$ for all $x, y \in X$. Moreover, a relaxed pseudo-metric d is a relaxed metric provided that it satisfies the following separation property for all $x, y \in X$:

- (iii) $d(x, x) = d(x, y) = d(y, y) \Rightarrow x = y$.

Furthermore, a relaxed (pseudo-)metric d on X will be called a relaxed (pseudo-)ultrametric if satisfies in addition, for all x, y, z , the following inequality:

- (iv) $d(x, y) \leq \max\{d(x, z), d(z, y)\}$.

The following example gives an instance of a relaxed pseudo-metric which is not a pseudo-metric.

Example 1 Let $X = \{1, 2, 3\}$. Define the function $d_X : X \times X \rightarrow [0, 1]$ by

$$d_X(a, b) = \begin{cases} 1 & \text{if } (a, b) = (1, 2) \text{ or } (a, b) = (2, 1), \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

A straightforward computation shows that d_X is a relaxed pseudo-metric on X which satisfies the SSD property. However, d_X is not a pseudo-metric, since $d_X(x, x) = \frac{1}{2}$. Observe that defining $\hat{d}_X(1, 1) = 1$ and $\hat{d}_X(x, y) = d_X(x, y)$ otherwise, we obtain an instance of relaxed pseudo-metric which is not a pseudo-metric and, in addition, it does not satisfy the SSD property.

Recently, it has been discussed that the notion of indistinguishability operator and relaxed metric are closely related. Indeed, in [5, 6] it has been stated that the logical counterpart for relaxed metrics is, in some sense, a generalized indistinguishability operator. On account of [5] (see also [14]), the notion of generalized indistinguishability operator related to relaxed metrics can be formulated as follows:

Definition 2 Let X be a non-empty set and let $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be a t-norm. A relaxed T -indistinguishability operator E on X is a fuzzy relation $E : X \times X \rightarrow [0, 1]$ satisfying the following properties for any $x, y, z \in X$:

- (i) $E(x, y) = E(y, x)$,
- (ii) $T(E(x, z), E(z, y)) \leq E(x, y)$.

Moreover, a relaxed T -indistinguishability operator E satisfies the small-self indistinguishability (SSI for short) property provided that

- (i) $E(x, y) \leq E(x, x)$.

for all $x, y \in X$. Furthermore, a relaxed T -indistinguishability operator E is said to separate points provided that $E(x, y) = E(x, x) = E(y, y) \Rightarrow x = y$ for all $x, y \in X$.

Notice that the notion of T -indistinguishability operator is retrieved as a particular case of relaxed T -indistinguishability operator whenever the relaxed T -indistinguishability operator satisfies also the reflexivity. In fact, a relaxed indistinguishability operator is an indistinguishability operator if and only if it is reflexive. The same occurs when we consider T -indistinguishability operators that separate points. Furthermore, it must be stressed that M -valued equalities are exactly relaxed T -indistinguishability operators with the SSI property when the t-norm T is left-continuous and the underlying GL-monoid M is exactly $([0, 1], \leq, T)$ (see, for instance, [13, 14]).

The next examples give instances of relaxed indistinguishability operators which are not indistinguishability operators.

Example 2 Fix $k \in]0, 1[$. Consider the fuzzy binary relation $E_k : \mathbb{R}^+ \rightarrow [0, 1]$ defined by

$$E(x, y) = k$$

for $x, y \in \mathbb{R}^+$. It is obvious that E_k is a relaxed T_{Min} -indistinguishability operator which is not a T_{Min} -indistinguishability operator because $E(x, x) = k \neq 1$ for each $x \in \mathbb{R}^+$. Notice that E_k satisfies the SSI property but it does not separate points.

Example 3 Let Σ be a nonempty alphabet. Denote by Σ^∞ the set of all finite and infinite sequences over Σ . Given $v \in \Sigma^\infty$ denote by $l(v)$ the length of v . Thus $l(v) \in \mathbb{N} \cup \{\infty\}$ for all $v \in \Sigma^\infty$. Moreover, if $\Sigma_F = \{v \in \Sigma^\infty : l(v) \in \mathbb{N}\}$ and $\Sigma_\infty = \{v \in \Sigma^\infty : l(v) = \infty\}$, then $\Sigma^\infty = \Sigma_F \cup \Sigma_\infty$. Define the fuzzy binary relation $E_\Sigma : \Sigma^\infty \times \Sigma^\infty \rightarrow [0, 1]$ by

$$E_\Sigma(u, v) = 1 - 2^{-l(v, w)}$$

for all $u, v \in \Sigma^\infty$, where $l(v, w)$ denotes the longest common prefix between v and w . Of course we have adopted the convention that $2^{-\infty} = 0$. Then it is not hard to check that E_Σ is a relaxed T_{Min} -indistinguishability operator which is not a T_{Min} -indistinguishability operator. Notice that $E_\Sigma(u, u) = 1 - \frac{1}{2^{l(u)}}$ for all $u \in \Sigma^\infty$ and that $E_\Sigma(u, u) = 1 \Leftrightarrow u \in \Sigma_\infty$. It is clear that E_Σ satisfies the SSI property and separates points. Clearly E_Σ is not a T_{Min} -indistinguishability operator because $E_\Sigma(u, u) < 1$ for each $x \in \Sigma_F$.

The following example shows that there are relaxed indistinguishability operators that do not satisfy the SSI property.

Example 4 Let X be the set considered in Example 1. Define the fuzzy binary relation $E_X : X \times X \rightarrow [0, 1]$ by

$$E_X(a, b) = \begin{cases} \frac{1}{4} & \text{if } (a, b) = (1, 1), \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

A straightforward computation yields that E_X is a relaxed T_P -indistinguishability operator which does not separate points, where T_P denotes the product t-norm. Moreover, E_X does not satisfy the SSI property, since $E_X(1, 3) = \frac{1}{2}$ and $E_X(1, 1) = \frac{1}{4}$.

