On the standard fuzzy metric: generalizations and application to model estimation

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***Abstract*—Different approaches to obtain a notion of metric in the context of fuzzy setting can be found in the literature. In this paper, we deal with the concept due to George and Veeramani, which is defined by means of continuous triangular norms. Different authors have addressed the study of such a concept from a theoretical point of view. In this paper, we provide a new methodology to induce fuzzy metrics which generalize the celebrated standard fuzzy metric. The aforementioned methodology allows us to approach some questions related to the continuous triangular norms from which such fuzzy metrics are defined. Morever, we show the applicability of the new fuzzy metrics to an engineering problem. More specifically, we address successfully robust model estimation through a variant of the well-known estimator RANSAC. By way of illustration of the performance of the approach, we report on the accuracy achieved by the new estimator and other RANSAC variants for a benchmark involving a specific model estimation problem and a large number of datasets with varying proportion of outliers and different levels of noise. The resulting estimator is shown able to outperform the classical counterparts considered.**

***Index Terms*—Fuzzy metric; continuous t-norm; Dombi t-norm; standard fuzzy metric; model estimation; RANSAC**

I. Introduction and preliminaries

In 1965, L. A. Zadeh introduced the notion of fuzzy set in [1]. Since then, such a concept has constituted the grounds of many lines of research in different fields, such as Mathematics, Computer Science, Economics. In Mathematics and, in particular, in Topology, an interesting issue consists in providing a notion of metric, in the fuzzy setting, in accordance with the essence of the classical concept. With this aim, in [2], I. Kramosil and J. Michalek introduced a notion of fuzzy metric space by adapting the concept of statistical metric due to Menger (see [3]) to the fuzzy context. Later on, in [4], A. George and P. Veeramani slightly modified the notion of Kramosil and Michalek with the aim of obtaining a more faithful adaptation to the fuzzy setting of the classical concept of metric. In both cases, the concept of fuzzy metric is defined by means of continuous t-norms (see [5] to find a deep treatment on t-norms). Following [4], a *fuzzy metric space* is a triplet $(X, M, *)$ where X is a non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X, \times]0, \infty[$ satisfying, for each $x, y, z \in X$ and $t, s \in]0, \infty[$, the following:

(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, y, t) = 1$ if and only if $x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(GV5) The assignment $M_{x,y} :]0, \infty[\rightarrow]0, 1]$ is a continuous function.

As usual, we say that $(M, *)$, or simply M if no confusion arises, is a fuzzy metric on X.

On account of the previous definition, the value of $M(x, y, t)$ can be interpreted as a degree of nearness between the point x and y of X with respect to the parameter $t \in]0, \infty[$. Then, the closer to 1 is such a value, the nearer the points x and y with respect to t are. Contrarily, values close to 0 indicate a lower degree of nearness. Thus, in this notion of fuzzy metric, 1 plays a similar role to 0 for the classical case, whereas 0 can be seen as ∞ in classical metrics. So, axiom (GV1) is justified by the fact that the degree of nearness with respect to a parameter never can be zero, just as in the classical case the distance between two points cannot become ∞.

One can easily identify (GV2), (GV3) and (GV4) as fuzzy versions of the axioms of, respectively, separation, symmetry and transitivity, which altogether define the notion of classical metric. Concretely, (GV2) means that, on the one hand, the degree of nearness between two points with respect to an arbitrary parameter only can be 1 whenever both points are the same. On the other hand, the degree of nearness between a point and itself is 1, with respect to an arbitrary parameter. Finally, (GV5) ensures that no drastic changes arise in the degree of nearness due to slight modifications of the parameter with respect to which it is being measured.

An immediate consequence of (GV4), which was pointed out by M. Grabiec for fuzzy metrics in the sense of Kramosil and Michaleck (see [6]), is that the degree of nearness between two points does not decrease when the parameter for which such a degree is relative increases, i.e. for each $x, y \in X$, we have that $M(x, y, t) \geq M(x, y, s)$ for each $t, s]0, \infty[$ with $t > s$.

This kind of fuzzy metric spaces has been studied by several authors from the mathematical point of view. Besides, they have been used successfully in engineering problems such as colour image filtering or perceptual colour difference (see [7]–[11]). Indeed, fuzzy metrics show some advantages with respect to the classical ones. On the one hand, the parameter t allows the fuzzy metric to be better adapted to context in which it is to be used. On the other hand, fuzzy

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metrics match perfectly with the employment of other fuzzy techniques, since the value given by a fuzzy metric, as pointed out before, can be directly interpreted as a fuzzy degree of nearness. So, providing useful techniques for generating fuzzy metrics becomes an interesting issue in order to provide a wider range of measurement tools in such a way that the fuzzy metric that best fits the problem being studied can be applied to solve it.

A celebrated example of fuzzy metric is the so-called standard fuzzy metric, which is defined from a classical metric (see [4]). Indeed, let (X, d) be a metric space and define the fuzzy set M_d on $X \times X \times]0, \infty[$ as follows:

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \text{ for each } x, y \in X, t \in]0, \infty[. \quad (1)$$

The *standard fuzzy metric* on X deduced from d is the pair $(M_d, *_M)$, where $*_M$ denotes the minimum t-norm (i.e. $a *_M b = \min\{a, b\}$ for each $a, b \in [0, 1]$).

