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Lyapunov exponents for the Lorentz transformations

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Abstract

We calculate the Lyapunov exponents of the complex stretch factor $f^{-}(z)=(1-z^2)^{-1/2}$ from the Lorentz transformations and of its reciprocal $f^{+}(z)=(1-z^2)^{+1/2}$ as well.

keywords: Lyapunov exponent, chaos, discrete dynamical system, Lorentz transformation

The most updated version of this white paper is available at
https://osf.io/jsncm/download
https://zenodo.org/record/4905180

Introduction

1. Very little is known regarding the connection of mathematical maps and physical systems.

2. We proposed two conjectures for this end, one relating quantum superposition with the logistic map [1], and the other giving an insight about the fundamental nature of time [2].

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Lorentz transformations

3. The warp of spacetime in the realm of special relativity (inertial observers) is due to the following function,

$$f(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2},$$

in the natural system of units (NSU), where light has velocity $c = 1$ [3].

4. In the NSU, x is the fraction of the light speed, i.e., $0 \leq x \leq 1$.

5. $f(x)$ accounts for both time dilation and space contraction; let's call it the stretch function.

Lyapunov exponent

6. A chaotic system exhibits sensitive dependence on initial conditions and henceforth has a positive Lyapunov exponent.

7. We summarize the idea of the Lyapunov exponent in the following; more details can be found in [4].

8. $x_0 :=$ initial condition

9. $\delta_0 :=$ a point nearby x_0 such that $|x_0 - \delta_0| << 1$

10. $\delta_n = |x_n - x_0|$ is the separation of x_n from x_0 after n iterates.

11. $|\delta_n| \approx |\delta_0| e^{n\lambda} \rightarrow \lambda :=$ Lyapunov exponent

12. $f'(x) = \frac{df(x)}{dx}$

13. The analytical form of the Lyapunov exponent is thus given by

$$\lambda = \lim_{n\to\infty} \left(\frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)| \right).$$

14. Although λ depends on x_0, *it is the same for all x_0 in the basin of attraction of a given attractor* [4].

15. A positive Lyapunov exponent is a signature of sensitive dependence on initial conditions; it does not necessarily mean the system is in a chaotic regime, as we will see in the next section.

Results

16. We calculate numerically the Lyapunov exponents of $|f^-(z)|$ and $|f^+(z)|$ for the rational interval $0 < z < 1$ from the following definitions.

17. $$f^- : \mathbb{C} \to \mathbb{C} \quad \text{such that} \quad f^-(z) = (1 - z^2)^{-1/2}$$

18. $$f^{-\prime}(z) = z(1 - z^2)^{-3/2}$$

19. $$f^+ : \mathbb{C} \to \mathbb{C} \quad \text{such that} \quad f^+(z) = (1 - z^2)^{+1/2}$$

