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Generating Pythagorean Triples and Magic Squares: Orders 3 to 31

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Abstract

*This paper shows how to create magic squares with a **perfect square number** for the total sum of their entries. This has been done in two ways: Firstly, by using the sum of **consecutive odd numbers**, and secondly, by using **consecutive natural numbers**. In the first case, for all orders of magic squares, one can always obtain a **perfect square entries sum**. In the second case, magic squares with perfect square magic sums do exist, but only for **odd order magic squares**. For the **even order magic squares**, such as, 4, 6, 8, etc. it is not possible to write **consecutive natural number magic squares** with perfect square entries sums. A simplified idea is introduced to check when it is possible to obtain **minimum perfect square entries sums**. Also, a uniform method is presented so that, if k is the order of a magic square, then the magic sum of the square is k^3, and the sum of all entries of the magic square is k^4. Based on these aspects, connections with Pythagorean triples are also made. The work is for the magic squares of orders 3 to 31. Further orders are given in [23]. In another work [22], the magic squares are generated based on **Pythagorean triples**.*

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1 Introduction

Recently, the author [5, 6, 7] worked on magic squares connected with Pythagorean triples. In these, works the sum of all entries of a magic square are always a perfect square. The work in Pythagorean triples and patterns are done by author in [8, 9, 10, 11, 12]. On the other side the author worked extensively on magic squares in different situations, such as, block-wise [13, 14, 18, 19], block-bordered [15, 16, 18], block-wise-bordered [19, 20, 21], etc. In this we shall try to extend and revise previous works done separately.

Mainly there are four points studied in this work. Below are the details of these points:

1. *Write magic squares with entries as* ***consecutive odd number*** *and/or* ***consecutive natural number*** *having the same sum. Both type of entries are possible for* ***odd order magic squares****, while for* ***even order magic squares*** *there are only* ***consecutive odd numbers*** *entries;*
2. *Write magic squares with* ***minimum prefect square entries sums****. In this case, the magic squares are of* ***consecutive natural number*** *entries. This happens only for the case of* ***odd order magic squares****;*
3. *Write* ***Pythagorean triples*** *based on magic squares studied in items 1 and 2;*
4. *Generate magic squares based on* ***Pythagorean triples*** *given in item 3. In this case the magic square are with* ***consecutive odd number*** *entries.*

The Section 2 give mathematical aspects of these points. From Section 3 onwards, order-wise study of magic square is given. The magic squares considered are of two types. First types are **block-wise** and of orders 8, 9, 12, 15, 16, 18, 20, 21, 24, 25, 27, 28 and 30 . The second types are **block-bordered** and of orders 10, 11, 13, 14, 17, 19, 22, 23, 26, 29 and 31. More details of magic squares of types **block-wise** and **block-bordered** can be seen in author's work [17]–[**?**].

2 Series and Magic Square Sums

This section presents some basic ideas of series and magic square sums.

2.1 Natural Number Series Sums

It is well-known that the positive natural number series sum is given by

$$T_n := 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2},\ n \geq 1 \tag{1}$$

The sequence T_n is also famous for showing **Pascal's triangle values**

2.2 Odd Number Series Sums

It is well-known that sum of **odd-number series** is given by

$$\boldsymbol{T}_n := 1 + 3 + 5 + ... + (2n-1) = n^2,\ n \geq 1. \tag{2}$$

Since we are working with magic squares, let us write $n = k^2$, then we have

$$\boldsymbol{T}_{k^2} := 1 + 3 + 5 + ... + (2k^2 - 1) = k^4,\ k \geq 1. \tag{3}$$

The number k is the order of a magic square. Thus for all $k \geq 3$ we can always have odd numbers for the entries of a magic square with the sum of all numbers being a perfect square. More precisely, forth power of order of magic square. It happens for all order magic squares.

2.3 Magic Square and Entries Sums

It is well known that the magic sum of a magic square of order k with elements $1, 2, 3, \ldots k^2$ is given by

$$\boldsymbol{S}_{k \times k} := \frac{k(k^2+1)}{2},\ k \geq 1. \tag{4}$$

The sum of all of the entries of a magic square is given by

$$\boldsymbol{T}_{k^2} := \frac{k^2\,(k^2+1)}{2},\ k \geq 1. \tag{5}$$

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2.4 Perfect Square Entries Sums

Below is a general formula for writing **perfect square entries sum magic squares** with consecutive natural numbers.

Let us consider

$$G := T(n) - T(n-1) = k\left(n - \frac{k-1}{2}\right),\ n \geq k. \tag{6}$$

a) For even order entries magic squares $k = 2p$:

$$G := k\left(n - \frac{2p-1}{2}\right). \tag{7}$$

In this case, the expression $n - \frac{2p-1}{2}$ is never a natural number. *Thus, there is no magic square of consecutive even numbers with a perfect square entries sum for all of its entries.*

b) For odd order magic squares, $k = 2p + 1$

$$G := T(n) - T(n-k) = k\left(n - \frac{2p+1-1}{2}\right) = k(n-p), \tag{8}$$

In this case, we can always find a natural number, such that $k(n-p)$ is a perfect square with $n - p \geq k$. By considering $n - p = k$, we obtain $G := k^2$, and the sum of all the entries is $\boldsymbol{T}_{k^2} := k^4$.

Result 1. *Thus, from equations (3) and (8), we conclude that there are at least two ways of writing magic squares with all entries summing to a perfect square:*

*(i) Magic squares with **consecutive odd numbers** starting from 1;*

*(ii) Magic squares with **consecutive natural numbers**.*

For the first case (i), the author [10] has recently calculated magic squares with odd numbers not starting from 1 using the idea of **Pythagorean triples** to obtain a **perfect square entries sum** magic squares.

For the second case (ii), the subsection below gives a procedure that can be used to obtain a perfect square entries sum of all the entries of a magic square.

2.5 Procedure

Before considering examples, let us analyse some simple formulas. After simplifying equation (6), one can write

$$\begin{aligned} G(n,k) &= T(n) - T(n-k) \\ &= (n-k+1) + (n-k+3) + \cdots + n \\ &= k\left(n - \frac{k-1}{2}\right),\ n \geq k, \\ &= \frac{k}{2}(2n-k+1),\ n\ \geq k \end{aligned} \tag{9}$$

Since we are working with magic squares and k is the total number of elements in a magic square it should also be the square of the order of the magic square. Therefore if p is the order of magic square then $k = p^2$. In order to obtain a perfect square for $G(n,k)$ we should have

$$\sqrt{G(n,k)} = t,\ t \in \mathbf{N}_+,\ n\ \geq k \tag{10}$$

For each case, we find a natural number t.

2.6 Consecutive Natural Numbers

From equation (8), we observe that the first element of the sequence is $n-k+1$ and the last is n. In case of uniformity always choose

$$n = k + \frac{k-1}{2} = \frac{3k-1}{2}$$

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This gives

$$G = T\left(\frac{3k-1}{2}\right) - T\left(\frac{k-1}{2}\right) = k^2,\ n \geq k$$

Since k is the total number of elements, let us consider p as the order of a magic square, and then the first and last members of the sequence are $\boldsymbol{T}_1 := \frac{p^2+1}{2}$ and $\boldsymbol{T}_{p^2} := \frac{3p^2-1}{2}$ respectively. In this case, the sum of the sequence is given by

$$\boldsymbol{T}_1 + \boldsymbol{T}_2 + \ldots + \boldsymbol{T}_{p^2} := p^4.$$

For simplicity, let's represent the **uniformity** as,

$$\left\langle p, p^2, p^3, p^4 \right\rangle \tag{11}$$

where

$$\begin{aligned} p &\rightarrow \text{order of a magic square;} \\ p^2 &\rightarrow \text{total number of entries;} \\ p^3 &\rightarrow \text{magic square sum;} \\ p^4 &\rightarrow \text{sum of all the entries of a magic square.} \end{aligned}$$

It happens with all the magic squares with **consecutive odd number** entries starting from 1. Also the same happens with **consecutive natural numbers** obtained based on odd order entries having the same magic sum.