Observe that $(X, M_d, *)$ is also a fuzzy metric space for each continuous t-norm $*$, since $*_M$ is the largest t-norm. Indeed, given a continuous t-norm $*$, the inequality $a *_M b \geq a * b$ is satisfied for each $a, b \in [0, 1]$.

From the topological point of view, the standard fuzzy metric enjoys outstanding properties. The topologies generated from the standard fuzzy metric and from the classical metric, from which it is induced, coincide. Besides, it fulfils some interesting properties which do not make sense in the classical context but they do in the fuzzy context. Among others, it should be stressed the property of being strong (see [12]). Let us recall that a fuzzy metric space $(X, M, *)$ is said to be *strong* if, in addition, M satisfies, for each $x, y, z \in X$ and $t \in]0, \infty[$, the next inequality:

$$M(x, z, t) \geq M(x, y, t) * M(y, z, t). \quad (2)$$

Observe that the preceding inequality is stronger than that given in the axiom (GV4).

It is a well-known fact that, given a metric space (X, d), then the standard fuzzy metric space $(X, M_d, *_P)$ is strong, where $*_P$ denotes the usual product t-norm, i.e. $a *_P b = a \cdot b$ for each $a, b \in [0, 1]$. Nevertheless, $(X, M_d, *_M)$ is not a strong fuzzy metric space in general, as pointed out in [12]. In view of the preceding fact, an interesting question arises: there exists a continuous t-norm $*$, different from $*_P$, with $* \geq *_P$ and such that $(X, M_d, *)$ is a strong fuzzy metric space for each metric space (X, d)?

In [13], a generalization of the fuzzy set M_d given by (1) was introduced defining, for each $x, y \in X$ and $t \in]0, \infty[$, the next fuzzy set:

$$M_d^{g,m}(x, y, t) = \frac{g(t)}{g(t) + m \cdot d(x, y)}, \quad (3)$$

where $m \in]0, \infty[$ and $g :]0, \infty[\to]0, \infty[$ is a non-decreasing continuous function. According to [13], $(X, M_d^{g,m}, *_P)$ is a strong fuzzy metric space. Nevertheless, an extra condition on g is required so that $(X, M_d^{g,m}, *_M)$ is a fuzzy metric space for any arbitrary metric space (X, d). Indeed, if the function g is not superadditive, i.e. $g(t+s) \geq g(t) + g(s)$ for each $t, s \in]0, \infty[$, then $(X, M_d^{g,m}, *_M)$ is not, in general, a fuzzy metric space. Again, similar to the case of the standard fuzzy metric, it seems natural to wonder whether there exits a continuous t-norm, different from $*_P$, with $* \geq *_P$ such that $(X, M_d^{g,m}, *)$ is a fuzzy metric space for each metric space (X, d) without requiring any extra condition on g.

Coming back to the applicability of fuzzy metrics, in most problems, we are interested in measuring some kind of difference or similarity between objects. Therefore, fuzzy metrics can be good candidates to evaluate such a measurement. Concretely, the fuzzy set M is used to provide the aforementioned difference or similarity. However, the t-norm that defines M as a fuzzy metric does not play any role in the way in which such a measure is provided and, thus, it does not contribute anything that can make the fuzzy metric better fit for the problem under consideration. Since the fuzzy set $M_d^{g,m}$ given by expression (3) depends on more elements than the standard fuzzy metric M_d, $M_d^{g,m}$ allows to get more flexibility to obtain a measurement tool that fits better to the problem under consideration than M_d. So, providing a fuzzy set that generalizes expression (3) could improve the potential applicability of fuzzy metrics, even though such a generalization does not become a fuzzy metric for the same class of t-norms for which $M_d^{g,m}$ is so.

In the light of the exposed facts, the aim of this paper is twofold. On the one hand, we focus our efforts on obtaining a fuzzy set that generalizes expression (3) and on finding a family of continuous t-norms for which this new fuzzy set becomes a fuzzy metric. Moreover, we are interested in the study of those continuous t-norms for which this new fuzzy metric fulfils the property of being strong. Such a study allows us to approach the two questions posed above. On the other hand, we address a model estimation problem as an example of engineering application to illustrate the applicability of the new fuzzy metric proposed in Section II.

II. The generalized standard fuzzy metric

In this section, we build a new fuzzy metric which generalizes, in some sense, the standard fuzzy metric and the fuzzy metric given by expression (3). To this end, we recall a well-known family of continuous t-norms introduced by J. Dombi in [14].

Given $\lambda \in]0, \infty[$ the t-norm $*_{Dom}^{\lambda}$ is defined, for each $a, b \in [0, 1]$, by the following expression:

$$a *_{Dom}^{\lambda} b = \begin{cases} 0, & \text{if } a = b = 0 \\ \frac{1}{1+\left(\left(\frac{1-a}{a}\right)^{\lambda}+\left(\frac{1-b}{b}\right)^{\lambda}\right)^{\frac{1}{\lambda}}} & \text{otherwise} \end{cases} \quad (4)$$

The construction of the promised fuzzy metric can be found in the next result:

Theorem 2.1: Let (X, d) be a metric space, $m, n \in]0, \infty[$ and $g :]0, \infty[\to]0, \infty[$ be a non-decreasing continuous function. Define the fuzzy set $M_d^{m,n,g}$ on $X \times X \times]0, \infty[$ as:

$$\tilde{M}_d^{g,m,n}(x, y, t) = \frac{g(t)}{g(t) + m \cdot d^n(x, y)}, \quad (5)$$

where $d^n(x,y)$ denotes $(d(x,y))^n$. Then, $(X, \tilde{M}_d^{g,m,n}, *)$ is a fuzzy metric space for each continuous t-norm $*$ satisfying $* \leq *_{Dom}^{\frac{1}{n}}$.