20. $$f^{+\prime}(z) = -z(1 - z^2)^{-1/2}$$

21. Figs. 1–4 show the Lyapunov exponents of $|f^-(z)|$ and $|f^+(z)|$, respectively.

22. Figs. 5–12 show the iterations (time series) of $|f^-(z)|$ and $|f^+(z)|$, respectively.

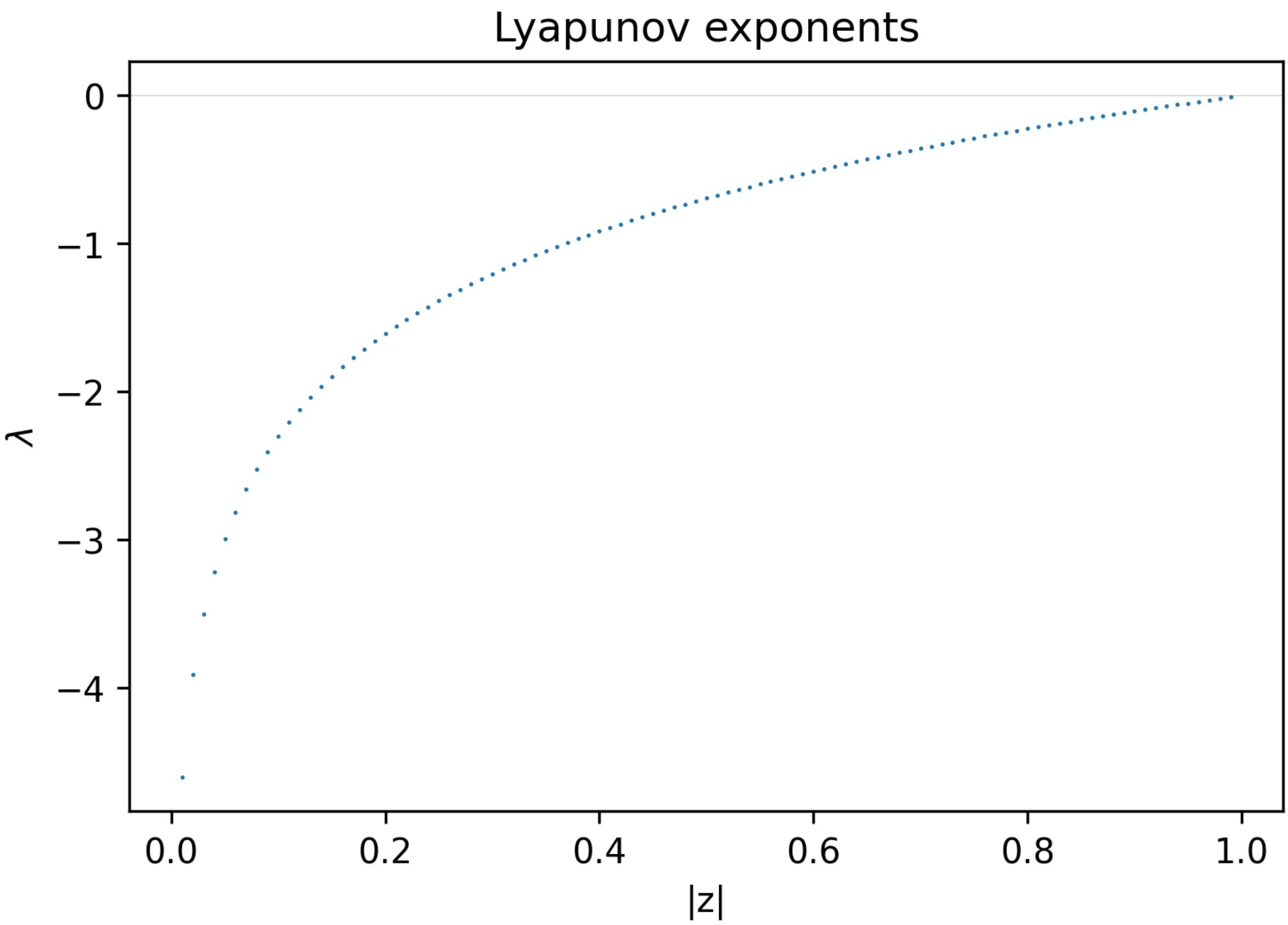

Figure 1: The Lyapunov exponents of $f^{-}(z) = (1 - z^2)^{-1/2}$ plotted for $z_0 \in \{0.01, 0.02, 0.03, ..., 0.99\}$.

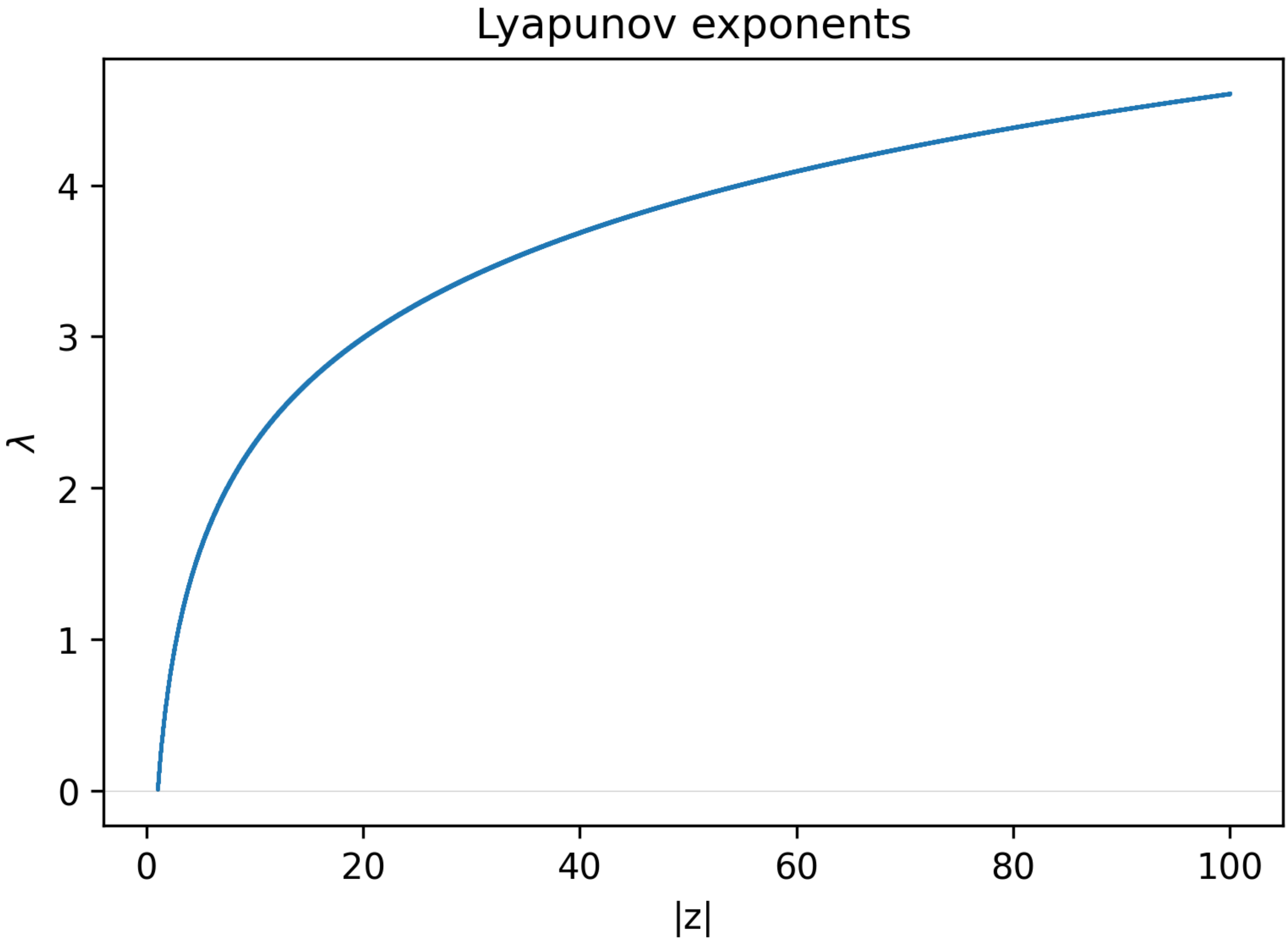

Figure 2: The Lyapunov exponents of $f^{-}(z) = (1 - z^2)^{-1/2}$ plotted for $z_0 \in \{1.01, 1.02, 1.03, ..., 99.99\}$.