2.7 Minimum Perfect Square Entries Sum

Let p be an order of a magic square, i.e., $k = p^2$. Then from equations (9) or (10), we have

$$p\sqrt{\frac{2n-p^2+1}{2}} = p\sqrt{n - \frac{p^2-1}{2}} = t,\ t \in \mathbf{N}_+,\ n\ \geq p^2,$$

The sum of all entries of a magic square is given by

$$\boldsymbol{T}_1 + \boldsymbol{T}_1 + \ldots + \boldsymbol{T}_{p^2}$$

where $\boldsymbol{T}_1 = n - p^2 + 1$ and $\boldsymbol{T}_{p^2} := n$.

Again, let's consider, $n = m^2 + \frac{p^2-1}{2}$, and then we have

$$G(m,p) := p\sqrt{\left(m^2 + \frac{p^2-1}{2}\right) - \left(\frac{p^2-1}{2}\right)} = t,\ t \in \mathbf{N}_+,\ n\ \geq p^2. \tag{12}$$

In this case, how do we find the minimum value of m? For simplicity, let's write,

$$L(n,k) := \left(\sqrt{k}, n-k+1, n, \frac{G(n,k)^2}{\sqrt{k}}, G(n,k)^2, \sqrt{G(n,k)}\right), \tag{13}$$

where $n = m^2 + \frac{p^2-1}{2}$, $k = p^2$ and p is the order of a magic square.

From above we observe that, if $\sqrt{G(n,k)}$ equals the order of a magic square then, we have a **uniformity case**. This only happens when $m = p$. And, first positive entry obtained from m give us a **minimum perfect square entries sum**.

For simplicity, let's write $L\,(a,b,c,d,e,f)$, for the representations given in expression (13), where

$a \to$ Order of magic square;

$b \to$ First member of a sequence;

$c \to$ Last member of a sequence;

$d \to$ Sum of magic square;

$e \to$ Sum of all entries of a magic square;

$f \to$ Uniformity, if $a = f$.

$$\ldots \tag{14}$$

Remark 2.1. *In all cases, we use **consecutive natural numbers** starting from 1. Amongst the magic squares studied here the magic square of order 7 is the only ones starting with the number 1 that yield a **minimum perfect square entries sum**. According to Sloane [3], the next magic square with **consecutive natural numbers** entries starting with the number 1 that yields a **minimum perfect square entries sum** are of orders, 7, 41, 239, etc. The magic square of order 3 is the only one where the first value of m gives a **minimum perfect square entries sum** and is with **uniformity property**. It should be noted that we are only examining positive entries for the magic squares. We can also obtain perfect square entries sum magic squares with some negative entries, but these cases are not under study.*

2.8 Generating Perfect Square Entries Sum Magic Squares from Pythagorean Triples

In view of Equations (2) and (3), we have

$$\boldsymbol{T}_m - \boldsymbol{T}_n := (2n+1) + (2n+3) + ... + (2m-1) = m^2 - n^2$$

The total number of terms are given by

$$T_{terms} := \frac{(2m-1)-(2n+3)}{2} + 1 = m - n$$

Result 2. *In a **Pythagorean triple**,* $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ *if any of the difference*

$$c - b \text{ or } c - a, \; c > b, \; c > a,$$

*is a **perfect square** greater than or equal to 9, then we can always write a **perfect square entries sum magic square**.*

Let's consider following representation,

$$
\begin{aligned}
(\mathbf{a},\mathbf{b},\mathbf{c}) \Rightarrow \{&\text{order of magic square},\\
&\text{first member of sequence},\\
&\text{last member of sequence},\\
&\text{magic sum},\\
&\text{sum of all members of a magic square}\}. \qquad (15)
\end{aligned}
$$

For example, let's consider a **Pythagorean triple** $(\mathbf{8},\mathbf{15},\mathbf{17})$, then according to notations given in (15), we have

$$(\mathbf{8},\mathbf{15},\mathbf{17}) \Rightarrow \{3,17,33,75,225\}. \qquad (16)$$

The above representations of numbers is as follows:

$$
\begin{aligned}
3 &\rightarrow \text{Order of a magic square;}\\
17 &\rightarrow \text{First member of the sequence;}\\
33 &\rightarrow \text{Last member of the squence;}\\
75 &\rightarrow \text{Magic sum;}\\
225 &\rightarrow \text{Sum of all entries the magic square, a perfect square, } 225 = 15^2.
\end{aligned}
$$

In this case, 9 **consecutive odd numbers** generates a magic square for the entries: $\{17,19,\ldots,31,33\}$

Result 3. *The following formula is used to reach the result appearing in equation* (16)*:*

$$(\mathbf{a},\mathbf{b},\mathbf{c}) \Rightarrow \left\{\left\{\frac{c^2-a^2}{\sqrt{c-a}},\, c^2-a^2,\, 2a+1, 2c-1,\, \sqrt{c-a}\right\}, \left\{\frac{c^2-b^2}{\sqrt{c-b}},\, c^2-b^2,\, 2b+1,\, 2c-1,\, \sqrt{c-b}\right\}\right\} \qquad (17)$$

Remark 2.2. *Result 2 is applied to test the existence of a magic square with* ***perfect square entries sum****. The Result 3 is useful for knowing the details of a magic square. Result 3 may also be used to text the existence of a magic square, but it give more work.*

Below are magic squares of orders 3 to 31 with **consecutive odd order entries** starting from 1. In all these case, we get a magic sum as a **third power** of order of a magic square, i.e., if the magic square is of order k, then the magic sum is k^3. In case of odd order magic squares, there are two more possibilities are written. One with **consecutive natural number entries** and another with **minimum perfect square magic sum**. In each case, the **Pythagorean triples** are given. Then these **Pythagorean triples** are analysed to check further existence of different magic squares.

3 Magic Square of Order 3

According to equations (3) and (8), one can obtain **perfect square entries sum magic squares** of order 3 in two ways: first using **consecutive odd numbers**, and the second using **consecutive natural numbers**. For the **consecutive natural numbers** there are again two ways: the first one is of **uniformity**, and the second is of **minimum perfect square entries sum**.

3.1 Consecutive Odd Numbers

Taking $k=3$ in (5), we get

$$\begin{aligned} &1+3+5+\ldots+(2\times 3^2-1)=3^4 \\ &\Rightarrow 1+3+5+\ldots+17=9^2=3^4 \end{aligned} \tag{18}$$

Example 3.1. *For the* ***consecutive odd number*** *entries* $\{1,3,5,\ldots,15,17\}$*, the magic square of order 3 is given by*

			27
3	13	11	27
17	9	1	27
7	5	15	27
27	27	27	27

In this case, we have a magic sum $S_{3\times 3} = 27 = 3^3$ and the sum of all entries $\boldsymbol{T}_9 = 3 \times 27 = 81 = 9^2 = 3^4$. This satisfies the uniformity property 11, i.e., $\left\langle 3, 3^2, 3^3, 3^4 \right\rangle$.

3.2 Consecutive Natural Numbers

According to equation (8),

$$G := \frac{n(n+1)}{2} - \frac{(n-9)(n-8)}{2} = 9(n-4)$$

Taking $n = 13$, we get a perfect square, i.e.,

$$G := T(13) - T(4) = \frac{13 \times 14}{2} - \frac{4 \times 5}{2} = 9 \times 8 = 81$$

Simplifying, we get

$$5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 = 81$$

Example 3.2. *According to the values given above, there is a perfect sum magic square of order 3 given by*

			27
6	11	10	27
13	9	5	27
8	7	12	27
27	27	27	27

In this case also, we have a magic sum $S_{3\times 3} = 27 = 3^3$ and the sum of all entries $T_9 = 3 \times 27 = 81 = 9^2 = 3^4$. The perfect square entries sum obtained above satisfy the uniformity property (11), but it is not minimum. See following subsection for the minimum perfect square entries sum magic square.

3.3 Minimum Perfect Square Entries Sum

Choosing $m = 1, 2, 3$ and 4, and $p = 3$ in equation (13), we get

$$1.\quad L\left(1^2 + \frac{3^2-1}{2}, 3^2\right) \to (3, -3, 5, 3, 9)$$

$$2.\quad L\left(2^2 + \frac{3^2-1}{2}, 3^2\right) \to (3, 0, 8, 12, 36)$$

$$3.\quad L\left(3^2 + \frac{3^2-1}{2}, 3^2\right) \to (3, 5, 13, 27, 81) \Rightarrow \left\langle 3, 3^2, 3^3, 3^4 \right\rangle$$

$$4.\quad L\left(4^2 + \frac{3^2-1}{2}, 3^2\right) \to (3, 12, 20, 48, 144)$$

The values written above are for $m = 1, 2, 3, 4$. See below the details:

(i) For $m = 1$, the magic square is formed by entries $\{-3, -2, -1, 0, 1, 2, 3, 4, 5\}$ giving a **perfect square entries sum** as 9. Since it is formed by negative entries, it is excluded form our study.