Proof.

Let $*$ be a continuous t-norm such that $* \leq *_{Dom}^{\frac{1}{n}}$. It is not hard to check that $\tilde{M}_d^{g,m,n}$ satisfies axioms (GV1), (GV2), (GV3) and (GV5). It remains to prove that (GV4) also holds.

Let $x, y, z \in X$ and $t, s \in]0, \infty[$. We will see that

$$\tilde{M}_d^{g,m,n}(x,z,t+s) \geq \tilde{M}_d^{g,m,n}(x,y,t) * \tilde{M}_d^{g,m,n}(y,z,s).$$

Set $\alpha = \max\{g(t), g(s)\}$. Observe that

$$\tilde{M}_d^{g,m,n}(x,z,t+s) \geq \frac{\alpha}{\alpha + m \cdot d^n(x,z)},$$

$$\tilde{M}_d^{g,m,n}(x,y,t) \leq \frac{\alpha}{\alpha + m \cdot d^n(x,y)}$$

and

$$\tilde{M}_d^{g,m,n}(y,z,s) \leq \frac{\alpha}{\alpha + m \cdot d^n(y,z)}.$$

So, since $* \leq *_{Dom}^{\frac{1}{n}}$, we have that

$$\tilde{M}_d^{g,m,n}(x,y,t) * \tilde{M}_d^{g,m,n}(y,z,s) \leq$$

$$\leq \tilde{M}_d^{g,m,n}(x,y,t) *_{Dom}^{\frac{1}{n}} \tilde{M}_d^{g,m,n}(y,z,s) \leq$$

$$\leq \frac{\alpha}{\alpha + m \cdot d^n(x,y)} *_{Dom}^{\frac{1}{n}} \frac{\alpha}{\alpha + m \cdot d^n(y,z)} =$$

$$= \frac{1}{1 + \frac{m \cdot (d(x,y) + d(y,z))^n}{\alpha}} = \frac{\alpha}{\alpha + m \cdot (d(x,y) + d(y,z))^n} \leq$$

$$\leq \frac{\alpha}{\alpha + m \cdot d^n(x,z)} \leq \tilde{M}_d^{g,m,n}(x,z,t+s).$$

Therefore, for each $x, y, z \in X$ and $t, s \in]0, \infty[$, $\tilde{M}_d^{g,m,n}$ satisfies (GV4) for $*$ and we conclude that $(X, \tilde{M}_d^{g,m,n}, *)$ is a fuzzy metric space. □

It must be stressed that $(X, \tilde{M}_d^{g,m,n}, *)$ is not a fuzzy metric space, in general, when $*$ does not satisfy the condition $* \leq *_{Dom}^{\frac{1}{n}}$, as the next example shows.

Example 2.2: Let $(\mathbb{R}, d_u)$ be the metric space where d_u is the Euclidean metric on $\mathbb{R}$, i.e. $d_u(x,y) = |x-y|$. Consider the non-decreasing continuous function $g_1 :]0, \infty[\to]0, \infty[$ given by $g(t) = 1$, for each $t \in]0, \infty[$, and $m = n = 1$. Then, the fuzzy set $\tilde{M}_{d_u}^{g_1,1,1}$ is given by expression (5) as follows:

$$\tilde{M}_{d_u}^{g_1,1,1}(x,y,t) = \frac{1}{1 + d_u(x,y)}, \text{ for each } x, y \in \mathbb{R}, t \in]0, \infty[.$$

Let $*$ be a continuous t-norm such that $* \not\leq *_{Dom}^{1}$. Then, there exists $a, b \in]0,1[$ such that $a * b > a *_{Dom}^{\frac{1}{n}} b$.

Consider $x = \frac{a-1}{a}$, $y = 0$, $z = \frac{1-b}{b}$ and $t, s \in]0, \infty[$. Then,

$$\tilde{M}_{d_u}^{g_1,1,1}(x,z,t+s) = \frac{1}{1 + \frac{1-a}{a} + \frac{1-b}{b}} = a *_{Dom}^{1} b,$$

$$\tilde{M}_{d_u}^{g_1,1,1}(x,y,t) = a \text{ and } \tilde{M}_{d_u}^{g_1,1,1}(y,z,s) = b.$$

Therefore, $\tilde{M}_{d_u}^{g_1,1,1}(x,y,t) * \tilde{M}_{d_u}^{g_1,1,1}(y,z,s) = a * b > a *_{Dom}^{1} b = \tilde{M}_{d_u}^{g_1,1,1}(x,z,t+s)$, and so $\tilde{M}_{d_u}^{g_1,1,1}$ does not satisfy (GV4).

On account of Theorem 2.1 and the preceding example, we conclude that $*_{Dom}^{\frac{1}{n}}$ is the largest (continuous) t-norm for which $\tilde{M}_d^{g,m,n}$ is a fuzzy metric on X, for each arbitrary metric space (X,d), each non-decreasing continuous function $g :]0, \infty[\to]0, \infty[$ and each $m, n \in]0, \infty[$. Such a conclusion allows us to approach the two questions posed in Section I.