Motivated, on the one hand, by the exposed facts and, on the other hand, by the utility of generalized metrics in Computer Science and Artificial Intelligence, the target of this paper is to study deeply the relationship between both concepts, relaxed indistinguishability operators and relaxed metrics, and try to extend the methods given in Theorems 1 and 2 to this new context.

2 From relaxed indistinguishability operators to relaxed metrics

In this section we focus our work on the possibility of extending Theorem 1 to the relaxed framework. To this end, we will structure our study in two subsections. The first one, Subsection 2.1, will be devoted to make clear the relationship between relaxed metrics and relaxed T_{Min} -indistinguishability operators, where T_{Min} stands for the minimum t-norm. The second one, Subsection 2.2, will be devoted to specify the correspondence between relaxed T -indistinguishability operators and relaxed metrics whenever one considers t-norms T with additive generator.

2.1 Relaxed T_{Min} indistinguishability

According to [24] (see also [19]), the relationship between T_{Min} -indistinguishability operators and metrics is given by the next result.

Proposition 1 *Let X be a nonempty set and let $E : X \times X \rightarrow [0, 1]$ be a fuzzy relation. Then the following assertions are equivalent:*

- 1) E is a T_{Min} -indistinguishability operator.
- 2) The function d_E is a pseudo-ultrametric on X , where $d_E(x, y) = 1 - E(x, y)$ for all $x, y \in X$.

Moreover, E separates points if and only if d_E is a ultrametric on X .

Next we show that the preceding result can be easily extended to our new context.

Proposition 2 *Let X be a nonempty set and let E be a fuzzy relation on X . Then the following assertions are equivalent:*

- 1) E is a relaxed T_{Min} -indistinguishability operator.
- 2) The function d_E is a relaxed pseudo-ultrametric on X , where $d_E(x, y) = 1 - E(x, y)$ for all $x, y \in X$.

Moreover, E separates points if and only if d_E is a relaxed ultrametric on X .

Proof 1) \Leftrightarrow 2). It is clear that

$$E(x, y) = E(y, x) \Leftrightarrow 1 - E(x, y) = 1 - E(y, x) \Leftrightarrow d_E(x, y) = d_E(y, x)$$

for all $x, y \in X$. Next fix $x, y, z \in X$. Then we have that the next inequality

$$\min\{E(x, z), E(z, y)\} \leq E(x, y)$$

is equivalent to

$$1 - \max\{1 - E(x, z), 1 - E(z, y)\} \leq E(x, y).$$

Moreover, the preceding inequality is equivalent to the next one:

$$-1 + \max\{1 - E(x, z), 1 - E(z, y)\} \geq -E(x, y).$$

At the same time the above inequality is equivalent to the following one:

$$\max\{1 - E(x, z), 1 - E(z, y)\} \geq 1 - E(x, y).$$

Furthermore, the last inequality is equivalent to

$$\max\{d_E(x, z), d_E(z, y)\} \geq d_E(x, y).$$

Therefore E is a relaxed T_{Min} -indistinguishability operator if and only if d_E is a relaxed pseudo-ultrametric on X .

Finally, it is clear that

$$E(x, y) = E(x, x) = E(y, y) \Leftrightarrow 1 - E(x, y) = 1 - E(x, x) = 1 - E(y, y)$$

$$\Leftrightarrow d_E(x, y) = d_E(x, x) = d_E(y, y)$$

for all $x, y \in X$. It follows that E separates points if and only if d_E is a relaxed ultrametric on X . \square

Corollary 1 *Let X be a nonempty set and let E be a T_{Min} -indistinguishability operator on X . Then E fulfills the SSI property if and only if d_E fulfills the SSD property.*

Proof Since E is a T_{Min} -indistinguishability operator we have that

$$\min\{E(x, z), E(z, y)\} \leq E(x, y)$$

for all $x, y \in X$. So taking $x = y$ in the preceding inequality we obtain that

$$E(x, z) = \min\{E(x, z), E(z, x)\} \leq E(x, x)$$

for all $x, z \in X$. Thus every T_{Min} -indistinguishability operator satisfies the SSI property. Clearly

$$E(x, y) \leq E(x, x) \Leftrightarrow 1 - E(x, x) \leq 1 - E(x, y) \Leftrightarrow d_E(x, x) \leq d_E(x, y)$$

for all $x, y \in X$. Therefore E fulfills the SSI property if and only if d_E fulfills the SSD property.

It must be pointed out that relaxed T_{Min} -indistinguishability operators match up with Ω -valued equalities in the sense of [7] when the underlying Heyting algebra is exactly $([0, 1], \leq)$.

The next example illustrates Theorem 2.

Example 5 Consider the relaxed T_{Min} -indistinguishability operator E_Σ introduced in Example 3. Proposition 2 guarantees that the function d_{E_Σ} given by

$$d_{E_\Sigma}(u, v) = 1 - E_\Sigma(u, v)$$

for all $u, v \in \Sigma^\infty$ is a relaxed pseudo-ultrametric. Since E_Σ satisfies the SSI property and separates points we get by Corollary 1 that the relaxed pseudo-ultrametric d_{E_Σ} is, in fact, a relaxed ultrametric which fulfills the SSD property. Observe that the preceding facts agree with those pointed out in [5] (see also [16]).

2.2 T-norms with additive generator and relaxed T -indistinguishabilities

In this section we study the duality relationship that exists between relaxed metrics and relaxed indistinguishability operators when the t-norm under consideration admits an additive generator. Notice that the study developed in Subsection 2.1 considers relaxed T_{Min} -indistinguishability operators and that the t-norm T_{Min} does not admit additive generator. In particular we wonder whether Theorem 1 can be stated in our more general framework. The next result provides an affirmative answer to the posed question. Although few parts of the proof run following the same arguments to those given in the proof of Theorem 1, we have included all of them for the sake of completeness.