(ii) For $m = 2$, we have **minimum perfect square entries sum** magic square.

(iii) For $m = 3$, we have a uniformity property given in (11). This case is already studied in Example 3.1.

(iv) For $m = 4$, we have a magic square of order 3. It is just written as an example without any special specification.

Remark 3.1. *The formula* (13) *works only for odd order magic squares. The entries are* ***consecutive natural numbers****. From now onward, the magic squares with negative values are not shown. The first value of* m *for positive entries always give* ***minimum perfect square entries sum****. Magic squares with* ***uniformity property*** (11) *are also written. It happens when* $m = p$.

Let's write below **minimum perfect square entries sum** as given for $m = 2$:

Example 3.3. *A magic square of order 3 with perfect square entries sum is given by*

			12
1	6	5	12
8	4	0	12
3	2	7	12
12	12	12	12

In this case, the magic sum is $S_{3\times 3} = 12$, and the sum of all entries is $T_9 := 36 = 6^2$.

3.4 Pythagorean Triples

According to Examples 3.1, 3.2 and 3.3, we have two perfect square entries sums, i.e., $T_9 = 81 = 9^2$ and $T_9 := 36 = 6^2$. Below are **Pythagorean triples** for the numbers 9 and 6:

$$\begin{aligned} \mathbf{9^2} + 40^2 &:= 41^2 \\ \mathbf{9^2} + 12^2 &:= 15^2 \\ \mathbf{6^2} + 8^2 &:= 10^2 \end{aligned}$$

Remark 3.2. *Let's consider following differences:*

$$\begin{aligned} (\mathbf{9, 40, 41}) &\Rightarrow 41 - 9 = 32 \quad \text{and} \quad 41 - 40 = 1 \\ (\mathbf{9, 12, 15}) &\Rightarrow 15 - 9 = 6 \quad \text{and} \quad 15 - 12 = 3 \\ (\mathbf{6, 8, 10}) &\Rightarrow 10 - 6 = 4 \quad \text{and} \quad 10 - 8 = 2 \end{aligned}$$

According to Result 2, the above three ***Pythagorean triples*** *with numbers 9 and 6 don't generates any magic square.*

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4 Magic Squares of Order 4

According to equation (7), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (3), we can only obtain a magic square of order 4 with odd number entries. Taking $k = 4$ in equation (3), we get

$$1 + 3 + 5 + \ldots + 27 + 29 + 31 = 256 = 16^2$$

Example 4.1. *For the* ***consecutive odd number*** *entries* $\{colorblue1, 3, 5, \ldots, 29, 31\}$, *the* ***pandiagonal*** *magic square of order 4 is given by*

		64	64	64	64
	13	23	1	27	64
64	3	25	15	21	64
64	31	5	19	9	64
64	17	11	29	7	64
	64	64	64	64	64

In this case, the magic sum is $S_{4\times4} := 64 = 4^3$, and the sum of the entries is $T_{16} := 256 = 16^2 = 4^4$. This satisfies the uniformity property 11, i.e., $\left\langle 4, 4^2, 4^3, 4^4 \right\rangle$.

4.1 Pythagorean Triples

According to Example 4.1, we have a perfect square entries sums, i.e., $T_{16} := 256 = 16^2$. Below are **Pythagorean triples** for the number 16:

$$\mathbf{16^2} + 12^2 := 20^2$$
$$\mathbf{16^2} + 30^2 := 34^2$$
$$\mathbf{16^2} + 63^2 := 65^2$$

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Remark 4.1. *Let's consider following differences:*

$$
\begin{aligned}
(\mathbf{16},\mathbf{12},\mathbf{20}) &\Rightarrow 20-16=4 & \textit{and} \quad & 20-12=8\\
(\mathbf{16},\mathbf{30},\mathbf{34}) &\Rightarrow 34-30=4 & \textit{and} \quad & 34-16=18\\
(\mathbf{16},\mathbf{63},\mathbf{65}) &\Rightarrow 65-16=49=7^2 & \textit{and} \quad & 65-63=2
\end{aligned}
$$

According to Result 2, among above three ***Pythagorean triples*** *there is only one triple* $(\mathbf{16},\mathbf{63},\mathbf{65})$ *generating a* ***perfect square entries sum*** *magic square of order 7. From Result 3, the* ***odd number entries***, $\{33,35,....,127,129$ *magic square with magic sum,* $\boldsymbol{S}_{7\times 7}:=567$. *The entries total sum is* $\boldsymbol{T}_{49}:=7\times 567=3969=63^2$.

5 Magic Squares of Order 5

According to equations (3) and (8), one can obtain **perfect square entries sum magic squares** of order 5 in two ways: first using **consecutive odd numbers**, and the second using **consecutive natural numbers**. For the **consecutive natural numbers** there are again two ways: the first one is of **uniformity**, and the second is of **minimum perfect square entries sum**.

5.1 Consecutive Odd Numbers

Taking $k=5$ in equation (3), we get

$$
\begin{aligned}
1+3+5+\ldots+(2\times 5^2-1) &= 5^4\\
\Rightarrow 1+3+5+\ldots+47+49 &= 625=25^2=5^4
\end{aligned}
$$

Example 5.1. *For the* ***consecutive odd number*** *entries* $\{1,3,5,\ldots,47,49\}$, *the* ***pandiagonal*** *magic square of order 5 is given by*

		125	125	125	125	125
	1	17	23	39	45	125
125	33	49	5	11	27	125
125	15	21	37	43	9	125
125	47	3	19	25	31	125
125	29	35	41	7	13	125
	125	125	125	125	125	125

In this case, the magic sum is $S_{5\times 5} := 125 = 5^3$, and the sum of the entries is $\boldsymbol{T}_{25} := 625 = 25^2 = 5^4$.

5.2 Consecutive Natural Numbers

According to equation (8),

$$G := \frac{n(n+1)}{2} - \frac{(n-25)(n-24)}{2} = 25(n-12)$$

Taking $n = 37$, we get a perfect square, i.e.,

$$G := T(37) - T(12) = \frac{37 \times 38}{2} - \frac{12 \times 13}{2} = 37 \times 19 - 6 \times 13$$
$$= 703 - 78 = 625 = 25^2 = 4 = 5^4.$$

Simplifying, we get

$$13 + 14 + 15 + \ldots + 36 + 37 = 625 = 25^2 = 5^4$$

Example 5.2. *For the* ***consecutive natural number*** *entries* $\{13, 14, 15, \ldots, 36, 37\}$, *the* ***pandiagonal*** *magic square of order 5 is given by*

		125	125	125	125	125
	13	21	24	32	35	125
125	29	37	15	18	26	125
125	20	23	31	34	17	125
125	36	14	22	25	18	125
125	27	30	33	16	19	125
	125	125	125	125	125	125

In this case also, the magic sum is $S_{5\times 5} = 125 = 5^3$, and the sum of all the entries is $\boldsymbol{T}_{25} := 625 = 5^4$.

Both the examples 5.1 and 5.2 satisfy the uniformity property (11), i.e., $\left\langle 5, 5^2, 5^3, 5^4 \right\rangle$.

5.3 Minimum Perfect Square Entries Sum

Choosing $m = 4, 5$ and 6, and $p = 5$ in equation (13), we get

$$1. \quad L\left(4^2 + \frac{5^2-1}{2}, 5^2\right) \to (5, 4, 28, 80, 400)$$

$$2. \quad L\left(5^2 + \frac{5^2-1}{2}, 5^2\right) \to (5, 13, 37, 125, 625) \Rightarrow \left\langle 5, 5^2, 5^3, 5^4 \right\rangle$$

$$3. \quad L\left(6^2 + \frac{5^2-1}{2}, 5^2\right) \to (5, 24, 48, 180, 900)$$

The values written above are for $m = 4, 5$ and 6. The second value for $m = 5$ satisfy the uniformity property (11). This case is already studied in Example 5.1. The first value for $m = 4$ give **minimum perfect square** magic square. In this case, we have magic square of order 5 with a perfect square entries sum $400 = 20^2$, but it does not satisfy the uniformity property (11). See below a **pandiagonal** magic square of order 5, where the total entries sum is **minimum perfect square**.