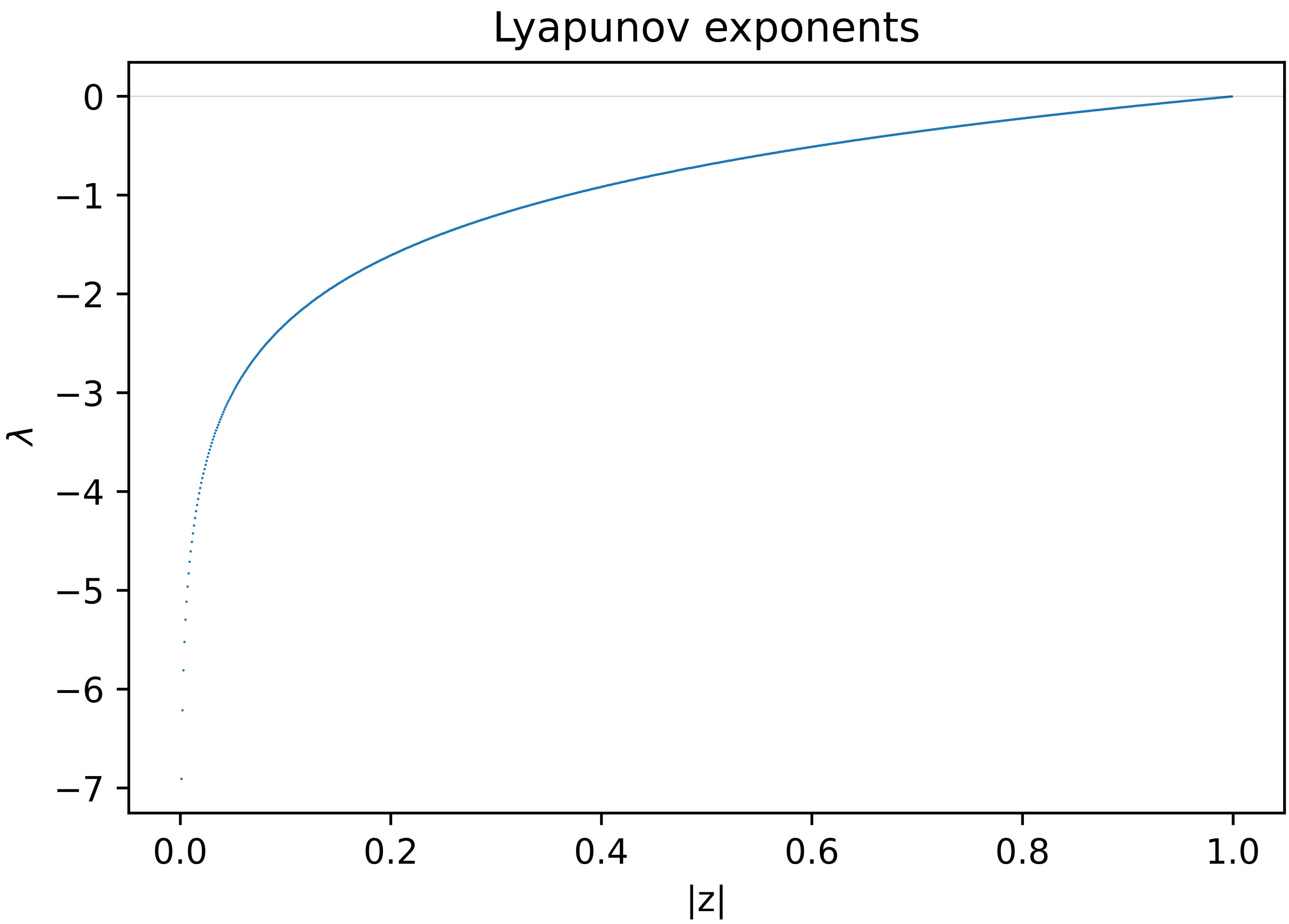

Figure 3: The Lyapunov exponents of $f^{+}(z) = (1 - z^2)^{+1/2}$ plotted for $z_0 \in \{0.01, 0.02, 0.03, ..., 0.99\}$.

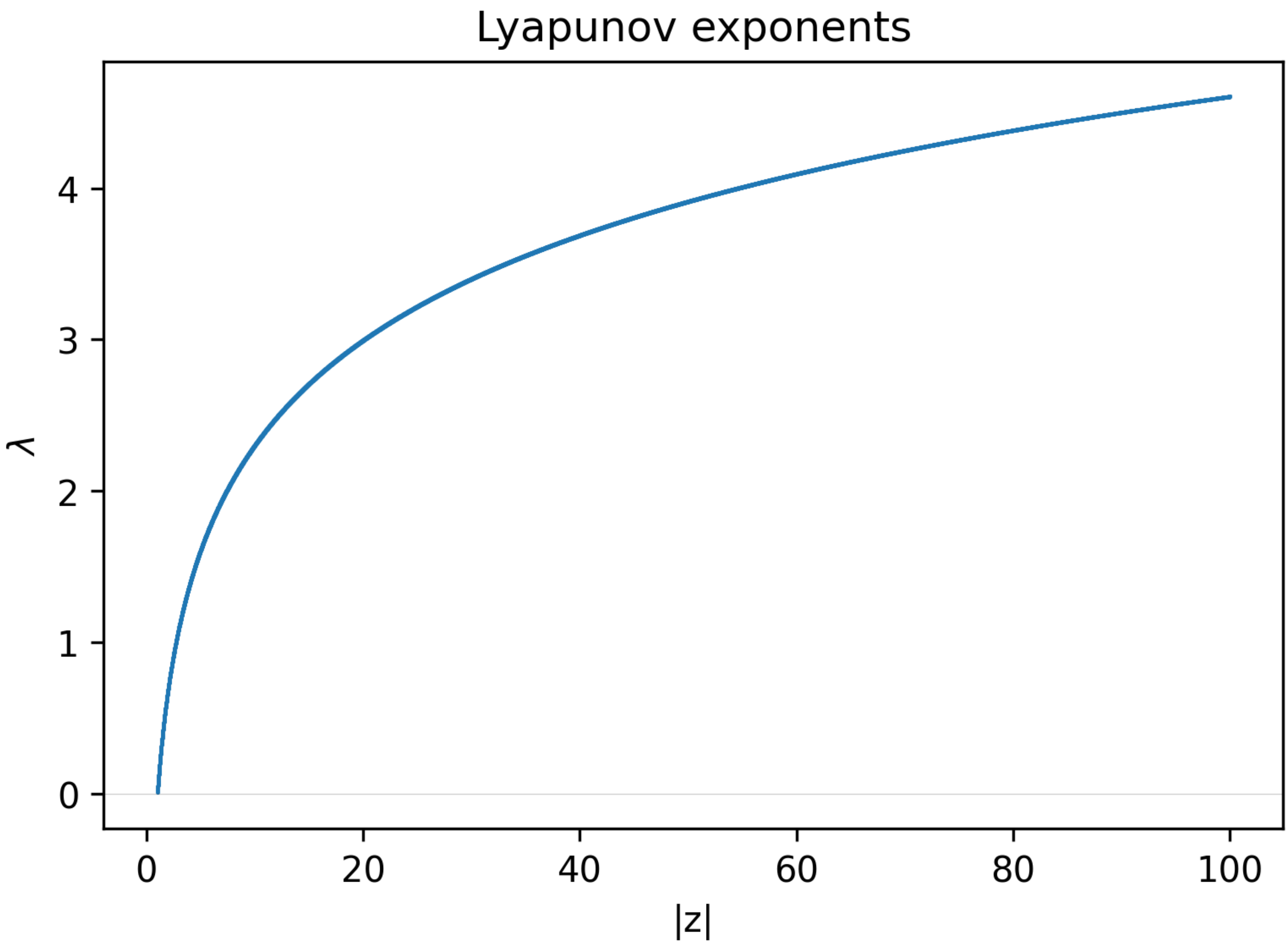

Figure 4: The Lyapunov exponents of $f^{+}(z) = (1 - z^2)^{+1/2}$ plotted for $z_0 \in \{1.01, 1.02, 1.03, \ldots, 99.99\}$.