Theorem 3 *Let X be a nonempty set and let T^* be a t-norm with additive generator $f_{T^*} : [0, 1] \rightarrow [0, \infty]$. Given a fuzzy relation E , let d_E be a function defined by*

$$d_E^{f_{T^*}}(x, y) = f_{T^*}(E(x, y))$$

for all $x, y \in X$. If T is a t-norm, then the following assertions are equivalent:

- 1) $T^* \leq T$.
- 2) For any relaxed T -indistinguishability operator E on X the function $d_E^{f_{T^*}}$ is a relaxed pseudo-metric on X .
- 3) For any relaxed T -indistinguishability operator E on X that separates points the function $d_E^{f_{T^*}}$ is a relaxed metric on X .

Proof 1) \Rightarrow 2). Since $E(x, y) = E(y, x)$ for all $x, y \in X$ we have $d_E^{f_{T^*}}(x, y) = d_E^{f_{T^*}}(y, x)$ for all $x, y \in X$. Next we show that

$$d_E^{f_{T^*}}(x, z) \leq d_E^{f_{T^*}}(x, y) + d_E^{f_{T^*}}(y, z)$$

for all $x, y \in X$. To this end note that if E is a relaxed T -indistinguishability operator, then E is also a relaxed T^* -indistinguishability operator. Thus we have that

$$T^*(E(x, y), E(y, z)) \leq E(x, z)$$

for all $x, y, z \in X$.

Since f_{T^*} is an additive generator of the t-norm T^* we have that

$$T^*(u, v) = f_{T^*}^{(-1)}(f_{T^*}(u) + f_{T^*}(v))$$

for all $u, v \in [0, 1]$. It follows that

$$T^*(E(x, y), E(y, z)) = f_{T^*}^{(-1)}(f_{T^*}(E(x, y)) + f_{T^*}(E(y, z)))$$

for all $x, y, z \in X$. Whence we have that

$$f_{T^*}^{(-1)}(f_{T^*}(E(x, y)) + f_{T^*}(E(y, z))) \leq E(x, z)$$

for all $x, y, z \in X$.

Since f_{T^*} is decreasing we deduce, from the preceding inequality, that

$$f_{T^*}\left(f_{T^*}^{(-1)}(f_{T^*}(E(x, y)) + f_{T^*}(E(y, z)))\right) \geq f_{T^*}(E(x, z)).$$

Next we distinguish two possible cases:

Case 1. $f_{T^*}(E(x, y)) + f_{T^*}(E(y, z)) \in \text{Ran}(f_{T^*})$, where we have $\text{Ran}(f_{T^*}) = \{f_{T^*}(x) : x \in [0, 1]\}$. Then there exists $u \in [0, 1]$ such that

$$f_{T^*}(u) = f_{T^*}(E(x, y)) + f_{T^*}(E(y, z)).$$

Hence we obtain that

$$\begin{aligned} f_{T^*}\left(f_{T^*}^{(-1)}(f_{T^*}(E(x, y)) + f_{T^*}(E(y, z)))\right) &= f_{T^*}\left(f_{T^*}^{(-1)}(f_{T^*}(u))\right) \\ &= f_{T^*}(u) \\ &= f_{T^*}(E(x, y)) + f_{T^*}(E(y, z)). \end{aligned}$$

Therefore

$$\begin{aligned} d_E^{f_{T^*}}(x, y) + d_E^{f_{T^*}}(y, z) &= f_{T^*}(E(x, y)) + f_{T^*}(E(y, z)) \geq \\ f_{T^*}(E(x, z)) &= d_E^{f_{T^*}}(x, z). \end{aligned}$$

Case 2. $f_{T^*}(E(x, y)) + f_{T^*}(E(y, z)) \notin \text{Ran}(f_{T^*})$. Then the fact that f_{T^*} is an additive generator of the t-norm T^* , and thus that $\text{Ran}(f_{T^*})$ is relatively closed, implies that $f_{T^*}(E(x, y)) + f_{T^*}(E(y, z)) > f_{T^*}(0)$. Thus, taking into account that f_{T^*} is decreasing we have that $f_{T^*}(0) \geq f_{T^*}(E(x, z))$. Whence we deduce that

$$\begin{aligned} d_E^{f_{T^*}}(x, y) + d_E^{f_{T^*}}(y, z) &= f_{T^*}(E(x, y)) + f_{T^*}(E(y, z)) \geq \\ f_{T^*}(E(x, z)) &= d_E^{f_{T^*}}(x, z). \end{aligned}$$

Therefore we conclude that the function $d_E^{f_{T^*}}$ is a relaxed pseudo-metric on X .

2) \Rightarrow 3). Since E is a T -indistinguishability operator we have that the function d_E is a relaxed pseudo-metric on X . Moreover, on the one hand, the fact that E separates points guarantees that $E(x, y) = E(x, x) = E(y, y) \Leftrightarrow x = y$. On the other hand, the fact that f_{T^*} is an additive generator of the t -norm T^* implies that f_{T^*} is strictly decreasing and, hence, injective. So we obtain that

$$\begin{aligned} d_E^{f_{T^*}}(x, y) = d_E^{f_{T^*}}(x, x) = d_E^{f_{T^*}}(y, y) &\Leftrightarrow \\ f_{T^*}(E(x, y)) = f_{T^*}(E(x, x)) = f_{T^*}(E(y, y)) &\Leftrightarrow \\ E(x, y) = E(x, x) = E(y, y) &\Leftrightarrow x = y. \end{aligned}$$

Consequently the function $d_E^{f_{T^*}}$ is a relaxed metric on X .