Example 5.3. *For the **consecutive natural number** entries $\{4, 5, 6, \ldots, 27, 28\}$, the **pandiagonal** magic square of order 5 is given by*

		80	80	80	80	80
	4	10	16	22	28	80
80	21	27	8	9	15	80
80	13	14	20	26	7	80
80	25	6	12	18	19	80
80	17	23	24	5	11	80
	80	80	80	80	80	80

In this case the magic sum is $S_{5\times 5} = 80$, and the sum of all entries is $T_{25} := 400 = 20^2$. It is **minimum perfect square entries sum**.

5.4 Pythagorean Triples

According to Examples 5.1, 5.2 and 5.3, we have two perfect square entries sums, i.e., $T_{25} := 625 = 25^2$ and $T_{25} := 400 = 20^2$. Below are **Pythagorean triples** for the numbers 25 and 20:

$$25^2 + 60^2 := 65^2 \qquad 20^2 + 21^2 := 29^2$$
$$25^2 + 312^2 := 313^2 \qquad 20^2 + 48^2 := 52^2$$
$$20^2 + 15^2 := 25^2 \qquad 20^2 + 99^2 := 101^2$$

Remark 5.1. *The above six **Pythagorean triples** are with numbers 25 and 20 due to Examples 5.1 and 5.3. According to Result 2, there are only 2 triples generating magic squares with **perfect square entries sum**:*

$$(\mathbf{20,21,29}) \Rightarrow 29 - 20 = 9 = 3^2 \quad \text{and} \quad 29 - 21 = 8$$
$$(\mathbf{20,99,101}) \Rightarrow 101 - 20 = 81 = 9^2 \quad \text{and} \quad 101 - 99 = 2$$

Below are details of magic squares calculated according expression (17) *given in Result 3:*

1. *The triple* $(20, 21, 29)$ *generating a **perfect square entries sum** magic square of order 3 with the **odd number***

entries, $\{41, 43, ..., 145, 147\}$ *with magic sum,* $S_{3\times 3} := 147$*. The entries total sum is* $T_9 := 3 \times 147 = 441 = 21^2$.

2. *The triple* $(20, 99, 101)$ *generating a* ***perfect square entries sum*** *magic square of order 9 with the* ***odd number entries***, $\{41, 43, ..., 199, 201\}$ *with magic sum,* $S_{9\times 9} := 1089$*. The entries total sum is* $T_{81} := 9 \times 1089 = 9801 = 99^2$.

6 Magic Squares of Order 6

Let's consider $k = 6$ in Equation (3). Then, we have

$$\begin{aligned} &1 + 3 + 5 + \ldots + (2 \times 6^2 - 1) = 6^4 \\ \Rightarrow &1 + 3 + 5 + \ldots + 69 + 71 \qquad = 1296 = 36^2 = 6^4. \end{aligned}$$

Example 6.1. *For the* ***consecutive odd number*** *entries* $\{1, 3, 5, \ldots, 69, 71\}$*, a magic square of order 6 is given by*

						216
1	45	55	67	33	15	216
57	13	69	27	41	9	216
23	11	25	53	61	43	216
63	31	7	47	19	49	216
37	65	21	5	59	29	216
35	51	39	17	3	71	216
216	216	216	216	216	216	216

In this case, the magic sum is $S_{6\times 6} := 216 = 6^3$, and the sum of the entries is $T_{36} := 1296 = 36^2 = 6^4$. This satisfies the Property 11 of uniformity, i.e., $\left\langle 6, 6^2, 6^3, 6^4 \right\rangle$.

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6.1 Pythagorean Triples

According to Example 6.1, we have perfect square entries sum, i.e., $\boldsymbol{T}_{36} := 1296 = 36^2$. Below are **Pythagorean triples** for the number 36:

$$\mathbf{36^2} + 15^2 := 39^2$$
$$\mathbf{36^2} + 27^2 := 45^2$$
$$\mathbf{36^2} + 48^2 := 60^2$$
$$\mathbf{36^2} + 77^2 := 85^2$$

$$\mathbf{36^2} + 323^2 := 325^2$$
$$\mathbf{36^2} + 160^2 := 164^2$$
$$\mathbf{36^2} + 105^2 := 111^2$$

Remark 6.1. *The above seven **Pythagorean triples** are with number 36. This number is due to Example 6.1. According to Result 2, there are only 3 triples generating magic squares with **perfect square entries sum**:*

$$(\mathbf{36,27,45}) \Rightarrow 45-36 = 9 = 3^2$$
$$\mathbf{36,77,85}) \Rightarrow 85-36 = 49 = 7^2$$
$$(\mathbf{36,323,325}) \Rightarrow 325-36 = 289 = 17^2.$$

Below are details of magic squares calculated according expression (17) *given in Result 3:*

1. *The triple* $(36,27,45)$ *generating a **perfect square entries sum** magic square of order 3 with the **odd number entries**,* $\{73,75,....,87,89\}$ *with magic sum,* $\boldsymbol{S}_{3\times 3} := 243$*. The entries total sum is* $\boldsymbol{T}_9 := 3\times 243 = 729 = 27^2$.

2. *The triple* $(36,77,85)$ *generating a **perfect square entries sum** magic square of order 7 with the **odd number entries**,* $\{73,75,....,167,169\}$ *with magic sum,* $\boldsymbol{S}_{7\times 7} := 847$*. The entries total sum is* $\boldsymbol{T}_{49} := 7\times 847 = 5929 = 77^2$.

3. *The triple* $(36,323,325)$ *generating a **perfect square entries sum** magic square of order 17 with the **odd number entries**,* $\{73,75,....,647,649\}$ *with magic sum,* $\boldsymbol{S}_{17\times 17} := 6137$*. The entries total sum is* $\boldsymbol{T}_{289} := 17\times 6137 = 104329 = 323^2$.

7 Magic Squares of Order 7

According to equations (3) and (8), one can obtain **perfect square entries sum magic squares** of order 7 in two ways: first using **consecutive odd numbers**, and the second using **consecutive natural numbers**. For the **consecutive natural numbers** there are again two ways: the first one is of **uniformity**, and the second is of **minimum perfect square entries sum**.

7.1 Consecutive Odd Numbers

Taking $k = 7$ in equation (3), we get

$$\begin{aligned} &1+3+5+\ldots+(2\times 7^2-1) = 7^4 \\ \Rightarrow &1+3+5+\ldots+95+97 \qquad = 2401 = 49^2 = 7^4 \end{aligned}$$

Example 7.1. *For the **consecutive odd number** entries* $\{1,3,5,\ldots,95,97\}$*, a **pandiagonal** magic square of order 7 is given by*

		343	343	343	343	343	343	343
	1	17	33	49	65	81	97	343
343	79	95	13	15	31	47	63	343
343	45	61	77	93	11	27	29	343
343	25	41	43	59	75	91	9	343
343	89	7	23	39	55	57	73	343
343	69	71	87	5	21	37	53	343
343	35	51	67	83	85	3	19	343
	343	343	343	343	343	343	343	343

7.2 Consecutive Natural Numbers

According to equation (8),

$$G := \frac{n(n+1)}{2} - \frac{(n-49)(n-48)}{2} = 49(n-24)$$

Taking $n = 73$, we get a perfect square, i.e.,

$$\begin{aligned} G := T(73) - T(24) = \frac{73 \times 74}{2} - \frac{24 \times 25}{2} &= 73 \times 37 - 12 \times 25 \\ &= 2701 - 300 = 2401 = 49^2 = 7^4. \end{aligned}$$

Simplifying, we get

$$25 + 26 + 27 + \ldots + 72 + 73 = 2401 = 49^2 = 7^4$$

This gives a perfect square entries sum for the 49 **consecutive natural numbers** from 25 to 73.

