

## Quantum Wavefunction and Fluctuation of Classical Action

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The classical action:  $\int_0^T dt L(v(t), x(t), t)$  may be varied to find a stationary solution which yields Newton's second law, a standard procedure in classical mechanics.

Is it possible to obtain quantum mechanical motion from this same classical action?

We argue that one may write the action for a relativistic/nonrelativistic particle moving at constant speed in terms of  $X$  and  $T$  i.e.  $v = \text{velocity} = \text{constant} = X/T$ . Varying with respect to  $X$  and  $T$  yields:

$d/dT \text{ Action} = -E$  and  $d/dX \text{ Action} = p$ . Thus, changing  $X$  by  $\delta X$  yields a change in the action of  $p \delta X$  which holds for all  $X$  because of the constant velocity. If one envisions a periodic function such as  $\exp(i \text{ Action})$ , this behaves as  $\exp(ipx)$  in terms of  $-id/dX$ . Thus, there is an intrinsic wavelength related to  $1/p$ . This follows directly from small changes of the Action.

Alternatively, one may create a differential equation such as the Klein-Gordon or Schrodinger equation from classical conservation of energy-momentum using  $id/dT \exp(i \text{ Action}) = E \exp(i \text{ Action})$  and  $-id/dX \exp(i \text{ Action}) = p \exp(i \text{ Action})$ .

We note that  $\exp(ipx)$  which describes the fluctuational behaviour of the classical Action may have positive and negative values showing that fluctuations may be both positive and negative. These negative and positive fluctuations are, however, periodic so there is some systematic behaviour occurring, it seems.. We argue that  $\exp(ipx)$  is statistical as its modulus is 1, indicating that a free particle has equal probability to be at any  $x$  point. For a system with a potential  $V(x)$ , one may create an ensemble describing fluctuations of all possible  $p$  values i.e.  $W(x) = \text{wavefunction} = \sum_p a(p) \exp(ipx)$ . The fluctuations are regular positive and negative, but there is different wavelength behaviour for each  $p$  i.e.  $1/p$  so there is interference to yield an average fluctuation pattern i.e. the wavefunction.

### Classical Action

The classical action:

$$\int_0^T dt L(v(t), x(t), t) \quad ((1))$$

may be varied with respect to the form of  $v(t)$  and  $x(t)$  in order to find a stationary solution i.e. is Newton's second law as is known from classical mechanics.

It would be interesting if one could obtain quantum mechanical information from this same classical action. One approach is to use the Feynman path integral which is based on  $\exp(i \text{ Action})$ , a function introduced by P. Dirac. In a previous note (1), we argued that one may write the Action for a particle moving at constant velocity  $v$  in terms of  $v = X/T$  and then vary with respect to  $X$  and  $T$ . This approach does not find a stationary solution by varying  $v(t)$  and  $x(t)$ , but finds changes due to fluctuations in  $X$  and  $T$ . Interestingly, these are proportional to  $E$  and  $p$ :

$$d/dT \text{ Action} = -E \quad \text{and} \quad d/dX \text{ Action} = p \quad ((2))$$

As an example, one may use:  $L = -m_0 \sqrt{1-v^2}$  where  $c=1$  and  $v=X/T$  ((3)).

This is the relativistic Lagrangian.  $L = \frac{1}{2} m_0 v^2$  for the nonrelativistic case also yields ((2)). Thus, one may form  $\exp(i \text{ Action})$  and create two differential equations, namely the Klein-Gordon and bound state Schrodinger equation with ((2)). These equations have  $\exp(ipx - iEt)$  as a solution.

An important feature of ((2)) is that:

$$\text{Action}(X+dX) = \text{Action}(X) + dX p \quad ((3))$$

Thus, the fluctuation depends on the combination of  $p$   $dX$ . Given that one has a constantly moving particle, a fluctuation with respect to  $X$  should be the same at any  $X$ . This leads to the idea of a periodic function  $\exp(ipx)$ . (This result also follows formally from the solution to the Klein-Gordon or bound Schrodinger equation.)

Thus, the response of the Action (based largely on the Lagrangian) to a fluctuation in  $X$  yields the idea of a wavelength dependent on  $p$  i.e. proportional to  $1/p$ .  $\exp(ipx)$  is periodic in a regular way i.e. has both positive and negative values (i.e.  $\cos(px)$  and  $\sin(px)$ ). This appears to be linked to positive and negative fluctuations in  $dX$ , but in a regular, periodic manner. If one considers  $\exp(ipx)$  as a statistical quantity indicating regular periodic fluctuations (positive followed by negative etc), then for the case of  $V(x)$ , one may imagine stochastic hits which yield various  $p$  values. In such a case, one has an ensemble of fluctuations i.e.  $W(x) = \sum_p a(p) \exp(ipx)$ . Given different wavelengths ( $1/p$ ) and the positive/negative  $X$  fluctuation pattern, an overall "average" fluctuation result is created by "interference" (the combination of positive and negative values).