3) \Rightarrow 1). Let $(a, b) \in [0, 1]^2$. We have to show that $T^*(a, b) \leq T(a, b)$. Of course, $T^*(a, b) = T(a, b)$ whenever $a = 1$ or $b = 1$. So we can assume that $a, b \in [0, 1[$. Now fix three different elements $x, y, z \in X$ and define the fuzzy binary relation E on X as follows:

$$E(u, v) = \begin{cases} T(a, b) & \text{if } u = x \text{ and } v = y \\ a & \text{if } u = x \text{ and } v = z, \\ b & \text{if } u = y \text{ and } v = z \end{cases}$$

$E(u, v) = E(v, u)$ for all $u, v \in \{x, y, z\}$, $E(u, u) = 1$ for all $u \in X$ and $E(u, v) = 0$ otherwise. A simple computation shows that E is a relaxed T -indistinguishability operator which separates points. Thus $d_E^{f_{T^*}}$ is a relaxed metric on X . Consequently

$$f_{T^*}(E(x, y)) \leq f_{T^*}(E(x, z)) + f_{T^*}(E(z, y)).$$

Taking into account that

$$T^*(u, v) = f_{T^*}^{(-1)}(f_{T^*}(u) + f_{T^*}(v))$$

for all $u, v \in [0, 1]$ and that $f_{T^*}^{(-1)}$ is decreasing, we deduce that

$$\begin{aligned} f_{T^*}^{(-1)}(f_{T^*}(E(x, y))) &\geq \\ f_{T^*}^{(-1)}(f_{T^*}(E(x, z)) + f_{T^*}(E(z, y))) &= T^*(E(x, z), E(z, y)). \end{aligned}$$

Since $f_{T^*}^{(-1)}(f_{T^*}(E(x, y))) = E(x, y)$, we conclude that

$$T(a, b) = E(x, y) \geq T^*(E(x, z), E(z, y)) = T^*(a, b),$$

as we claim. \square

It is worth pointing out that Theorems 1 and 3 disclose a surprising connection (equivalence) between indistinguishability operators and the relaxed ones.

In [4,8,23] (see also [2,3]), the subsequent characterization was given in order to establish the relationship between indistinguishability operators and (pseudo-)metrics. Concretely, the aforesaid characterization states the following.

Theorem 4 *Let X be a nonempty set and let E be a fuzzy binary relation on X . Let d_E be the function defined by $d_E^{f_{T_L}}(x, y) = 1 - E(x, y)$ for all $x, y \in X$. If T is a t -norm, then the following assertions are equivalent:*

- 1) $T_L \leq T$, where T_L denotes the Lukasiewicz t -norm.
- 2) For any T -indistinguishability operator the function $d_E^{f_{T_L}}$ is a pseudo-metric on X .
- 3) For any T -indistinguishability operator that separates points the function $d_E^{f_{T_L}}$ is a metric on X .

Taking in Theorem 3, T^* as the Lukasiewicz t -norm T_L and the function f_{T^*} as the function $f_{T_L} : [0, 1] \rightarrow [0, \infty]$ given by $f_{T_L}(x) = 1 - x$ for all $x \in [0, 1]$ we obtain as a particular case the following results, one of them, Corollary 2, providing an extension of Theorem 4.

Corollary 2 *Let X be a nonempty set and let $E : X \times X \rightarrow [0, 1]$ be a fuzzy relation. Let $d_E : X \times X \rightarrow \mathbb{R}^+$ be the function defined by $d_E(x, y) = 1 - E(x, y)$ for all $x, y \in X$. If $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a t -norm, then the following assertions are equivalent:*

- 1) $T_L \leq T$.
- 2) For any relaxed T -indistinguishability operator the function d_E is a relaxed pseudo-metric on X .
- 3) For any relaxed T -indistinguishability operator that separates points the function d_E is a relaxed metric on X .

When we consider in Theorem 3 the t -norm T as the minimum t -norm T_M and the function f_{T^*} as an additive generator of any t -norm T^* we retrieve as a particular case the following result.

Corollary 3 *Let X be a nonempty set and let E be a relaxed T_{Min} -indistinguishability operator on X . Then the function $d_E^{f_{T^*}}$ is a relaxed pseudo-metric on X for any additive generator f_{T^*} of a t -norm T^* .*

Of course the preceding results agree with Theorem 1 because every relaxed pseudo-ultrametric is a relaxed pseudo-metric.

If we consider in Theorem 3 the t -norm T^* as the Drastic product T_D and the function f_{T^*} the additive generator of T_D given by $f_{T_D}(x) = 2 - x$ if $x \in [0, 1[$ and $f(1) = 0$, then we get as a consequence the following result.

Corollary 4 *Let X be a nonempty set and let T be a t -norm. If E is a relaxed T -indistinguishability operator on X , then the function $d_E^{f_{T^D}}$ is a relaxed pseudo-metric on X .*

Clearly if we consider in Corollaries 3 and 4 indistinguishability operators that separate points then the obtained relaxed pseudo-metrics become relaxed metrics.

Clearly Theorem 3 provides a technique to generate relaxed pseudo-metrics from relaxed indistinguishability operators. Observe that in spite of the aforementioned equivalence between Theorems 1 and 3, the new technique gives instances of relaxed pseudo-metric which are not pseudo-metrics. The following examples illustrate the exposed facts.