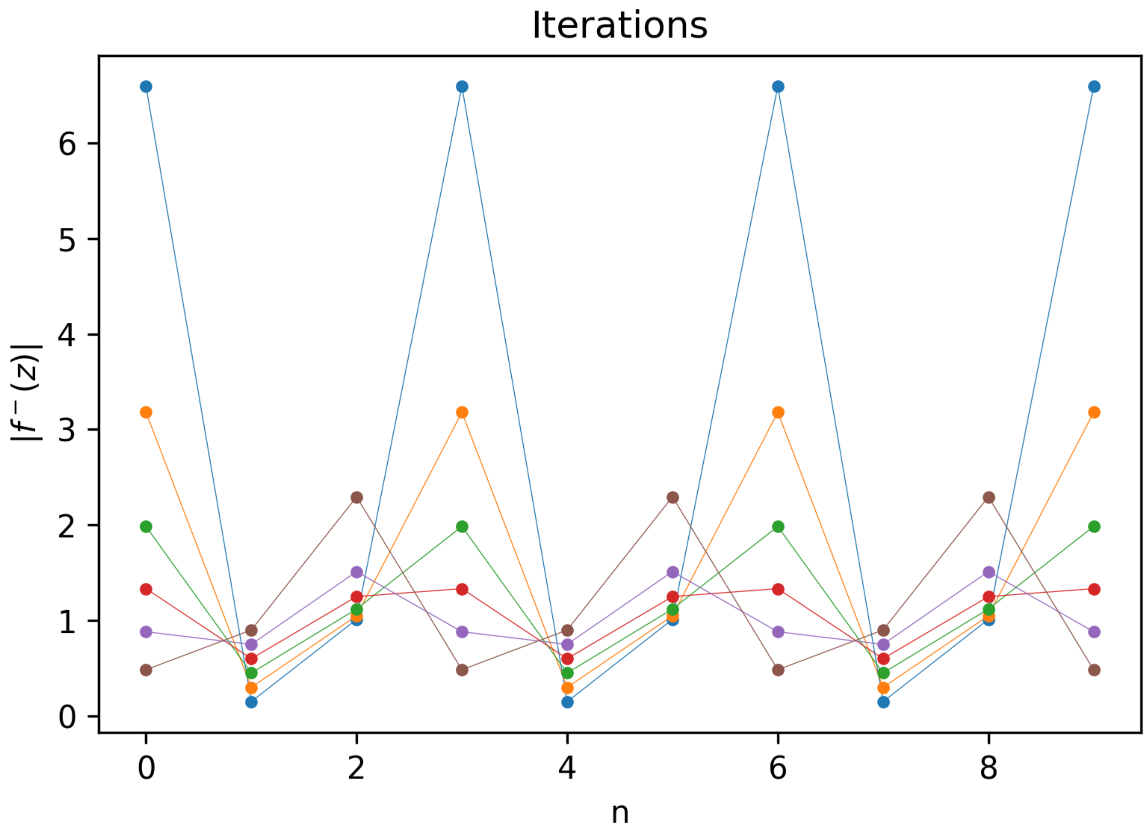

Figure 5: Iterations (time series) of $|f^-(z)| = |(1-z^2)^{-1/2}|$ for the initial conditions $Z_0 = \{0.15, 0.30, 0.45, 0.60, 0.75, 0.90\}$. Each color represents one different initial condition $z_0 \in Z_0$. Note that all conditions led to three attractors.

23. The attractors a_i of the system in Fig. 5 are the following.

z_0	a_1	a_2	a_3
0.15	6.59	0.15	1.01
0.30	3.18	0.30	1.05
0.45	1.98	0.45	1.12
0.60	1.33	0.60	1.25
0.75	0.88	0.75	1.51
0.90	0.48	0.90	2.29