Example 7.2. *For the* ***consecutive natural number*** *entries* $\{25, 26, 27, \ldots, 72, 73\}$, *a* ***pandiagonal*** *magic square of order 7 is given by*

		343	343	343	343	343	343	343
	25	33	41	49	57	65	73	343
343	64	72	31	32	40	48	56	343
343	47	55	63	71	30	38	39	343
343	37	45	46	54	62	70	29	343
343	69	28	36	44	52	53	61	343
343	59	60	68	27	35	43	51	343
343	42	50	58	66	67	26	34	343
	343	343	343	343	343	343	343	343

In both the Examples 7.1 and 7.2, the magic sum is $S_{7\times 7} = 343 = 7^3$, and the sum of all the entries is $T_{49} := 2401 = 49^2 = 7^4$, and satisfy the uniformity property (11), i.e., $\left\langle 7, 7^2, 7^3, 7^4 \right\rangle$.

7.3 Minimum Perfect Square Entries Sum

Choosing $m = 5, 6$ and 7, and $p = 7$ in equation (13), we get

$$1.\quad L\left(5^2 + \frac{7^2-1}{2}, 7^2\right) \to (7, 1, 49, 175, 1225)$$

$$2.\quad L\left(6^2 + \frac{7^2-1}{2}, 7^2\right) \to (7, 12, 60, 252, 1764)$$

$$3.\quad L\left(7^2 + \frac{7^2-1}{2}, 7^2\right) \to (7, 25, 73, 343, 2401) \Rightarrow \left\langle 7, 7^2, 7^3, 7^4 \right\rangle$$

The values written above are for $m = 5, 6$ and 7. The second value for $m = 6$ satisfy the uniformity property (11). This case is already studied in Example 7.1. The first value for $m = 5$ give **minimum perfect square** magic square. In this case, we have a magic square of order 7 with a **perfect square entries sum**, i.e., $1225 = 175^2$, but it does not satisfy the uniformity property (11). The **pandiagonal** magic square of order 7 arising due to **minimum perfect square entries sum** is given in example below.

Example 7.3. *For the* ***consecutive natural number*** *entries* $\{1, 2, 3, \ldots, 48, 49\}$, *a* ***pandiagonal*** *magic square of order 7 is given by*

		175	*175*	*175*	*175*	*175*	*175*	*175*
	1	*9*	*17*	*25*	*33*	*41*	*49*	*175*
175	*40*	*48*	*7*	*8*	*16*	*24*	*32*	*175*
175	*23*	*31*	*39*	*47*	*6*	*14*	*15*	*175*
175	*13*	*21*	*22*	*30*	*38*	*46*	*5*	*175*
175	*45*	*4*	*12*	*20*	*28*	*29*	*37*	*175*
175	*35*	*36*	*44*	*3*	*11*	*19*	*27*	*175*
175	*18*	*26*	*34*	*42*	*43*	*2*	*10*	*175*
	175	*175*	*175*	*175*	*175*	*175*	*175*	*175*

In this case the magic sum is $S_{7\times 7} = 175$, and the sum of all entries is $\boldsymbol{T}_{49} := 7 \times 175 = 1225 = 35^2$. This is the only example (amongst the orders 3 to 51) of a **minimum perfect square entries sum** with entries starting from the number 1.

7.4 Pythagorean Triples

According to Examples 7.1, 7.2 and 7.3, we have two perfect square entries sums, i.e., $\boldsymbol{T}_{49} := 2401 = 49^2$ and $\boldsymbol{T}_{49} := 7 \times 175 = 1225 = 35^2$. Below are **Pythagorean triples** for the numbers 49 and 35:

$$\mathbf{49^2} + 168^2 := 175^2 \qquad \mathbf{35^2} + 84^2 := 91^2$$
$$\mathbf{49^2} + 1200^2 := 1201^2 \qquad \mathbf{35^2} + 120^2 := 125^2$$
$$\mathbf{35^2} + 12^2 := 37^2 \qquad \mathbf{35^2} + 612^2 := 613^2$$

Remark 7.1. *The above six* ***Pythagorean triples*** *are with numbers 49 and 35. These numbers are due to Examples 7.1 and 7.3. According to Result 2, there is only one triples generating magic square with* ***perfect square entries sum:***

$$(35, 12, 37) \Rightarrow 37 - 12 = 25^2.$$

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Below are details of a magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(35, 12, 37)$ *generating a* ***perfect square entries sum*** *magic square of order 5 with the* ***odd number entries****,* $\{25, 27, ..., 71, 73\}$ *with magic sum,* $S_{5\times5} := 245$*. The entries total sum is* $T_{25} := 5 \times 245 = 1225 = 35^2$.

8 Bimagic Squares of Order 8

According to equation (7), we cannot obtain a consecutive natural number magic square where the sum of all the entries is a perfect square. According to equation (3), we can only obtain a magic square of order 8 with this property if we use odd number entries. Taking $k = 8$ in equation (3), we get

$$\begin{aligned} & 1 + 3 + 5 + \ldots + (2 \times 8^2 - 1) && = 8^4 \\ \Rightarrow & 1 + 3 + 5 + \ldots + 123 + 125 + 127 = 4096 = 64^2 = 8^4. \end{aligned}$$

Example 8.1. *For the* ***consecutive odd number*** *entries* $\{1, 3, 5, \ldots, 125, 127\}$*, a* ***pandiagonal*** *magic square of order 8 is given by*

		512	512	512	512	512	512	512	512
	31	81	71	9	53	123	109	35	512
512	51	125	107	37	25	87	65	15	512
512	1	79	89	23	43	101	115	61	512
512	45	99	117	59	7	73	95	17	512
512	75	5	19	93	97	47	57	119	512
512	103	41	63	113	77	3	21	91	512
512	85	27	13	67	127	49	39	105	512
512	121	55	33	111	83	29	11	69	512
	512	512	512	512	512	512	512	512	512

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In this case the magic sum is $S_{8\times 8} = 512$, and the sum of the entries is $T_{64} = 4096 = 64^2 = 8^4$. This magic square is also **bimagic** and has a bimagic sum $Sb_{8\times 8} = 43688$.

8.1 Pythagorean Triples

According to Example 8.1, we have perfect square entries sum, i.e., $T_{64} = 4096 = 64^2$. Below are **Pythagorean triples** for the numbers 64:

$$\begin{aligned} \mathbf{64^2} + 48^2 &:= 80^2 \\ \mathbf{64^2} + 120^2 &:= 136^2 \\ \mathbf{64^2} + 252^2 &:= 260^2 \end{aligned}$$

$$\begin{aligned} \mathbf{64^2} + 510^2 &:= 514^2 \\ \mathbf{64^2} + 1023^2 &:= 1025^2 \end{aligned}$$

Remark 8.1. *The above five* ***Pythagorean triples*** *are with number 64. This number is due to Example 8.1. According to Result 2, there are only four triples generating magic square with* ***perfect square entries sum****:*

$$\begin{aligned} (\mathbf{64, 48, 80}) &\Rightarrow 80 - 64 = 16 = 4^2 \\ (\mathbf{64, 120, 136}) &\Rightarrow 136 - 120 = 16 = 4^2 \\ (\mathbf{64, 252, 260}) &\Rightarrow 260 - 64 = 196 = 14^2 \\ (\mathbf{35, 12, 37}) &\Rightarrow 1025 - 64 = 961 = 31^2. \end{aligned}$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(64, 48, 80)$ *generating a* ***perfect square entries sum*** *magic square of order 4 with the* ***odd number entries****,* $\{129, 131, ..., 157, 159\}$ *with magic sum,* $S_{4\times 4} := 576$*. The entries total sum is* $T_{16} := 4 \times 576 = 2304 = 48^2$.

2. *The triple* $(64, 120, 136)$ *generating a* ***perfect square entries sum*** *magic square of order 4 with the* ***odd number entries****,* $\{241, 243, ..., 269, 271\}$ *with magic sum,* $S_{4\times 4} := 1024$*. The entries total sum is* $T_{16} := 4 \times 1024 = 4096 = 64^2$.

3. *The triple* $(64, 252, 260)$ *generating a* ***perfect square entries sum*** *magic square of order 14 with the* ***odd number entries***, $\{129, 131, ..., 517, 519\}$ *with magic sum,* $\boldsymbol{S}_{14\times 14} := 4536$. *The entries total sum is* $\boldsymbol{T}_{196} := 14 \times 4536 = 63504 = 252^2$.

4. *The triple* $(64, 1023, 1025)$ *generating a* ***perfect square entries sum*** *magic square of order 31 with the* ***odd number entries***, $\{129, 131, ..., 2047, 2049\}$ *with magic sum,* $\boldsymbol{S}_{31\times 31} := 33759$. *The entries total sum is* $\boldsymbol{T}_{961} := 17 \times 33759 = 1046529 = 1023^2$.