On the one hand, Theorem 2.1 introduces a generalization of the fuzzy set given by expression (3). Indeed, such a fuzzy set is obtained by considering $n = 1$ in the fuzzy set defined by expression (5), i.e. $M_d^{g,m} = \tilde{M}_d^{g,m,1}$. Besides, the aforementioned theorem establishes that $M_d^{g,m}$ is a fuzzy metric on X for each t-norm $*$ with $* \leq *_{Dom}^{1}$. This fact allows us to answer in affirmative way one of the questions that we wondered in Section I, which is whether there exists a continuous t-norm $* \geq *_P$ such that $(X, M_d, *)$ is a fuzzy metric space for each metric space (X,d) without requiring any extra condition on g.

First, observe that, for each $a, b \in]0,1]$, we have that

$$a *_{Dom}^{1} b = \frac{1}{1 + \frac{1-a}{a} + \frac{1-b}{b}} = \frac{ab}{a + b - ab}.$$

Now,

$$\frac{ab}{a+b-ab} \geq a \cdot b \Leftrightarrow 1 \geq a + b - ab = a(1-b) + b.$$

Taking into account that $a \leq 1$ we have that $1 = 1 - b + b \geq a(1-b) + b$. So, $a *_{Dom}^{1} b \geq a *_P b$ for each $a, b \in]0,1]$. Thus $*_{Dom}^{1} \geq *_P$ for each $a, b \in [0,1]$, since $a *_{Dom}^{1} b = 0 = a *_P b$ whenever $a = 0$ or $b = 0$.

Hence, we have found a continuous Archimedean t-norm greater that the product t-norm $*_P$ for which $M_d^{g,m}$ is a fuzzy metric on X, for each arbitrary metric space (X,d), each non-decreasing continuous function $g :]0, \infty[\to]0, \infty[$ and each $m \in]0, \infty[$. Furthermore, on account of Example 2.2 we conclude that $*_{Dom}^{1}$ is the largest t-norm for which $(X, M^{g,1}, *_{Dom}^{1})$ is a fuzzy metric space, in general.

On the other hand, $\tilde{M}_d^{g,m,n}$ becomes the standard fuzzy metric when we consider the non-decreasing continuous function $g(t) = t$ and $m = n = 1$. Under this remark and, based on the argument exposed in the proof of Theorem 2.1, we prove the next result which will be useful to answer the first question about the standard fuzzy metric set out in Section I.

Theorem 2.3: Let (X,d) be a metric space, $m, n \in]0, \infty[$ and $g :]0, \infty[\to]0, \infty[$ be a non-decreasing continuous function. Then the fuzzy metric space $(X, \tilde{M}_d^{g,m,n}, *)$, where $\tilde{M}_d^{g,m,n}$ is given by (5), is strong for each continuous t-norm satisfying $* \leq *_{Dom}^{\frac{1}{n}}$.

Proof.

Consider a continuous t-norm $*$ such that $* \leq *_{Dom}^{\frac{1}{n}}$. By Theorem 2.1 we conclude that $(X, \tilde{M}_d^{g,m,n}, *)$ is a fuzzy metric space. It remains to show that inequality (2) holds.

Let $x, y, z \in X$ and $t, s \in]0, \infty[$. Then,

$$\tilde{M}_d^{g,m,n}(x,z,t) = \frac{g(t)}{g(t) + m \cdot d^n(x,z)} \geq$$

$$\geq \frac{g(t)}{g(t) + m \cdot (d(x,y) + d(y,z))^n} =$$

$$= \tilde{M}_d^{g,m,n}(x,y,t) *_{Dom}^{\frac{1}{n}} \tilde{M}_d^{g,m,n}(y,z,t) \geq$$

$$\tilde{M}_d^{g,m,n}(x,y,t) * \tilde{M}_d^{g,m,n}(y,z,t).$$

Hence, $(X, \tilde{M}_d^{g,m,n}, *)$ is a strong fuzzy metric space. □

As a consequence of the previous theorem, we conclude that $(X, M_d, *_{Dom}^1)$ is a strong fuzzy metric space. This fact answers affirmatively the first question lay out in Section I by providing a continuous t-norm greater that the product t-norm $*_P$ for which the standard fuzzy metric is strong for any arbitrary metric space (X, d).

Even more, on account of Example 2.2 we conclude that the standard fuzzy metric is just a strong fuzzy metric in general for continuous t-norms less than $*_{Dom}^1$. Notice that the aforementioned example provides that the standard fuzzy metric $(\mathbb{R}, M_{d_u}, *)$ is not strong if $* \nleq *_{Dom}^1$. Indeed, let $a, b \in]0, 1[$ such that $a * b > a *_{Dom}^1 b$. Then, take $x = \frac{a-1}{a}$, $y = 0$ and $z = \frac{1-b}{b}$. Therefore,

$$M_{d_u}(x,z,1) = \frac{1}{1 + \frac{1-a}{a} + \frac{1-b}{b}} = a *_{Dom}^1 b < a * b =$$

$$= M_{d_u}(x,y,1) * M_{d_u}(y,z,1).$$

III. Application Case: Robust Model Estimation

Solving model estimation problems is a fundamental component of numerous applications involving perception tasks. Nowadays, facing this kind of problem requires to cope with new challenges due to an increased use of poor, low-cost sensors, and the ever growing deployment of robotic devices which may operate in potentially unknown environments. Generally speaking, the underlying algorithms have to be robust against uncertain data that besides may be corrupted by outliers, i.e. data items which are not consistent with the original model due to an arbitrary bias affecting them. A *robust estimator* is able to correctly find the original model that supposedly the input data fits to under the aforementioned conditions [15]. The Random Sample Consensus algorithm (RANSAC) [16] is one of these robust estimation techniques, which is widely used nowadays, so much that it has become common in robotics and computer vision.