Example 6 Consider the fuzzy binary relation E_{Min} on $[0, 1]$ given by

$$E_{Min}(x, y) = \min\{x, y\}$$

for all $x, y \in [0, 1]$. Let $f_{T_P} : [0, 1] \rightarrow [0, \infty]$ be the additive generator of the product t -norm T_P given by $f_{T_P}(x) = -\log(x)$ for all $x \in [0, 1]$. Theorem 3 yields that the function $d_{E_{Min}}^{f_{T_P}}$ is a relaxed metric on $[0, 1]$, since E_{Min} is a T_P -indistinguishability operator that separates points. Notice that

$$d_{E_{Min}}^{f_{T_P}}(x, y) = -\log(\min\{x, y\})$$

for all $x, y \in [0, 1]$. It is clear that E_{Min} does not satisfy the reflexivity and, thus, that $d_{E_{Min}}^{f_{T_P}}$ is not a (pseudo-)metric on X .

Example 7 Consider the set X introduced in Example 1. Define the fuzzy binary relation $\widehat{E}_X : X \times X \rightarrow [0, 1]$ by

$$\widehat{E}_X(a, b) = \begin{cases} \frac{1}{4} & \text{if } (a, b) = (1, 2) \text{ or } (a, b) = (2, 1), \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

A straightforward computation yields that \widehat{E}_X is a relaxed T_P -indistinguishability operator which does not separate points. Moreover, assertion 2) in Theorem 3 (or assertion 2) in Corollary 2) gives that the function $d_{\widehat{E}_X}^{f_{T_L}}$ defined by $d_{\widehat{E}_X}^{f_{T_L}}(x, y) = 1 - \widehat{E}_X(x, y)$ for all $x, y \in X$ is a relaxed pseudo-metric on X . Of course, since \widehat{E}_X does not separate points Theorem 3 (or Corollary 2) provides that $d_{\widehat{E}_X}$ is a relaxed pseudo-metric that is not a relaxed metric. Moreover, it is clear that \widehat{E}_X is not reflexive and, thus, that $d_{\widehat{E}_X}^{f_{T_L}}$ is not a pseudo-metric on X .

It must be stressed that there are relaxed T -indistinguishability operators that do not satisfy the SSI property such as Example 4 shows. Taking into account the aforementioned fact we obtain, from Theorem 3, the result below.

Corollary 5 *Let X be a nonempty set and let T^* be a t -norm with additive generator $f_{T^*} : [0, 1] \rightarrow [0, \infty]$. Moreover, let E be a relaxed T -indistinguishability operator on X such that $T^* \leq T$. Then E fulfills the SSI property if and only if $d_E^{f_{T^*}}$ fulfills the SSD property.*

Proof Theorem 3 guarantees that $d_E^{f_{T^*}}$ is a relaxed pseudo-metric on X . Since

$$d_E^{f_{T^*}}(x, y) = f_{T^*}(E(x, y))$$

for all $x, y \in X$ and f_{T^*} is strictly increasing we have that

$$d_E^{f_{T^*}}(x, x) \leq d_E(x, y) \Leftrightarrow f_{T^*}(E(x, x)) \leq f_{T^*}(E(x, y)) \Leftrightarrow E(x, y) \leq E(x, x)$$

for all $x, y \in X$. Therefore E fulfills the SSI property if and only if $d_E^{f_{T^*}}$ fulfills the SSD property. \square

In the light of Corollary 5 we can elucidate that the relaxed pseudo-metric $d_{\widehat{E}_X}^{f_{T_L}}$ given in Example 7 does not satisfy the SSD property, since the relaxed T_L -indistinguishability operator \widehat{E}_X does not satisfy the SSI property. However, Corollary 5 guarantees that the relaxed pseudo-metric, introduced in Example 6, $d_{E_{Min}}^{f_{T_P}}$ induced by the T_P -indistinguishability operator E_X fulfills the SSD property, since E_{Min} fulfills the SSI property.

3 From relaxed metrics to relaxed indistinguishability operators

This section is devoted to explore the possibility of developing a technique that allows to induce relaxed indistinguishability operators from relaxed pseudo-metrics. In particular we inquire if Theorem 2 can be stated in our more general framework. The next result answers the framed question. Although the proof of assertion 1) in the aforesaid result runs following the same arguments to those given in the proof of Theorem 2, we have included it for the sake of completeness.

Theorem 5 *Let X be a nonempty set and let T be a continuous Archimedean t -norm with additive generator $f_T : [0, 1] \rightarrow [0, \infty]$. If d is a relaxed pseudo-metric on X and E_d is the binary fuzzy relation defined by*

$$E_d(x, y) = f_T^{(-1)}(d(x, y))$$

for all $x, y \in X$, then the following assertions hold:

- 1) E_d is a relaxed T -indistinguishability operator.
- 2) If E_d is a relaxed T -indistinguishability operator that separates points for every relaxed pseudo-metric d on X , then T is strict.
- 3) E_d is a relaxed T -indistinguishability operator that separates points provided that d is a relaxed metric on X if and only if T is strict.

- 4) d is a relaxed metric on X provided that E_d is a relaxed T -indistinguishability operator that separates points.
- 5) If d is a relaxed metric such that $d(x, y) \leq f_T(0)$ for all $x, y \in X$, then E_d is a relaxed T -indistinguishability operator that separates points.
- 6) E_d is a relaxed T -indistinguishability operator which satisfies the SSI property provided that the relaxed pseudo-metric d on X satisfies the SSD property.