It seems that this may lead to density peaks and troughs, but density must still be defined.  $W(x)$  is the wavefunction based on  $\exp(ipx)$ . What is  $\exp(ipx)$ ? It measures a fluctuation pattern with respect to  $X$  in the Action. Its modulus is 1, so one may postulate that the modulus yields the physical spatial density. A free particle has equal probability to be at any  $X$ , so a constant describes its density (even though it may be at an  $X$  point at a particular time  $t$ , time and position have been separated in this statistical approach). Thus, interference yields an overall average  $X$  fluctuation pattern  $W(x)$  and one might expect  $W^*(x)W(x)$  to be spatial density.

A second approach to the density is to consider  $\exp(ipx)$  to be a conditional probability  $P(p/x)$  with the particle subject to stochastic hits to other  $p$  values. Then, one creates an AND situation i.e. there is a probability to be at  $x$  in state  $p$  and a probability to be knocked to  $p_1$  while still at  $x$ . This yields the product:  $\exp(ipx)a(p) a(p_1)\exp(ip_1x)$  ((4)). For a bound state, one has real wavefunction, so for a symmetric situation one may replace  $\exp(ipx)$  with  $\cos(px)$ . Then, ((4)) summed over  $p_1$  and  $p$  yields  $W(x)W(x)$ .

## Conclusion

In conclusion, we try to argue that one may consider fluctuations in  $X$  and  $T$  in a relativistic or nonrelativistic classical Action for a free particle. The classical Action is the starting point for the

Lagrangian approach to finding stationary solutions (i.e. Newton's second law), so it is interesting to consider how it fluctuates with  $X$  and  $T$ . (An alternative approach has been taken by Feynman who established the path integral method with P. Dirac suggesting the use of  $\exp(i \text{Action})$ .)

In this note, we observe that fluctuations in the action i.e.  $d/dT \text{Action} = -E$  and  $d/dX \text{Action} = p$ . Thus a change in  $\Delta X$  leads to a change of  $\Delta X p$ . This suggests a function in  $px$ . We consider a periodic function  $\exp(ipx)$  which allows for alternating positive and negative  $X$  fluctuations and imposes the idea of a wavelength proportional to  $1/p$ . Thus, the fluctuations in  $X$  for different  $p$  values differ. A high  $p$  value means sensitivity to very small  $X$  distances.

Furthermore,  $\exp(ipx)$  may be taken as a statistical result because its modulus is 1, indicating equal probability for a particle with constant velocity to be at any  $X$  (ignoring  $T$ ). For the case of a potential  $V(x)$ , an ensemble of periodic fluctuations  $\exp(ipx)$  may be created i.e.  $W(x) = \text{Sum over } p \ a(p) \exp(ipx)$ . This creates an average "fluctuation" with interference occurring i.e. positive  $\cos(px)$  or  $\sin(px)$  values combining with negative  $\cos(p_1 x) / \sin(p_1 x)$  values.

To find the overall spatial density, we suggest taking the modulus of  $W(x)$  (following the idea of the modulus of  $\exp(ipx)$ .) Alternatively, one may view  $\exp(ipx)$  as  $P(p/x)$  and consider an AND situation i.e. a probability to have  $p$  at  $X$  and to be knocked stochastically to  $p_1$  at  $X$ . This product  $a(p)\cos(px)a(p_1)\cos(p_1x)$  ( $\exp(ipx)$  replaced with  $\cos(px)$  for a bound symmetric solution) yields  $W(x)W(x)$  as well.

This note repeats many ideas expressed in previous notes, especially (1), but we here we wish to indicate that the idea of wavelength (proportional to  $1/p$ ) follows from the behaviour of the classical action for a particle with constant velocity  $= X/T$  to a variation in  $X$ . This change  $d/dX \text{Action} = p$  suggests that a combined variable  $px$  is important in the Action and if periodic behaviour occurs in  $X$ , then  $p$  governs the wavelength of this behaviour. Furthermore, we stress that the idea of  $\cos(px)$  from  $\exp(ipx)$  being both positive and negative (at different  $X$  values) is in keeping with fluctuations in  $X$  being positive or negative. For a periodic function, however, these positive and negative fluctuations are regular and not random and lead to the interference pattern observed in a wavefunction.

A question that arises is: Why should one consider fluctuations in  $X$  and  $T$  in a quantum realm and a variational solution  $d \text{Action} = 0$  in the classical Newton's equation result. The quantum picture seems to be one of stochastic hits or motion. Velocity which is constant is written as  $X/T$  and then varied with respect to  $X$  and  $T$ . After the variation  $X/T$  is set back to  $v$ . Thus, the constant motion of  $v$  is directly linked to variations (but these occur in a periodic manner). A potential is seen as rendering stochastic hits which create different free  $p$  states which have their own periodic fluctuations and interfere. Interestingly, the fluctuation picture is consistent with conservation of average energy at each  $x$  and so matches the classical mechanics result in this manner. In other words, the  $v(x)$  (velocity) obtained from the variation of the Action is the root mean square velocity which follows from the quantum momentum distribution  $W$  (which yields  $KE(x)_{\text{classical}} = -1/2m \ d/dx \ d/dx W / W$ ).

## References

1. Ruggeri, Francesco R. Klein Gordon Equation,  $\exp(i \text{Action})$  Propagator and Fluctuations (preprint, zenodo, 2021) <https://www.zenodo.org/record/4706004#.YL-2aWjYoWU>