Briefly speaking, RANSAC tries to achieve a maximum consensus in the input dataset in order to deduce the inliers by generating random hypotheses on the model parameters through a *hypothesize-and-verify* approach. That is to say, instead of using every sample in the dataset to perform the estimation as in traditional regression techniques, RANSAC tests many random sets of samples and outputs the one leading to the best fitting. Since picking an extra point decreases exponentially the probability of selecting an outlier-free sample [17], RANSAC takes the Minimum Sample Set size (MSS) to determine a unique candidate model, thus increasing its chances of finding an all-inlier sample set. This model is assigned a score based on the cardinality of its consensus set. Finally, RANSAC returns the hypothesis that has achieved the highest consensus and the set of inliers, which are used next to estimate the ultimate model by regression.

Searching for an all-inlier sample, RANSAC typically runs for N iterations:

$$N = \frac{\log(1-\rho)}{\log(1-(1-\omega)^s)} \tag{6}$$

where ρ is the desired probability of success, i.e. at least one of the considered random sets is outlier-free, s is the size of the MSS for the problem at hand and ω is the ratio of outliers. See [16] for the details on Eq. (6).

Algorithm 1 outlines FM-based RANSAC, a variant of RANSAC described in [18] that avoids discriminating between inliers and outliers by means of the use of a fuzzy metric that encodes as a similarity the compatibility of each sample to the currently hypothesized model. In this work, we particularize FM-based RANSAC for the fuzzy metric $M_d^{g,m,n}$ introduced as Eq. (5) in Section II. $M_d^{g,m,n}$ is also incorporated into the final model refinement step that follows the main hypothesis selection loop. Finally, in Section IV, we report on the accuracy achieved by FM-based RANSAC for a specific model estimation problem when using $M_d^{g,m,n}$ for different values of m and n. The assessment involves a comparison with RANSAC and MSAC [19] for a benchmark comprising a large number of datasets with varying proportion of outliers and different levels of noise.

We detail next the features of FM-based RANSAC:

1) **Samples classification**. In the original RANSAC, for every model considered, data samples are classified into inliers and outliers by comparing the fitting error with a threshold τ_I related to data noise. As already mentioned, FM-based RANSAC does not distinguish between inliers and outliers, but makes use of a compatibility value $\phi \in [0,1]$ between each sample x_j and the current model $\mathcal{M}_{\widehat{\Theta}_k}$, given the fitting error $\epsilon(x_j; \mathcal{M}_{\widehat{\Theta}_k})$. Such compatibility value derives from the fuzzy metric $M_d^{g,m,n}$ once parameterized by (d, Φ) with $\Phi = (n, m, g)$. Since in the following we contemplate the use of an only, specific distance d, i.e. the Euclidean metric, and g is set to the constant function θ^n as a reference of noise scale[1], we denote the fuzzy metric as $M^{m,n}$ eliminating the allusion to d and g. From now on, the value of $M^{m,n}$ will be denoted by $\phi(\epsilon; \Phi)$.
2) **Model scoring**. The individual compatibility values $\phi(\epsilon; \Phi)$ are aggregated by simple summation to obtain the model score (step 6 in Alg. 1) and hence the *so-far-the-best-model* is given by the maximum score found up to the current iteration (steps 7 - 9 of Alg. 1).

[1]In this regard, we refer to the form $M_d^{g,m,n}(x,y) = \frac{1}{1+m\cdot(d(x,y)/\theta)^n}$.

Algorithm 1 FM-based RANSAC

Input: D - dataset comprising samples $\{x_j\}$
$\phi(\epsilon\,;\,\Phi)$ - FM compatibility value for fitting error
k_{max} - maximum number of iterations of the main loop
t_{max} - maximum number of iterations of the refinement stage

Output: $\mathcal{M}_{\widehat{\Theta}}$ - estimated model

1: $k := 0,\ \varphi_{\text{max}} := -\infty$
2: **for** $k := 1$ **to** k_{max} **do** ▷ find best consensus model $\mathcal{M}_{\widehat{\Theta}}$
3: select randomly a minimal sample set S_k of size s
4: estimate model $\mathcal{M}_{\widehat{\Theta}_k}$ from S_k
5: calculate fitting errors $\epsilon(x_j;\mathcal{M}_{\widehat{\Theta}_k}), \forall x_j \in \text{D}$
6: find model score $\varphi_k := \sum_{x_j\in\text{D}} \phi(\,\epsilon(x_j;\mathcal{M}_{\widehat{\Theta}_k})\,;\,\Phi\,)$
7: **if** $\varphi_k > \varphi_{\text{max}}$ **then**
8: $\varphi_{\text{max}} := \varphi_k,\ \mathcal{M}^0_{\widehat{\Theta}} := \mathcal{M}_{\widehat{\Theta}_k}$
9: **end if**
10: **end for**
11: $t := 0$
12: **repeat** ▷ refine model $\mathcal{M}_{\widehat{\Theta}}$
13: calculate fitting errors $\epsilon(x_j;\mathcal{M}^t_{\widehat{\Theta}}), \forall x_j \in \text{D}$
14: estimate model $\mathcal{M}^{t+1}_{\widehat{\Theta}}$ using weights $\phi(\epsilon(x_j;\mathcal{M}^t_{\widehat{\Theta}})\,;\,\Phi\,)$
15: $t := t + 1$
16: **until** convergence **or** $t \geq t_{\text{max}}$
17: **return** $\mathcal{M}^t_{\widehat{\Theta}}$

3) **Model refinement**. Once a sufficient number of models have been considered, we re-estimate the winning model using iterative weighted least squares, where the compatibility values $\phi(\epsilon;\Phi)$, calculated for the fitting errors resulting from the current model, are used as weights for the new, refined model (steps 12 - 16 of Alg. 1). The loop iterates until changes in the estimated parameters of the model $\widehat{\Theta}$ are negligible (or after t_{max} iterations).