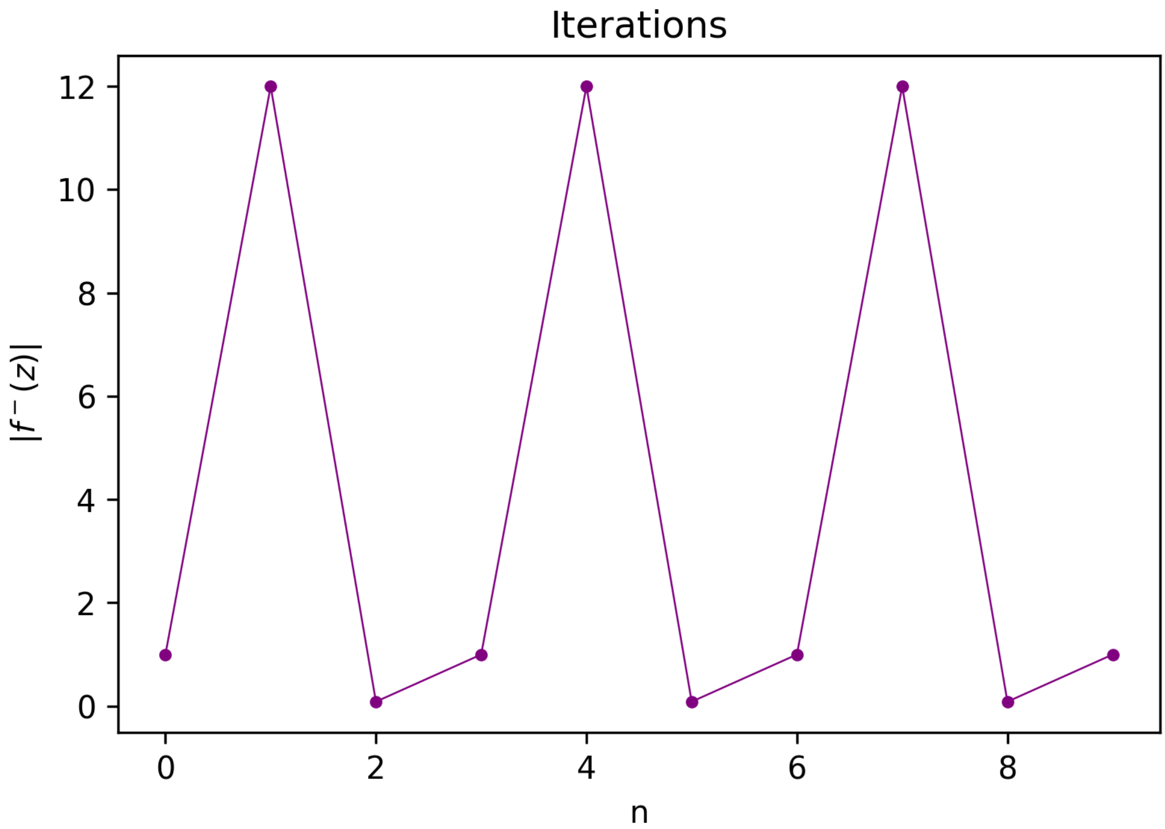

Figure 6: Iterations (time series) of $|f^{-}(z)| = |(1-z^2)^{-1/2}|$ for the initial condition $z_0 = 12$. The attractors are 11.999999999999925, 0.0836242010007096, and 0.9965217285917831.

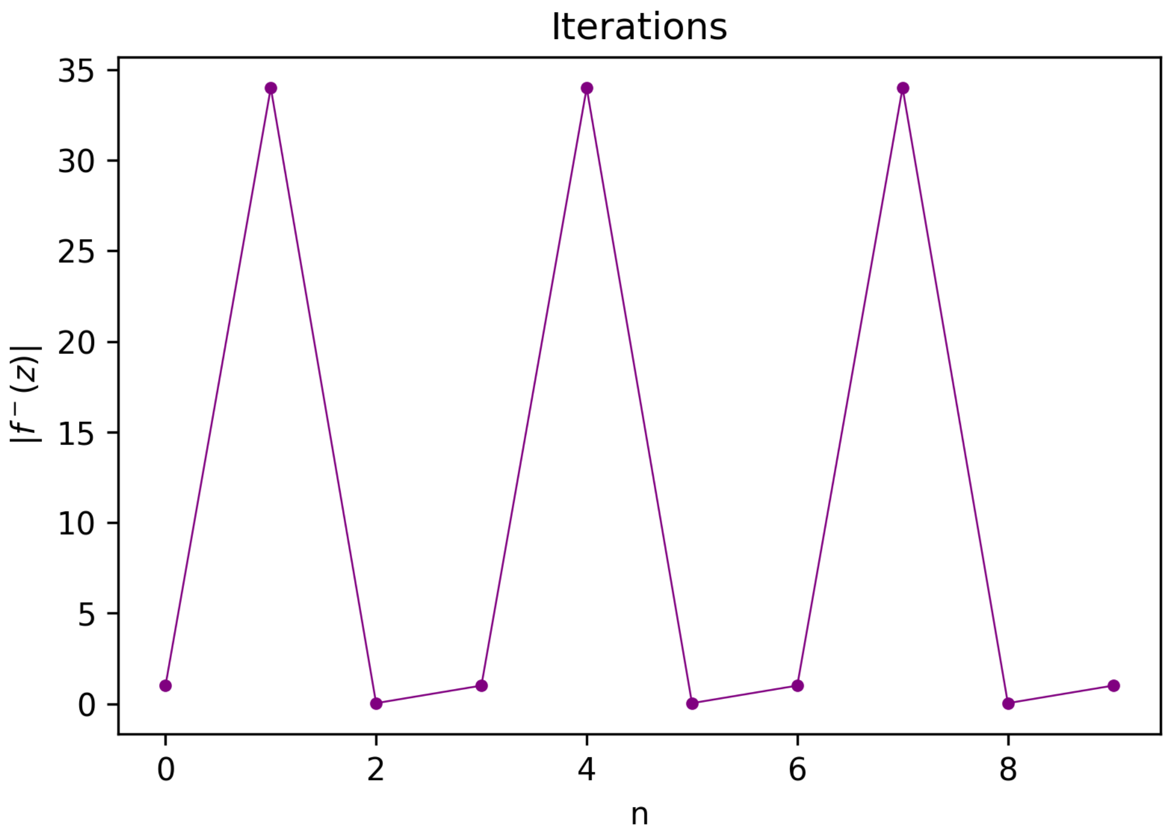

Figure 7: Iterations (time series) of $|f^{-}(z)| = |(1-z^2)^{-1/2}|$ for the initial condition $z_0 = 34$. The attractors are 0.9995673804686134, 33.99999999999647, and 0.029424494316828042.