Proof 1). Clearly $E_d(x, y) = E_d(y, x)$ for all $x, y \in X$. Next we show that

$$T(E_d(x, y), E_d(y, z)) \leq E_d(x, z)$$

for all $x, y, z \in X$. To this end, fix $x, y, z \in X$. Since T is a continuous Archimedean t -norm the continuity of the additive generator f_T is guaranteed (and, thus, left-continuous at 1). So f_T is strictly decreasing and continuous. Hence there exists f_T^{-1} which is also strictly decreasing on $\text{Ran}(f_T)$. Taking into account the preceding facts we have that

$$f_T^{(-1)}(x) = f_T^{-1}(\min\{f_T(0), x\})$$

for all $x \in [0, \infty]$. Therefore

$$E_d(x, y) = f_T^{(-1)}(d(x, y)) = f_T^{-1}(\min\{f_T(0), d(x, y)\})$$

for all $x, y \in X$. Since

$$d(x, z) \leq d(x, y) + d(y, z)$$

we have that

$$f_T^{-1}(\min\{f_T(0), d(x, z) + d(z, y)\}) \leq f_T^{-1}(\min\{f_T(0), d(x, z)\}) = E_d(x, z).$$

A straightforward computation gives that

$$\min\{f_T(0), f_T(E_d(x, y)) + f_T(E_d(y, z))\} = \min\{f_T(0), d(x, y) + d(y, z)\},$$

where

$$f_T(E_d(x, y)) = \min\{f_T(0), d(x, y)\}$$

and

$$f_T(E_d(y, z)) = \min\{f_T(0), d(y, z)\}.$$

Thus we obtain that

$$f_T^{(-1)}(f_T(E_d(x, y)) + f_T(E_d(y, z))) =$$

$$f_T^{-1}(\min\{f_T(0), f_T(E_d(x, y)) + f_T(E_d(y, z))\}) \leq E_d(x, z).$$

Since f_T is an additive generator of T we have that

$$T(E_d(x, y), E_d(y, z)) = f_T^{(-1)}(f_T(E_d(x, y)) + f_T(E_d(y, z)))$$

and, thus, we have deduced that

$$T(E_d(x, y), E_d(y, z)) \leq E_d(x, z).$$

Therefore we have shown that E_d is a relaxed T -indistinguishability operator on X .

2). For the purpose of contradiction suppose that T is not strict. Then $f_T(0) < \infty$. Consider the relaxed metric $d_+^{f_T} : [1, \infty[\times [1, \infty[\rightarrow [0, \infty]$ defined by $d_+^{f_T}(x, y) = f_T(0)(x + y)$ for all $x, y \in [1, \infty[$. Notice that $0 < d_+^{f_T}(x, y)$ for all $x, y \in [1, \infty[$, since $0 = f_T(1) < f_T(0)$. By hypothesis we have that $E_{d_+^{f_T}}$ is a relaxed T -indistinguishability operator that separates points. However, $E_{d_+^{f_T}}(1, 2) = E_{d_+^{f_T}}(1, 1) = E_{d_+^{f_T}}(2, 2) = 0$ and $1 \neq 2$.

3). Assertion 2) yields that if E_d is a relaxed T -indistinguishability operator that separates points, then T is strict. Now we show the converse. Assume that T is strict and that d is a relaxed metric on X . By assertion 1) we deduce that E_d is a relaxed T -indistinguishability operator. We only need to prove that E_d separates points. Since T is strict we have that f_T is bijective. Thus the fact that f_T is strictly decreasing and continuous yields that there exists f_T^{-1} which is strictly decreasing and $f_T^{-1}(f_T(x)) = x$ for all $x \in [0, \infty]$. Assume that there exists $x, y \in X$ such that $E_d(x, y) = E_d(x, x) = E_d(y, y)$. Then

$$f_T^{-1}(d(x, y)) = f_T^{-1}(d(x, x)) = f_T^{-1}(d(y, y)).$$

Hence

$$f_T^{-1}(d(x, y)) = f_T^{-1}(d(x, x)) = f_T^{-1}(d(y, y))$$

and, consequently, $d(x, y) = d(x, x) = d(y, y)$. The fact that d is a relaxed metric gives that $x = y$. Therefore the relaxed T -indistinguishability operator E_d separates points.

4). Suppose that there exist $x, y \in X$ such that $d(x, y) = d(x, x) = d(y, y)$. Then

$$f_T^{-1}(d(x, y)) = f_T^{-1}(d(x, x)) = f_T^{-1}(d(y, y)).$$

Thus

$$E_d(x, y) = E_d(x, x) = E_d(y, y).$$

Since the T -indistinguishability operator E_d separates points we deduce that $x = y$ and, thus, that d is a relaxed metric on X .

5). Since $d(x, y) \leq f_T(0)$ for all $x, y \in X$ we have that

$$E_d(x, y) = f_T^{-1}(\min\{f_T(0), d(x, y)\}) = f_T^{-1}(d(x, y))$$

for all $x, y \in X$. Consider $x, y \in X$ such that

$$E_d(x, y) = E_d(x, x) = E_d(y, y).$$

Since f_T^{-1} is strictly decreasing on $Ran(f_T)$ we have that

$$d(x, y) = d(x, x) = d(y, y).$$

The fact that d is a relaxed metric yields that $x = y$. Therefore the relaxed T -indistinguishability operator E_d separates points.

6). Assume that d is a relaxed pseudo-metric on X that fulfills the SSD property. Then $d(x, x) \leq d(x, y)$ for all $x, y \in X$. Since $f_T^{(-1)}$ is decreasing we have that

$$E_d(x, y) = f_T^{(-1)}(d(x, y)) \leq f_T^{(-1)}(d(x, x)) = E_d(x, x)$$

for all $x, y \in X$. □

The following examples show that, in general, the relaxed T -indistinguishability operator E_d provided by Theorem 5 does not separate points and does not enjoy the SSI property.

Example 8 Consider the Lukasiewicz t-norm T_L , which is continuous and not strict Archimedean. Let f_{T_L} be the additive generator of T_L given by $f_{T_L}(x) = 1 - x$ for all $x \in [0, 1]$. Then, the pseudo-inverse $f_{T_L}^{(-1)}$ of f_{T_L} is given by $f_{T_L}^{(-1)}(x) = \max\{1 - x, 0\}$ for all $x \in [0, \infty]$. Consider the relaxed metric $d_+^{f_{T_L}}$ introduced in the proof of assertion 2) in Theorem 5. Then assertion 1) in Theorem 5 gives that E_{d_+} is a relaxed T_L -indistinguishability operator. Nevertheless, E_{d_+} does not separate points. Indeed, $E_{d_+}(3, 2) = E_{d_+}(3, 3) = E_{d_+}(2, 2) = 0$ but $3 \neq 2$.