IV. Experimental results

A. Experimental setup

For testing purposes, we consider a hyperplane model estimation problem for 2D (straight lines), 3D (planes) and 10D, the latter as a case of higher dimensionality. To this end, we generate synthetic datasets stemming from hyperplanes in random orientations and positions: 500 for 2D/3D hyperplanes and 250 for 10D hyperplanes. Given a 2D/3D/10D random point p belonging to a hyperplane with normal vector $\vec{n}$, an inlier p_I is generated by shifting p along $\vec{n}$ using a zero-mean Gaussian distribution with standard deviation σ, i.e. $p_I = p + \mathcal{N}(0,\sigma)\cdot\vec{n}$. Outliers p_O are uniformly generated within a rectangular area containing part of the hyperplane, ensuring that they lie out of a 3σ stripe at both sides of the hyperplane. Every pair (σ,ω) gives rise to a different dataset.

Regarding hypothesis generation within the main loop, in all experiments, the size of the MSS is always set to the minimum, i.e. $s = 2$, $s = 3$ and $s = 10$ for respectively 2D, 3D and 10D. Besides, the number of iterations k_{max} is calculated according to Eq. (6), with $\rho = 99\%$. The parameters of $\phi(\epsilon;\Phi)$, $\Phi = (n,m,g)$, are set as follows: $n,m \in \{1,2\}$, as indicated for each experiment; and g is the constant function θ^n, where $\theta = \kappa\cdot\sigma$. For RANSAC/MSAC $\tau_I = \kappa\cdot\sigma$. Different values for κ are considered for both θ and τ_I. Finally, to compare properly RANSAC, MSAC and our estimator, we make use of the same sequence of MSS's to avoid the effect of randomness.

B. Results and discussion

In the following, to measure the estimation accuracy for the hyperplane fitting problem, we make use of the average $\mu[\varepsilon]$ of the angle ε between the true and the estimated normal vector. We also report on the average number of iterations spent during model refinement $\mu[t]$.

Table I and Fig. 1 show performance results for the fuzzy metric $M^{m,n}$ and several outlier ratios ω and Gaussian noise magnitudes σ. In sight of these results, it is worth noting that: (1) the estimation accuracy for $M^{2,2}$ is above that of plain RANSAC and MSAC in all cases, while, for the other configurations of $M^{m,n}$, the accuracy is in general better than the classical counterparts though not in all cases; (2) the value of θ in $M^{m,n}$ does not seem to be critical, since the highest change in $\mu[\varepsilon]$ for θ with $\kappa \in \{1, 2, 2.5, 3, 4\}$ is less than 1°; (3) the distribution of the average error $\mu[\varepsilon]$ shows always larger errors for RANSAC/MSAC than for $M^{2,2}$ for all percentiles. As for the number of iterations of the refinement stage t: (4) in general, $\mu[t]$ is similar for every combination of $M^{m,n}$ when varying the noise parameters (σ,ω) and particularly higher for $M^{2,2}$ when κ is low; (5) lower values of κ allow the proposed estimator to perform a better refinement stage in terms of accuracy but at the expense of computational cost since more iterations are required. Regarding the fuzzy metric $M^{m,n}$, both $M^{2,n}$ and $M^{m,2}$ lead in general to higher accuracy, with a slight increase in the number of refining iterations for low κ values or higher noise (σ,ω).

V. Conclusions

This work introduces a methodology to induce fuzzy metrics that generalizes the celebrated standard fuzzy metric. Moreover, some questions related to the continuous triangular norms from which such fuzzy metrics are defined have been posed and answered. A concrete new fuzzy metric induced through the aforementioned methodology has been succesfully embedded within a revised version of RANSAC. By means of this metric, we avoid discriminating between inliers and outliers, to instead make use of a compatibility value to the current model for each sample. Experimental results show good performance, actually outperforming two classical counterparts, RANSAC and MSAC.

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TABLE I

2D/3D/10D HYPERPLANES ESTIMATION ACCURACY AND NUMBER OF ITERATIONS OF THE REFINEMENT STAGE FOR (TOP) DIFFERENT OUTLIER RATIOS ω, (MIDDLE) DIFFERENT NOISE MAGNITUDES σ AND (BOTTOM) DIFFERENT SETTINGS FOR $\tau_I, \theta = \kappa \cdot \sigma$. WHEN KEPT CONSTANT: $\sigma = 1$, $\omega = 0.4$, $\kappa = 3$. LIGHTER BACKGROUND MEANS HIGHER PERFORMANCE.