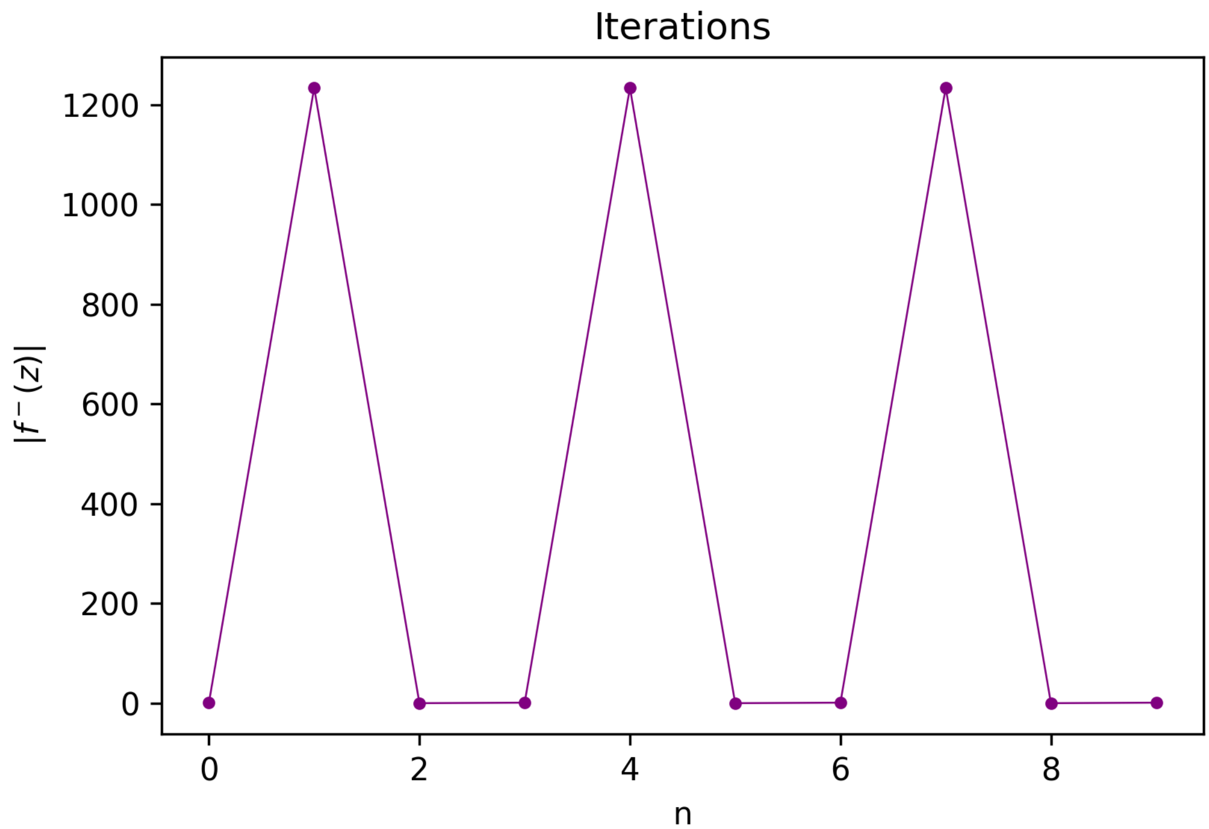

Figure 8: Iterations (time series) of $|f^{-}(z)| = |(1 - z^2)^{-1/2}|$ for the initial condition $z_0 = 1234$. The attractors are 0.9999996716479318, 1234.0000001380304, and 0.0008103730374718962.

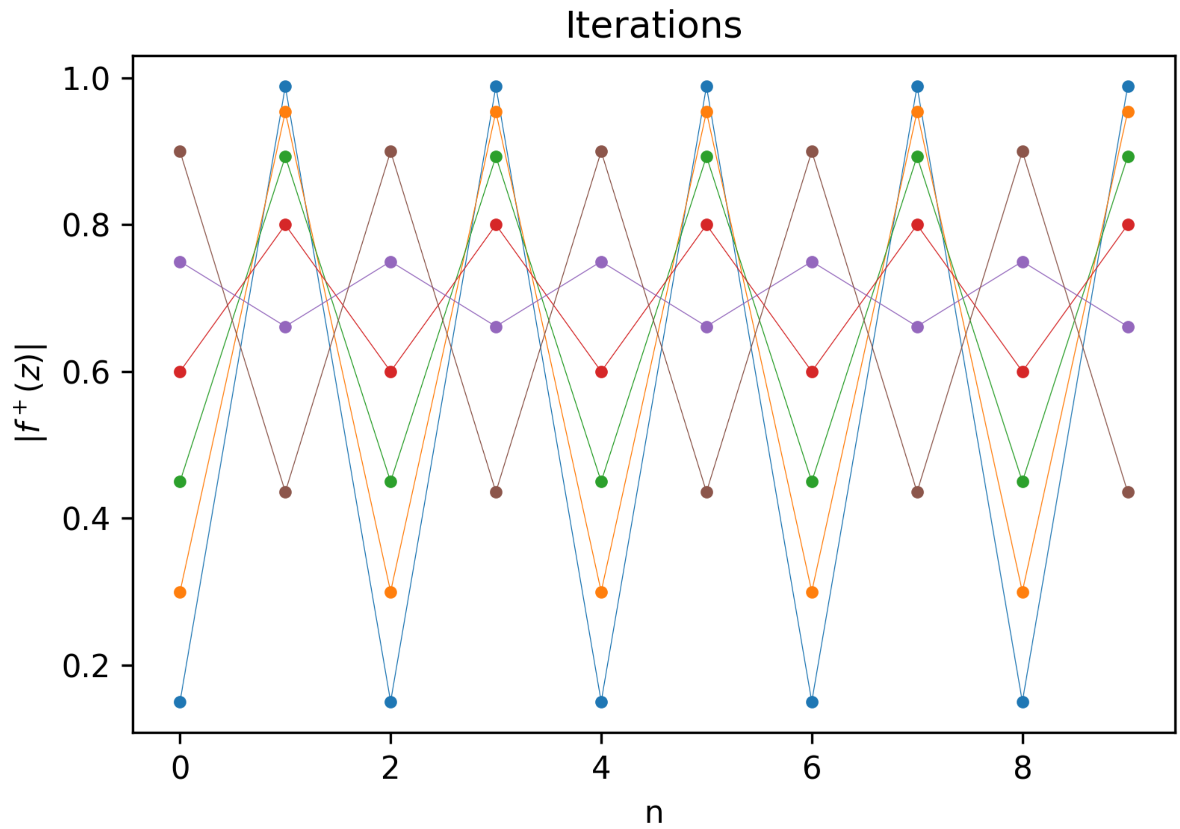

Figure 9: Iterations (time series) of $|f^{+}(z)| = |(1 - z^2)^{+1/2}|$ for the initial conditions $Z_0 = \{0.15, 0.30, 0.45, 0.60, 0.75, 0.90\}$. Each color represents one different initial condition $z_0 \in Z_0$. Note that all conditions led to two attractors.