Example 9 Consider the product t-norm T_P , which is T_P is continuous and Archimedean. Moreover, the pseudo-inverse $f_{T_P}^{(-1)}$ of the additive generator f_{T_P} (introduced in Example 6) is given by $f_{T_P}^{(-1)}(x) = e^{-x}$ for all $x \in [0, \infty]$. Consider the relaxed metric d^+ on $[0, \infty[$ given by $d^+(x, y) = x + y$ for all $x, y \in [0, \infty[$. Then assertion 1) in Theorem 5 gives that E_{d^+} is a relaxed T_P -indistinguishability operator. Nevertheless, the SSI property is not hold by E_{d^+} . Indeed, $E_{d^+}(2, 0) = e^{-2} \not\leq e^{-4} = E_{d^+}(2, 2)$.

In the next example we show that the continuity of the t-norm cannot be relaxed in Theorem 5 in order to guarantee the fact that the fuzzy binary relation is a relaxed indistinguishability operator.

Example 10 Consider the Drastic t-norm T_D , which is Archimedean and non continuous. Moreover, an additive generator f_{T_D} of T_D is given by

$$f_{T_D}(x) = \begin{cases} 2 - x & \text{if } x \in [0, 1[\\ 0 & \text{if } x = 1 \end{cases}.$$

Furthermore, the pseudo-inverse $f_{T_D}^{(-1)}$ of the above additive generator is given by

$$f_{T_D}^{(-1)}(x) = \begin{cases} 0 & \text{if } x \in [2, \infty] \\ 2 - x & \text{if } x \in]1, 2] \\ 1 & \text{if } x \in [0, 1] \end{cases}.$$

Consider the relaxed pseudo-metric d^+ on $[0, \infty]$ introduced in Example 9. It is clear that

$$0.5 = E_{d^+}(\frac{1}{2}, 1) < T_D(E_{d^+}(\frac{1}{2}, 0), E_{d^+}(0, 1)) = T_D(1, 1) = 1.$$

It follows that E_{d^+} is not a relaxed T_D -indistinguishability operator.

Notice that Example 11 (below) yields that the converse of assertion 6) in Theorem 5 is not satisfied in general. In the light of the preceding fact we assume an additional condition about the relaxed pseudo-metric, inspired by assertion 5) in Theorem 5 and by Example 9, in order to assure the converse of the aforesaid assertion.

Theorem 6 *Let X be a nonempty set and let T be a continuous Archimedean t -norm with additive generator $f_T : [0, 1] \rightarrow [0, \infty]$. Let d be a relaxed pseudo-metric on X such that $d(x, y) \leq f_T(0)$ for all $x, y \in X$. Then E_d , given as in Theorem 5, is a relaxed T -indistinguishability operator which fulfills the SSI property if and only if the relaxed pseudo-metric d on X fulfills the SSD property.*

Proof Assume that d is a relaxed pseudo-metric on X that fulfills the SSD property. By assertion 6) in Theorem 5, E_d is a relaxed T -indistinguishability operator which fulfills the SSI property. Suppose that E_d is a relaxed T -indistinguishability operator which fulfills the SSI property. Then

$$E_d(x, y) = f_T^{(-1)}(d(x, y)) \leq E_d(x, x) = f_T^{(-1)}(d(x, x))$$

for all $x, y \in X$. Then we have that

$$f_T^{-1}(\min\{f_T(0), d(x, y)\}) \leq f_T^{-1}(\min\{f_T(0), d(x, x)\})$$

for all $x, y \in X$. It follows that

$$\min\{f_T(0), d(x, x)\} \leq \min\{f_T(0), d(x, y)\},$$

since f_T is decreasing. The fact that $d(x, y) \leq f_T(0)$ implies that

$$d(x, x) = \min\{f_T(0), d(x, x)\} \leq \min\{f_T(0), d(x, y)\} = d(x, y).$$

Whence we conclude that the SSD property is not satisfied by the relaxed pseudo-metric. \square

It must be stressed that the boundness condition in Corollary 6 is always satisfied whenever the t -norm under consideration is strict.

We end the paper showing that the boundness of the relaxed pseudo-metric cannot be deleted in Theorem 6.

Example 11 Consider the pseudo-inverse $f_{T_L}^{(-1)}$ of the additive generator f_{T_L} of the Lukasiewicz t-norm as introduced in Example 8. Moreover, consider the relaxed metric $d_+^{f_{T_L}}$ introduced in Example 8. It is clear that $1 = f_{T_L}(0) < d_+^{f_{T_L}}(x, y)$ for all $x, y \in [1, \infty[$. Assertion 1) in Theorem 5 gives that $E_{d_+^{f_{T_L}}}$ is a relaxed T_L -indistinguishability operator. Besides $E_{d_+^{f_{T_L}}}$ satisfies the SSI property, since

$$E_{d_+^{f_{T_L}}}(x, y) = E_{d_+^{f_{T_L}}}(x, x) = 0$$

for all $x, y \in [1, \infty[$. Nonetheless, the relaxed metric $d_+^{f_{T_L}}$ does not satisfy the SSD property.

4 Conclusions and Future Work

In the last years many generalized metrics have been introduced in the literature with the purpose of developing mathematical tools for quantitative models in Computer Science and Artificial Intelligence. All the aforementioned metrics are particular cases of the notion of relaxed pseudo-metric. In this paper we have studied a duality relationship between relaxed pseudo-metrics and a new class of indistinguishability operators, that we have called relaxed indistinguishability operators, in such a way that the celebrated techniques to generate classical pseudo-metrics from indistinguishability operators, and vice-versa, can be retrieved as a particular case. A few differences between the classical framework and the new one have been exposed.