9 Magic Square of Order 9

According to equations (3) and (8), one can obtain **perfect square entries sum magic squares** of order 9 in two ways: first using **consecutive odd numbers**, and the second using **consecutive natural numbers**. For the **consecutive natural numbers** there are again two ways: the first one is of **uniformity**, and the second is of **minimum perfect square entries sum**.

9.1 Consecutive Odd Numbers

Taking $k = 9$ in equation (3), we get

$$\begin{aligned} 1 + 3 + 5 + \ldots + (2 \times 9^2 - 1) &= 9^4 \\ \Rightarrow 1 + 3 + 5 + \ldots + 159 + 161 &= 6561 = 81^2 = 9^4. \end{aligned}$$

Example 9.1. *For the* ***consecutive odd number*** *entries* $\{1, 3, 5, \ldots, 159, 161\}$, *a magic square of order 9 is given by*

									729
1	35	45	69	79	95	119	129	157	729
65	75	103	109	143	153	15	25	41	729
123	133	149	11	21	49	55	89	99	729
53	9	19	97	59	87	147	121	137	729
93	67	83	161	117	127	43	5	33	729
151	113	141	39	13	29	107	63	73	729
27	37	17	77	105	61	139	155	111	729
85	101	57	135	145	125	23	51	7	729
131	159	115	31	47	3	81	91	71	729
729	729	729	729	729	729	729	729	729	729

In this case, the magic sum is $S_{9\times 9}=729$, and the sum of the entries is $\boldsymbol{T}_{81}=6561=81^2=9^4$. This magic square is also **bimagic** and has a bimagic sum $Sb_{9\times 9}=78729$.

9.2 Consecutive Natural Numbers

According to equation (8),

$$G:=\frac{n(n+1)}{2}-\frac{(n-81)(n-80)}{2}=81(n-40)$$

Taking $n=121$, we get a perfect square, i.e.,

$$\begin{aligned}G:=T(121)-T(40)=\frac{121\times 122}{2}-\frac{40\times 42}{2}&=121\times 61-20\times 41\\&=7381-820=6561=81^2=9^4.\end{aligned}$$

Simplifying, we get

$$41+42+43+\ldots+120+121=6561=81^2=9^4$$

This gives a **perfect square entries sum** for the 81 entries using **consecutive natural numbers** from 41 to 121.

Example 9.2. *For the* ***consecutive natural number*** *entries* $\{41, 42, 43, \ldots, 120, 121\}$*, a magic square of order 9 is given by*

									729
41	58	63	75	80	88	100	105	119	729
73	78	92	95	112	117	48	53	61	729
102	107	115	46	51	65	68	85	90	729
67	45	50	89	70	84	114	101	109	729
87	74	82	121	99	104	62	43	57	729
116	97	111	60	47	55	94	72	77	729
54	59	49	79	93	71	110	118	96	729
83	91	69	108	113	103	52	66	44	729
106	120	98	56	64	42	81	86	76	729
729	729	729	729	729	729	729	729	729	729

In both the examples 9.1 and 9.2, the magic sum is $S_{9\times 9} = 729 = 9^3$, and the sum of all entries is $\boldsymbol{T}_{81} := 6561 = 81^2 = 9^4$. This magic square shown above is **bimagic** with a **bimagic sum**, $Sb_{9\times 9} = 63969$. We observe that although the Examples 9.1 and 9.2 show the same magic sums, but their **bimagic sums** are different.

Both the Examples 9.1 and 9.2 satisfy the uniformity property (11), i.e., $\left\langle 9, 9^2, 9^3, 9^4 \right\rangle$.

9.3 Minimum Perfect Square Entries Sum

The examples given in 9.1 and 9.2 satisfy the property (11) of uniformity, i.e., $\left\langle 9, 9^2, 9^3, 9^4 \right\rangle$. However, the total entries of the magic square given in Example 9.2 do not yield a **minimum perfect square entries sum**. Choosing $m = 7, 8, 9$ and $p = 9$ in equation (13), we get

1. $L\left(7^2+\frac{9^2-1}{2}, 9^2\right) \to (9, 9, 89, 441, 3969)$
2. $L\left(8^2+\frac{9^2-1}{2}, 9^2\right) \to (9, 24, 104, 576, 5184)$
3. $L\left(9^2+\frac{9^2-1}{2}, 9^2\right) \to (9, 41, 121, 729, 6561) \Rightarrow \left\langle 9, 9^2, 9^3, 9^4 \right\rangle$

The values written above are for $m = 7, 8, 9$. The values for $m = 9$ satisfy the uniformity property (11). This case is already studied in Example 9.1. The values for $m = 7$ give **minimum perfect square** magic square. In this case, we have magic square of order 9 with a **perfect square entries sum** with $3969 = 63^2$, but it does not satisfy the uniformity property (11). See below the magic square of order 9 constructed according values given in expression for $m = 7$:

Example 9.3. *For the* ***consecutive natural number*** *entries* $\{9, 10, 11, \ldots, 88, 89\}$, *a* ***pandiagonal*** *magic square of order 9 is given by*

		441	441	441	441	441	441	441	441	441
	10	26	30	41	45	61	69	76	83	441
441	42	49	56	64	80	84	14	18	34	441
441	68	72	88	15	22	29	37	53	57	441
441	28	17	21	59	36	52	87	67	74	441
441	60	40	47	82	71	75	32	9	25	441
441	86	63	79	33	13	20	55	44	48	441
441	19	35	12	50	54	43	78	85	65	441
441	51	58	38	73	89	66	23	27	16	441
441	77	81	70	24	31	11	46	62	39	441
	441	441	441	441	441	441	441	441	441	441

The above Example 9.3 is with magic sum $S_{9\times 9} = 441$, and the sum of all entries is $T_{81} := 9 \times 441 =$

$3969 = 63^2$. It is **pandiagonal minimum perfect square entries sum** magic square of order 9. In each case, the sum of each 3×3 block is the same as of each magic square in their respective way.

9.4 Pythagorean Triples

According to Examples 9.1, 9.2 and 9.3, we have two perfect square entries sums, i.e., $\boldsymbol{T}_{81} := 6561 = 81^2$ and $\boldsymbol{T}_{81} := 9 \times 441 = 3969 = 63^2$. Below are **Pythagorean triples** for the numbers 81 and 63:

$$\mathbf{81^2} + 108^2 := 135^2$$
$$\mathbf{81^2} + 360^2 := 369^2$$
$$\mathbf{81^2} + 1092^2 := 1095^2$$
$$\mathbf{81^2} + 3280^2 := 3281^2$$

$$\mathbf{63^2} + 16^2 := 65^2$$
$$\mathbf{63^2} + 60^2 := 87^2$$
$$\mathbf{63^2} + 84^2 := 105^2$$
$$\mathbf{63^2} + 216^2 := 225^2$$
$$\mathbf{63^2} + 280^2 := 287^2$$
$$\mathbf{63^2} + 660^2 := 663^2$$
$$\mathbf{63^2} + 1984^2 := 1985^2$$

Remark 9.1. *The above eleven **Pythagorean triples** are with numbers 81 and 63. These numbers are due to Examples 9.1 and 9.3. According to Result 2, there are only three triples generating magic square with **perfect square entries sum**:*

$$(\mathbf{81}, \mathbf{360}, \mathbf{369}) \Rightarrow 369 - 360 = 9 = 3^2$$
$$(\mathbf{63}, \mathbf{16}, \mathbf{65}) \Rightarrow 65 - 16 = 49 = 7^2$$
$$(\mathbf{63}, \mathbf{216}, \mathbf{225}) \Rightarrow 225 - 216 = 9 = 3^2.$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(81, 360, 369)$ *generating a **perfect square entries sum** magic square of order 3 with the **odd number entries**,* $\{721, 723, ..., 735, 737\}$ *with magic sum,* $S_{3\times 3} := 2187$*. The entries total sum is* $\boldsymbol{T}_9 :=$

$3 \times 2187 = 6561 = 81^2$.

2. *The triple* $(63, 16, 65)$ *generating a* ***perfect square entries sum*** *magic square of order 7 with the* ***odd number entries****,* $\{33, 35, ..., 127, 129\}$ *with magic sum,* $\boldsymbol{S}_{7\times 7} := 567$*. The entries total sum is* $\boldsymbol{T}_{49} := 7 \times 567 = 3969 = 63^2$.