2D	$\mu[\varepsilon]$ (°)							$\mu[t]$			
ω	**RANSAC**	**MSAC**	**FM-based RANSAC**				ω	**FM-based RANSAC**			
			$M^{1,1}$	$M^{1,2}$	$M^{2,1}$	$M^{2,2}$		$M^{1,1}$	$M^{1,2}$	$M^{2,1}$	$M^{2,2}$
0.60	4.43	3.14	4.63	4.10	3.97	3.10	**0.60**	9.11	10.44	10.79	11.05
0.50	3.03	2.33	2.61	2.33	2.28	1.88	**0.50**	7.28	7.93	8.57	8.58
0.40	2.13	1.81	1.77	1.59	1.57	1.33	**0.40**	6.47	6.71	7.68	7.40
0.20	1.58	1.53	0.96	0.88	0.89	0.80	**0.20**	5.68	5.51	6.82	6.32

σ	**RANSAC**	**MSAC**	**FM-based RANSAC**				σ	**FM-based RANSAC**			
			$M^{1,1}$	$M^{1,2}$	$M^{2,1}$	$M^{2,2}$		$M^{1,1}$	$M^{1,2}$	$M^{2,1}$	$M^{2,2}$
2.00	9.82	6.92	4.08	4.62	3.87	4.46	**2.00**	7.40	8.38	9.00	9.64
1.00	2.13	1.81	1.77	1.59	1.57	1.33	**1.00**	6.47	6.71	7.68	7.40
0.50	0.74	0.71	0.90	0.55	0.73	0.45	**0.50**	6.20	5.63	7.21	6.29
0.25	0.37	0.36	0.48	0.20	0.36	0.17	**0.25**	6.02	4.93	6.87	5.57

κ	**RANSAC**	**MSAC**	**FM-based RANSAC**				κ	**FM-based RANSAC**			
			$M^{1,1}$	$M^{1,2}$	$M^{2,1}$	$M^{2,2}$		$M^{1,1}$	$M^{1,2}$	$M^{2,1}$	$M^{2,2}$
4.00	2.85	2.09	1.85	1.84	1.65	1.54	**4.00**	6.05	6.26	7.14	6.81
3.00	2.13	1.81	1.77	1.59	1.57	1.33	**3.00**	6.47	6.71	7.68	7.40
2.50	2.03	1.88	1.71	1.45	1.52	1.23	**2.50**	6.79	7.02	8.03	7.94
2.00	2.18	2.18	1.65	1.29	1.47	1.13	**2.00**	7.14	7.54	8.61	8.78
1.00	3.60	3.58	1.47	1.04	1.35	1.01	**1.00**	8.61	10.65	10.66	13.56

3D	$\mu[\varepsilon]$ (°)							$\mu[t]$			
ω	**RANSAC**	**MSAC**	**FM-based RANSAC**				ω	**FM-based RANSAC**			
			$M^{1,1}$	$M^{1,2}$	$M^{2,1}$	$M^{2,2}$		$M^{1,1}$	$M^{1,2}$	$M^{2,1}$	$M^{2,2}$
0.60	6.14	4.58	5.11	4.00	4.24	3.04	**0.60**	10.02	10.87	11.69	11.47
0.50	4.07	3.48	2.90	2.34	2.46	1.87	**0.50**	7.55	8.10	9.05	8.83
0.40	3.13	3.01	1.95	1.61	1.69	1.35	**0.40**	6.69	6.91	8.01	7.67
0.20	2.31	2.29	1.07	0.92	0.98	0.84	**0.20**	5.73	5.55	6.87	6.41

σ	**RANSAC**	**MSAC**	**FM-based RANSAC**				σ	**FM-based RANSAC**			
			$M^{1,1}$	$M^{1,2}$	$M^{2,1}$	$M^{2,2}$		$M^{1,1}$	$M^{1,2}$	$M^{2,1}$	$M^{2,2}$
2.00	13.32	9.97	5.21	4.92	4.53	4.32	**2.00**	8.68	9.63	10.45	10.66
1.00	3.13	3.01	1.95	1.61	1.69	1.35	**1.00**	6.69	6.91	8.01	7.67
0.50	1.11	1.08	0.99	0.57	0.79	0.47	**0.50**	6.32	5.82	7.43	6.52
0.25	0.64	0.63	0.52	0.21	0.39	0.18	**0.25**	6.15	5.19	7.14	5.91

κ	**RANSAC**	**MSAC**	**FM-based RANSAC**				κ	**FM-based RANSAC**			
			$M^{1,1}$	$M^{1,2}$	$M^{2,1}$	$M^{2,2}$		$M^{1,1}$	$M^{1,2}$	$M^{2,1}$	$M^{2,2}$
4.00	3.71	2.95	2.07	1.87	1.79	1.56	**4.00**	6.23	6.47	7.40	7.04
3.00	3.13	3.01	1.95	1.61	1.69	1.35	**3.00**	6.69	6.91	8.01	7.67
2.50	3.23	3.22	1.88	1.46	1.63	1.25	**2.50**	7.00	7.28	8.45	8.25
2.00	3.75	3.75	1.79	1.31	1.57	1.15	**2.00**	7.40	7.87	9.04	9.17
1.00	5.73	5.83	1.57	1.07	1.43	1.06	**1.00**	9.04	11.25	11.33	14.38