24. The attractors a_i of the system in Fig. 9 are the following.

z_0	a_1	a_2
0.15	0.15	0.99
0.30	0.30	0.95
0.45	0.45	0.89
0.60	0.60	0.80
0.75	0.75	0.66
0.90	0.90	0.44

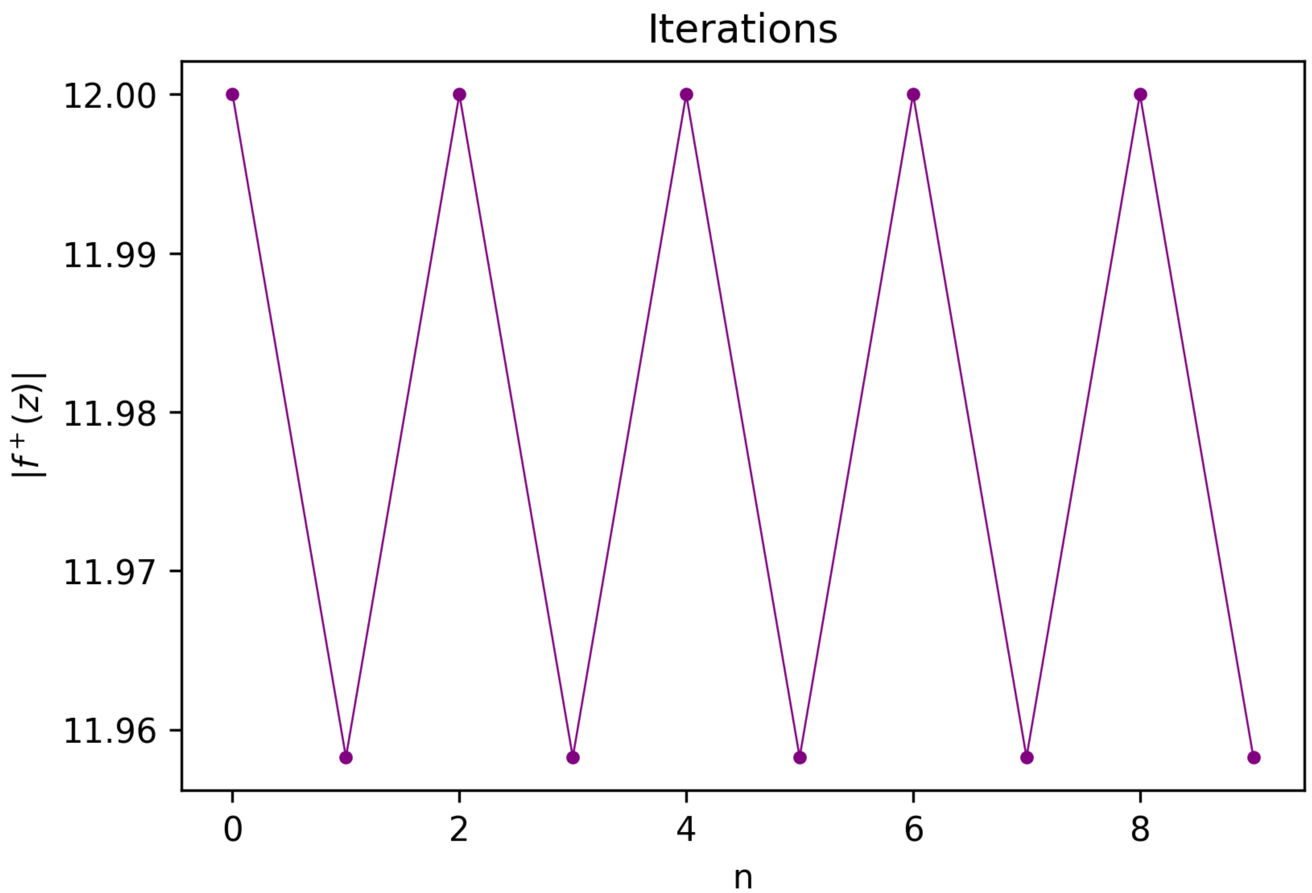

Figure 10: Iterations (time series) of $|f^{+}(z)| = |(1 - z^2)^{-1/2}|$ for the initial condition $z_0 = 12$. The attractors are 12.0 and 11.958260743101398.