3. *The triple* $(63, 216, 225)$ *generating a* ***perfect square entries sum*** *magic square of order 3 with the* ***odd number entries****,* $\{433, 435, ..., 447, 449\}$ *with magic sum,* $\boldsymbol{S}_{3\times 3} := 1323$*. The entries total sum is* $\boldsymbol{T}_9 := 3 \times 1323 = 3969 = 63^2$.

10 Magic Squares of Order 10

According to equation (7), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (3), we can only obtain a magic square of order 10 with this property if we use odd number entries. Taking $k = 10$ in equation (3), we get

$$\begin{aligned} & 1 + 3 + 5 + \ldots + (2 \times 10^2 - 1) = 10^4 \\ \Rightarrow & 1 + 3 + 5 + \ldots + 197 + 199 \qquad = 10000 = 100^2 = 10^4. \end{aligned} \tag{19}$$

Example 10.1. *For the* ***consecutive odd number*** *entries* $\{1, 3, 5, \ldots, 197, 199\}$*, a* ***block-bordered*** *magic square of order 10 is given by*

181	171	31	167	35	27	7	195	3	183
25	93	115	37	155	77	131	53	139	175
177	43	149	99	109	59	133	83	125	23
21	163	45	107	85	147	61	123	69	179
191	101	91	157	51	117	75	141	67	9
1	95	113	39	153	79	129	55	137	199
185	41	151	97	111	57	135	81	127	15
13	161	47	105	87	145	63	121	71	187
189	103	89	159	49	119	73	143	65	11
17	29	169	33	165	173	193	5	197	19

The magic sums of Example 10.1 is $S_{10\times 10} = 1000$, and the sum of all entries is $T_{100} := 10 \times 1000 = 10000 = 100^2 = 10^4$. It satisfies the uniformity property (11), i.e., $\left\langle 10, 10^2, 10^3, 10^4 \right\rangle$. Morever, the inner magic square is **pandiagonal** magic square of order 8 with equal sum blocks of **pandiagonal** magic square of order 4. The magic sums are $S_{8\times 8} = 800$ and $S_{4\times 4} = 400$

10.1 Pythagorean Triples

According to Example 10.1, we have perfect square entries sum, i.e., $T_{100} := 10 \times 1000 = 10000 = 100^2$. Below are **Pythagorean triples** for the number 100:

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$$\mathbf{100^2} + 75^2 := 125^2$$
$$\mathbf{100^2} + 105^2 := 145^2$$
$$\mathbf{100^2} + 240^2 := 260^2$$
$$\mathbf{100^2} + 495^2 := 505^2$$
$$\mathbf{100^2} + 621^2 := 629^2$$
$$\mathbf{100^2} + 1248^2 := 1252^2$$
$$\mathbf{100^2} + 2499^2 := 2501^2$$

Remark 10.1. *The above seven **Pythagorean triples** are with number 100. This number is due to Example 10.1. According to Result 2, there are only three triples generating magic square with **perfect square entries sum**:*

$$(\mathbf{100, 75, 125}) \Rightarrow 125 - 100 = 25 = 5^2$$
$$(\mathbf{100, 621, 629}) \Rightarrow 629 - 100 = 529 = 23^2$$
$$(\mathbf{100, 2499, 2501}) \Rightarrow 2501 - 100 = 2401 = 49^2.$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple $(100, 75, 125)$ generating a **perfect square entries sum** magic square of order 5 with the **odd number entries**, $\{201, 203, ..., 247, 249\}$ with magic sum, $S_{9\times 9} := 1125$. The entries total sum is $T_{25} := 5 \times 1125 = 5625 = 75^2$.*

2. *The triple $(100, 621, 629)$ generating a **perfect square entries sum** magic square of order 23 with the **odd number entries**, $\{201, 203, ..., 1255, 1257\}$ with magic sum, $S_{23\times 23} := 16767$. The entries total sum is $T_{529} := 23 \times 16767 = 385641 = 621^2$.*

3. *The triple $(100, 2499, 2501)$ generating a **perfect square entries sum** magic square of order 49 with the **odd number entries**, $\{201, 203, ..., 4999, 5001\}$ with magic sum, $S_{49\times 49} := 127449$. The entries total sum is $T_{2401} := 49 \times 127449 = 6245001 = 2499^2$.*

11 Magic Squares of Order 11

According to equations (3) and (8), one can obtain **perfect square entries sum magic squares** of order 11 in two ways: first using **consecutive odd numbers**, and the second using **consecutive natural numbers**.

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For the **consecutive natural numbers** there are again two ways: the first one is of **uniformity**, and the second is of **minimum perfect square entries sum**.

11.1 Consecutive Odd Numbers

Taking $k = 11$ in equation (3), we get

$$\begin{aligned} &1+3+5+\ldots+(2\times 11^2-1) = 11^4 \\ \Rightarrow &1+3+5+\ldots+239+241 \qquad = 14641 = 121^2 = 11^4 \end{aligned}$$

Example 11.1. *For the **consecutive odd number** entries* $\{1,3,5,\ldots,239,241\}$*, a **block-bordered** magic square of order 11 is given by*

23	39	35	31	27	225	227	231	235	239	19
241	83	181	99	93	167	103	79	177	107	1
237	109	81	173	95	85	183	105	89	169	5
233	171	101	91	175	111	77	179	97	87	9
229	119	55	189	129	41	193	115	51	197	13
21	199	117	47	185	121	57	195	125	43	221
25	45	191	127	49	201	113	53	187	123	217
29	155	145	63	165	131	67	151	141	71	213
33	73	153	137	59	157	147	69	161	133	209
37	135	65	163	139	75	149	143	61	159	205
223	203	207	211	215	17	15	11	7	3	219

11.2 Consecutive Natural Numbers

According to equation (8),

$$G := \frac{n(n+1)}{2} - \frac{(n-121)(n-120)}{2} = 121(n-60)$$

Taking $n = 181$, we get a perfect square, i.e.,

$$\begin{aligned} G := T(181) - T(60) = \frac{181 \times 182}{2} - \frac{60 \times 61}{2} &= 181 \times 91 - 30 \times 61 \\ &= 16471 - 1830 = 14641 = 121^2 = 11^4. \end{aligned}$$

Simplifying, we get

$$61 + 62 + 63 + \ldots + 180 + 181 = 14641 = 121^2 = 11^4.$$

This gives a perfect square entries sum for the 121 consecutive natural numbers from to 61 to 181.

Example 11.2. *For the **consecutive natural number** entries* $\{61, 62, 63, \ldots, 180, 181\}$, *a **block-bordered** magic square of order 11 is given by*

23	39	35	31	27	225	227	231	235	239	19
241	83	181	99	93	167	103	79	177	107	1
237	109	81	173	95	85	183	105	89	169	5
233	171	101	91	175	111	77	179	97	87	9
229	119	55	189	129	41	193	115	51	197	13
21	199	117	47	185	121	57	195	125	43	221
25	45	191	127	49	201	113	53	187	123	217
29	155	145	63	165	131	67	151	141	71	213
33	73	153	137	59	157	147	69	161	133	209
37	135	65	163	139	75	149	143	61	159	205
223	203	207	211	215	17	15	11	7	3	219

Both the Examples 11.1 and 11.2 are with equal magic sum, i.e., $S_{11\times 11} = 1331 = 11^3$, and the sum of all numbers is $T_{121} := 14641 = 121^2 = 11^4$. Moreover, the inner magic square of order 9 is **pandiagonal** with blocks of **semi-magic** squares of order 3 with equal **semi-magic** sums. The magic sums are $S_{9\times 9} = 1089$ and $Sm_{3\times 3} = 363$.

Both of the examples 11.1 and 11.2 satisfy the uniformity property (11), i.e., $\left\langle 11, 11^2, 11^3, 11^4 \right\rangle$.