10D	$\mu[\varepsilon]$ (°)							$\mu[t]$			
ω	**RANSAC**	**MSAC**	**FM-based RANSAC**				ω	**FM-based RANSAC**			
			$M^{1,1}$	$M^{1,2}$	$M^{2,1}$	$M^{2,2}$		$M^{1,1}$	$M^{1,2}$	$M^{2,1}$	$M^{2,2}$
0.60	10.69	10.18	7.46	4.19	5.53	3.00	**0.60**	14.12	14.53	15.61	14.52
0.50	8.82	8.77	3.56	2.26	2.83	1.78	**0.50**	8.85	9.50	10.67	10.34
0.40	6.92	6.94	2.29	1.54	1.88	1.29	**0.40**	7.30	7.60	8.88	8.50
0.20	5.66	5.70	1.17	0.90	1.03	0.84	**0.20**	6.02	5.98	7.28	6.88

σ	**RANSAC**	**MSAC**	**FM-based RANSAC**				σ	**FM-based RANSAC**			
			$M^{1,1}$	$M^{1,2}$	$M^{2,1}$	$M^{2,2}$		$M^{1,1}$	$M^{1,2}$	$M^{2,1}$	$M^{2,2}$
2.00	26.76	20.83	9.83	7.47	7.49	5.52	**2.00**	15.13	16.20	16.62	16.38
1.00	6.92	6.94	2.29	1.54	1.88	1.29	**1.00**	7.30	7.60	8.88	8.50
0.50	3.03	3.04	1.08	0.55	0.84	0.48	**0.50**	6.74	6.38	8.00	7.22
0.25	1.30	1.31	0.56	0.22	0.42	0.20	**0.25**	6.45	5.81	7.56	6.63

κ	**RANSAC**	**MSAC**	**FM-based RANSAC**				κ	**FM-based RANSAC**			
			$M^{1,1}$	$M^{1,2}$	$M^{2,1}$	$M^{2,2}$		$M^{1,1}$	$M^{1,2}$	$M^{2,1}$	$M^{2,2}$
4.00	6.26	6.09	2.50	1.85	2.03	1.49	**4.00**	6.96	7.16	8.14	7.74
3.00	6.92	6.94	2.29	1.54	1.88	1.29	**3.00**	7.30	7.60	8.88	8.50
2.50	7.75	7.86	2.16	1.39	1.80	1.20	**2.50**	7.78	8.03	9.28	9.11
2.00	9.03	9.13	2.03	1.26	1.71	1.14	**2.00**	8.14	8.79	10.06	10.06
1.00	12.61	12.61	1.71	1.10	1.52	1.12	**1.00**	10.06	12.41	12.58	15.87

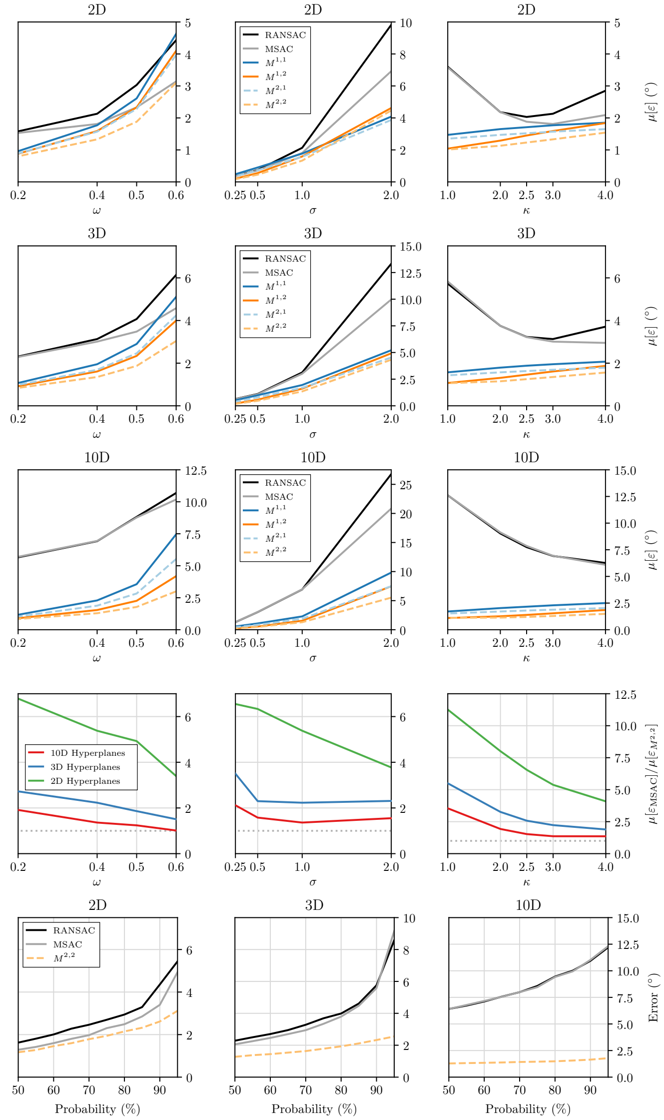

Fig. 1. (1st, 2nd, 3rd rows) Estimation accuracy for resp. 2D, 3D and 10D hyperplanes varying (left) the outlier ratio ω, (middle) the noise magnitude σ and (right) the setting of $\tau_I, \theta = \kappa \cdot \sigma$. (4th row) Ratio between the error of MSAC $\mu[\varepsilon_{\text{MSAC}}]$ and the error of FM-based RANSAC for $M^{2,2}$ $\mu[\varepsilon_{M^{2,2}}]$. (5th row) Comparison between FM-based RANSAC for $M^{2,2}$ and RANSAC/MSAC for different percentiles of the estimation error.

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