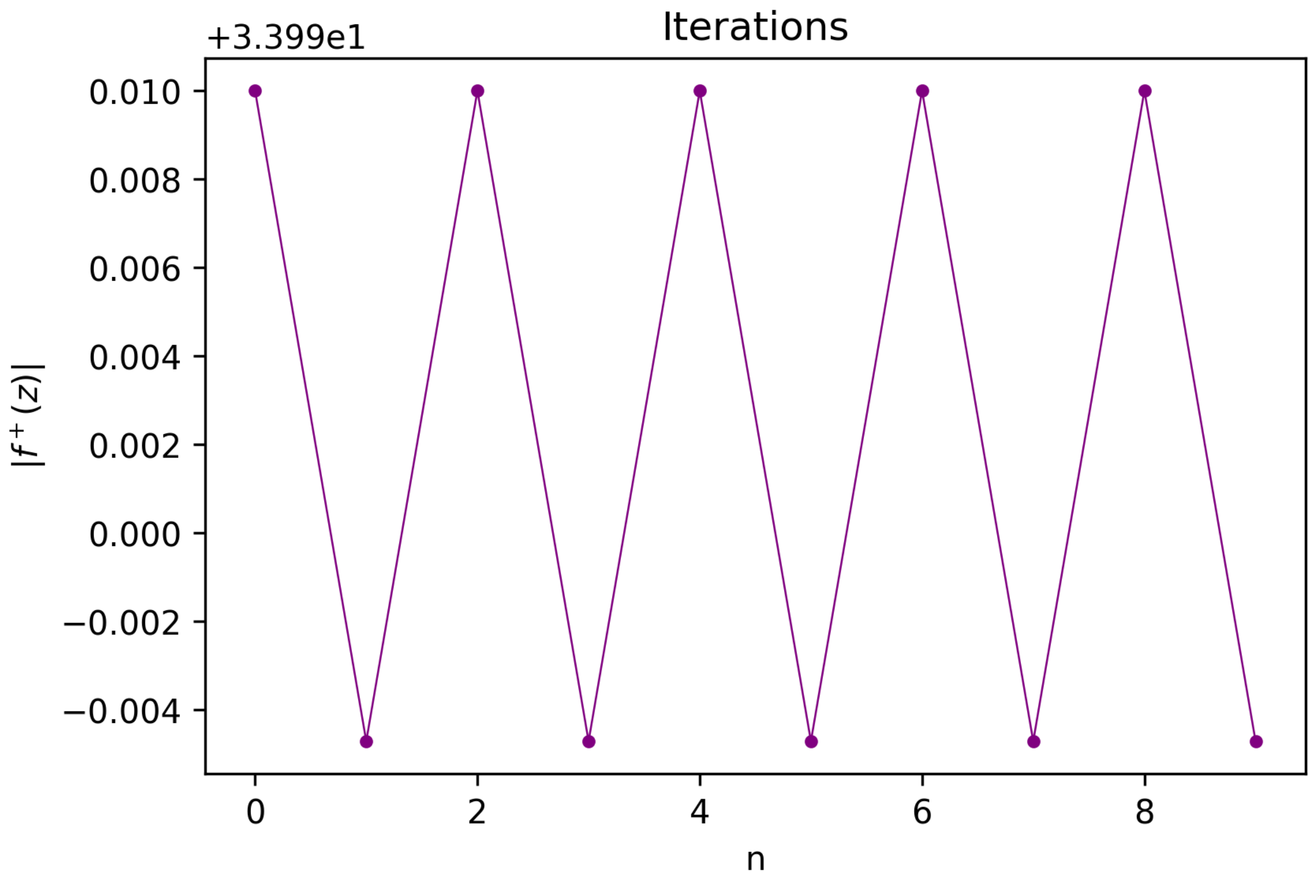

Figure 11: Iterations (time series) of $|f^+(z)| = |(1-z^2)^{-1/2}|$ for the initial condition $z_0 = 34$. The attractors are 34.0 and 33.98529093593286.

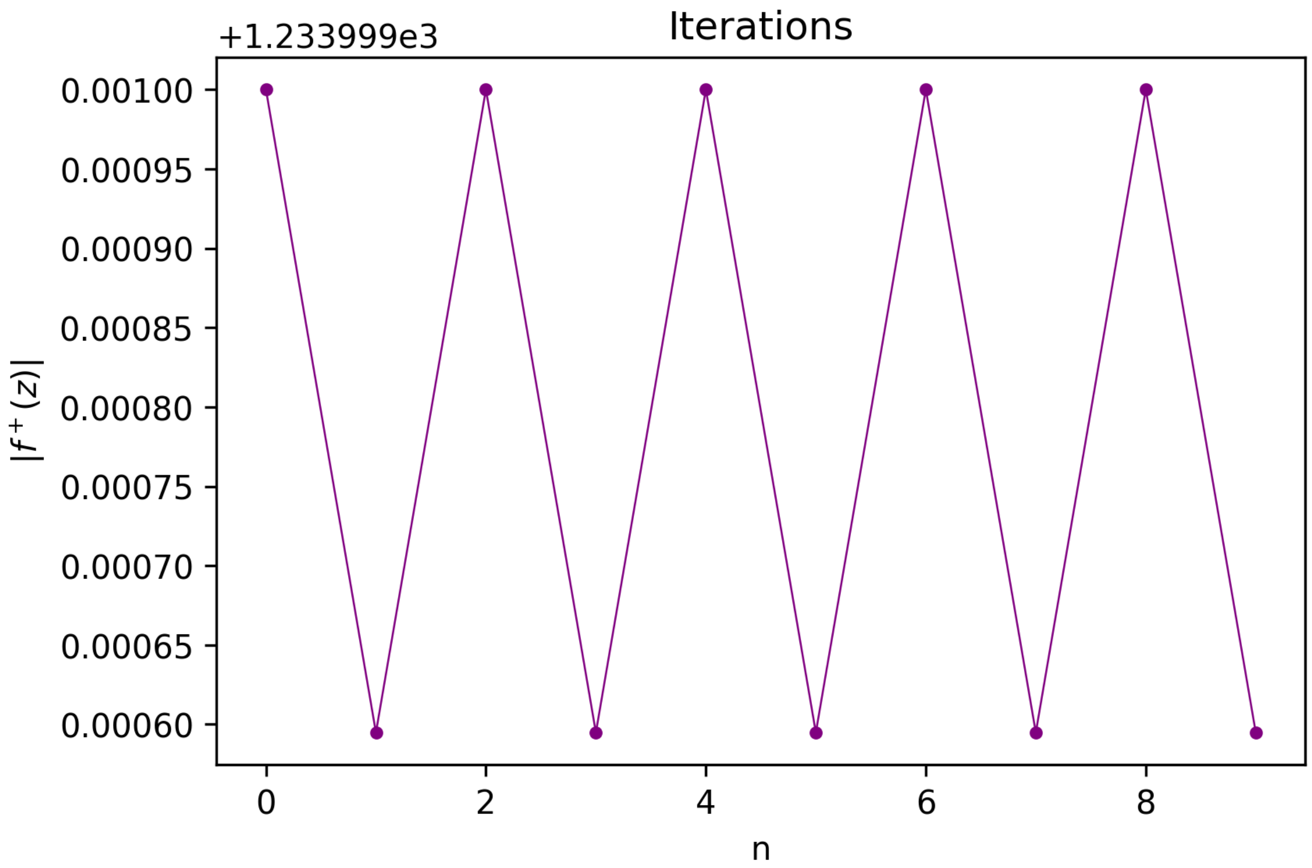

Figure 12: Iterations (time series) of $|f^+(z)| = |(1 - z^2)^{-1/2}|$ for the initial condition $z_0 = 1234$. The attractors are 1234.0 and 1233.9995948135477.

Python scripts

25. `https://osf.io/5cx4r`
26. `https://osf.io/rwg73`
27. `https://osf.io/rychk`

Final Remarks

28. We showed that the complex version of the Lorentz factor has both negative and positive Lyapunov exponents while preserving its periodic behavior under the dynamics of a certain subset of initial conditions.

29. The stretch function comprised in the Lorentz transformations might shed some light on the quantum nature of spacetime in case it follows such a discrete dynamical system in its innermost fundamental blocks [1, 2].

Open Invitation

Review, *add* content, and `co-author` this white paper [5, 6].
Join the **Open Mathematics Collaboration**.
Send your contribution to `mplobo@uft.edu.br`.

Open Science

The **latex file** for this *white paper* together with other *supplementary files* are available in [7, 8].

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Agreement

All authors agree with [6].

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