Among the aforesaid generalized metrics, it is worth mentioning the so-called partial pseudo-metrics which satisfy the SSD property and a modified triangle inequality. Concretely if p is a partial pseudo-metric on X , then

$$p(x, z) \leq p(x, y) + p(y, z) - p(y, y) \quad (1)$$

for all $x, y, z \in X$. Besides, in the literature, as we have mentioned in Section 1, a generalized indistinguishability operator, known as M -valued equality (in the sense of [13,14]), can be found. Specifically if E is an M -valued equality on X , then the following transitivity is hold:

$$T(E(x, y), E(y, z)) \rightarrow_T E(x, z), \quad (2)$$

for all $x, y, z \in X$ and where \rightarrow_T denotes the T -residuum. In [5,6], partial pseudo-metrics have been proposed as the logical counterpart for M -valued equalities. Motivated, on the one hand, by the preceding exposed fact and, on the other hand, by the fact that the triangle inequality (1) is a refinement of that one fulfilled by a relaxed pseudo-metric, it seems natural to try to explore in depth, as a future research, whether (1) and (2) are really dual and, thus, whether the techniques exposed in the present paper can be adapted in such a way that partial pseudo-metrics can be generated from M -valued equalities and vice-versa.

Acknowledgements The authors are very grateful to the reviewers for their valuable suggestions which have helped to improve the content of the paper.

Compliance with ethical standards:

Conflict of interest:

The authors declare that they has no conflict of interest.

Ethical approval:

This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. M.A. Alghamdi, N. Shahza, O. Valero, Fixed point theorems in generalized metric spaces with applications to Computer Science, *Fixed Point Theory A.* **2013**, 2013:118.
2. B. De Baets, R. Mesiar, Pseudo-metrics and T -equivalences, *Fuzzy. Math.* **5** (1997) 471-481.
3. B. De Baets, R. Mesiar, Metrics and T -equalities, *J. Math. Anal. Appl.* **267** (2002) 531-547.
4. J.C. Bezdek, J.D. Harris, Fuzzy partitions and relations: an axiomatic basis for clustering, *Fuzzy Set. Syst.* **256** (1978), 111-127.
5. M. Bukatin, R. Kopperman, S.G. Matthews, Some corollaries of the correspondence between partial metric and multivalued equalities, *Fuzzy Set. Syst.* **256** (2014), 57-72.
6. M. Demirci, The order-theoretic duality and relations between partial metrics and local equalities, *Fuzzy Set. Syst.* **192** (2011) 45-57.
7. M.P. Fourman, D.S. Scott, Sheaves and logic, *Applications of Sheaves, Lecture Notes in Mathematics* **753**, (M. Fourman et al., eds.), Springer Verlag, Berlin, 1979, pp. 302-401.
8. S. Gottwald, On t -norms which are related to distances of fuzzy sets, *BUSEFAL* **50** (1992), 25-30.
9. R. Heckmann, Approximation of metric spaces by partial metric spaces, *Appl. Categor. Struct.* **7** (1999), 71-83.
10. P. Hitzler, A.K. Seda, Generalized distance functions in the theory of computation, *Comput. J.* **53** (2010), 443-464.
11. P. Hitzler, A. Seda, *Mathematical Aspects of Logic Programming Semantics*, CRC Press, Boca Raton, 2011.
12. U. Höhle, Fuzzy equalities and indistinguishability, *Proc. of EUFIT'93, Vol. 1, Aachen, 1993* pp. 358-363.
13. U. Höhle, Many-valued equalities, singletons and fuzzy partitions, *Soft Comput.* **2** (1998), 134-140.
14. U. Höhle, M -valued sets and sheaves over integral, commutative cl -monoids, *Applications of Category Theory to Fuzzy Subsets* (S.E. Rodabaugh et al., eds.), Kluwer Academic Publishers, Dordrecht, 1992, pp. 33-72.
15. E.P. Klement, R. Mesiar, E. Pap, *Triangular Norms*, Kluwer, Dordrecht, 2000.
16. S.G. Matthews, Partial metric topology, *Ann. New York Acad. Sci.* **728** (1994), 183-197.
17. N. Shahza, O. Valero, On 0-complete partial metric spaces and quantitative fixed point techniques in Denotational Semantics, *Abstr. Appl. Anal.* **2013** (2013), Article ID 985095, 11 pages.
18. S. Ovchinnikov, Representation of transitive fuzzy relations, *Aspects of Vagueness* (H. Skala, S. Termini and E. Trillas, eds.), Reidel, Dordrecht, 1984, pp. 105-118.

19. J. Recasens, *Indistinguishability Operators: Modelling Fuzzy Equalities and Fuzzy Equivalence Relations*, Springer, Berlin, 2010.
20. S. Romaguera, O. Valero, A quantitative computational model for complete partial metric spaces via formal balls, *Math. Structures Comput. Sci.* **19** (2009), 541-563.
21. S. Romaguera, O. Valero, Complete partial metric spaces have partially metrizable computational models, *Int. J. Comput. Math.* **89** (2012), 284-290.
22. E. Trillas, Assaig sobre les relacions d'indistingibilitat, *Proc. Primer Congrés Català de Lògica Matemàtica*, Barcelona, 1982, pp. 51-59.
23. L. Valverde, On the structure of F-indistinguishability operators, *Fuzzy Set. Syst.* **17** (1985) 313-328.
24. L.A. Zadeh, Similarity relations and fuzzy orderings, *Inform. Sciences* **3** (1971), 177-200.