11.3 Minimum Perfect Square Entries Sum

The examples given in 11.1 and 11.2 satisfy the uniformity property (11), i.e., $\left\langle 11, 11^2, 11^3, 11^4 \right\rangle$. However, the sum of the magic square entries of 11.2 is not a **minimum perfect square entries sum**. Choosing $m = 8, 9, 10$ and 11, and $p = 11$ in equation (13), we get

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1. $L\left(8^2+\dfrac{11^2-1}{2}, 11^2\right) \rightarrow (11,4,124,704,7744)$
2. $L\left(9^2+\dfrac{11^2-1}{2}, 11^2\right) \rightarrow (11,21,141,891,9801)$
3. $L\left(10^2+\dfrac{11^2-1}{2}, 11^2\right) \rightarrow (11,40,160,1100,12100)$
4. $L\left(11^2+\dfrac{11^2-1}{2}, 11^2\right) \rightarrow (11,61,181,1331,14641) \Rightarrow \left\langle 11,11^2,11^3,11^4\right\rangle$

The values written above are for $m=8,9,10$ and 11. The forth value for $m=11$ satisfy the uniformity property (11). This case is already studied in Example 11.1. The first value for $m=8$ give **minimum perfect square** magic square. In this case, we have a magic square of order 11 with sum of all entries $7744=88^2$ which is a perfect square, but it does not satisfy the uniformity property (11). The magic square shown below has a **minimum perfect square entries sum**.

Example 11.3. *For the* ***consecutive natural number*** *entries* $\{4,5,6,\ldots,123,124\}$, *a* ***block-bordered*** *magic square of order 11 is given by*

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15	23	21	19	17	116	117	119	121	123	13
124	45	94	53	50	87	55	43	92	57	4
122	58	44	90	51	46	95	56	48	88	6
120	89	54	49	91	59	42	93	52	47	8
118	63	31	98	68	24	100	61	29	102	10
14	103	62	27	96	64	32	101	66	25	114
16	26	99	67	28	104	60	30	97	65	112
18	81	76	35	86	69	37	79	74	39	110
20	40	80	72	33	82	77	38	84	70	108
22	71	36	85	73	41	78	75	34	83	106
115	105	107	109	111	12	11	9	7	5	113

The Example 11.3 has a magic sum $S_{11\times 11} = 704$, and the sum of all entries is $T_{121} := 11 \times 704 = 7744 = 88^2$. It is **minimum perfect square entries sum** magic square of order 11. Moreover, the inner magic square of order 9 is **pandiagonal** with blocks of **semi-magic** squares of order 3 with equal **semi-magic** sums. The magic sums are $S_{9\times 9} = 576$ and $Sm_{3\times 3} = 192$.

11.4 Pythagorean Triples

According to Examples 11.1, 11.2 and 11.3, we have two perfect square entries sums, i.e., $T_{121} := 14641 = 121^2$ and $T_{121} := 11 \times 704 = 7744 = 88^2$. Below are **Pythagorean triples** for the numbers 121 and 88:

$$\mathbf{121^2} + 660^2 := 671^2$$
$$\mathbf{121^2} + 7320^2 := 7321^2$$
$$\mathbf{88^2} + 66^2 := 110^2$$
$$\mathbf{88^2} + 105^2 := 137^2$$

$$\mathbf{88^2} + 165^2 := 187^2$$
$$\mathbf{88^2} + 234^2 := 250^2$$
$$\mathbf{88^2} + 480^2 := 488^2$$
$$\mathbf{88^2} + 966^2 := 970^2$$
$$\mathbf{88^2} + 1935^2 := 1937^2$$

Remark 11.1. *The above nine* ***Pythagorean triples*** *are with numbers 121 and 88. These numbers are due to Examples 11.1 and 11.3. According to Result 2, there are only four triples generating magic square with* ***perfect square entries sum****:*

$$(\mathbf{88, 105, 137}) \Rightarrow 137 - 88 = 49 = 7^2$$
$$(\mathbf{88, 234, 250}) \Rightarrow 250 - 234 = 16 = 4^2$$
$$(\mathbf{88, 480, 488}) \Rightarrow 488 - 88 = 400 = 20^2$$
$$(\mathbf{88, 1935, 1937}) \Rightarrow 1937 - 88 = 1849 = 43^2.$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(88, 105, 137)$ *generating a* ***perfect square entries sum*** *magic square of order 7 with the* ***odd number entries****,* $\{177, 179, ..., 271, 273\}$ *with magic sum,* $S_{7\times 7} := 1575$*. The entries total sum is* $T_{49} := 7 \times 1575 = 11025 = 105^2$.

2. *The triple* $(88, 234, 250)$ *generating a* ***perfect square entries sum*** *magic square of order 4 with the* ***odd number entries****,* $\{469, 471, ..., 497, 499\}$ *with magic sum,* $S_{4\times 4} := 1936$*. The entries total sum is* $T_{16} := 4 \times 1936 = 7744 = 88^2$.

3. *The triple* $(88, 480, 488)$ *generating a* ***perfect square entries sum*** *magic square of order 20 with the* ***odd number entries****,* $\{177, 179, ..., 973, 975\}$ *with magic sum,* $S_{20\times 20} := 11520$*. The entries total sum is* $T_{400} := 20 \times 11520 = 230400 = 480^2$.

4. *The triple* $(88, 1935, 1937$ *generating a* ***perfect square entries sum*** *magic square of order 43 with the* ***odd number entries***, $\{177, 179, ..., 3871, 3973\}$ *with magic sum,* $\boldsymbol{S}_{43\times 43} := 87075$*. The entries total sum is* $\boldsymbol{T}_{1849} := 43 \times 87075 = 3744225 = 1935^2$.

12 Magic Squares of Order 12

According to equation (7), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (3), we can only obtain a magic square of order 12 with this property if we use odd number entries. Taking $k = 12$ in equation (3), we get

$$\begin{aligned} &1+3+5+\ldots+(2\times 12^2-1) = 12^4 \\ \Rightarrow &1+3+5+\ldots+285+287 \qquad = 20736 = 144^2 = 12^4. \end{aligned}$$

Example 12.1. *For the* ***consecutive odd number*** *entries* $\{1, 3, 5, \ldots, 285, 287\}$*, a* ***pandiagonal*** *magic square of order 12 is given by*

		1728	1728	1728	1728	1728	1728	1728	1728	1728	1728	1728	1728
	109	215	1	251	111	213	3	249	113	211	5	247	1728
1728	35	217	143	181	33	219	141	183	31	221	139	185	1728
1728	287	37	179	73	285	39	177	75	283	41	175	77	1728
1728	145	107	253	71	147	105	255	69	149	103	257	67	1728
1728	115	209	7	245	117	207	9	243	119	205	11	241	1728
1728	29	223	137	187	27	225	135	189	25	227	133	191	1728
1728	281	43	173	79	279	45	171	81	277	47	169	83	1728
1728	151	101	259	65	153	99	261	63	155	97	263	61	1728
1728	121	203	13	239	123	201	15	237	125	199	17	235	1728
1728	23	229	131	193	21	231	129	195	19	233	127	197	1728
1728	275	49	167	85	273	51	165	87	271	53	163	89	1728
1728	157	95	265	59	159	93	267	57	161	91	269	55	1728
	1728	1728	1728	1728	1728	1728	1728	1728	1728	1728	1728	1728	1728

The magic sum is $S_{12\times 12} = 1728 = 12^3$, and the sum of all entries is $T_{144} := 20736 = 144^2 = 12^4$. Thus , the example 12.1 satisfies the uniformity property (11), i.e., $\left\langle 12, 12^2, 12^3, 12^4 \right\rangle$. Moreover, this is a **block-wise** magic square, i.e., each 4×4 block is a **pandiagonal** magic square with equal magic sums, $S_{4\times 4} = 576$.

12.1 Pythagorean Triples

According to Example 12.1, we have perfect square entries sum, i.e., $T_{144} := 20736 = 144^2 = 12^4$. Below are **Pythagorean triples** for the number 144:

$$\mathbf{144^2} + 17^2 := 145^2 \qquad \mathbf{144^2} + 192^2 := 240^2 \qquad \mathbf{144^2} + 858^2 := 870^2$$
$$\mathbf{144^2} + 42^2 := 150^2 \qquad \mathbf{144^2} + 270^2 := 306^2 \qquad \mathbf{144^2} + 1292^2 := 1300^2$$
$$\mathbf{144^2} + 60^2 := 156^2 \qquad \mathbf{144^2} + 308^2 := 340^2 \qquad \mathbf{144^2} + 1725^2 := 1731^2$$
$$\mathbf{144^2} + 108^2 := 180^2 \qquad \mathbf{144^2} + 420^2 := 444^2 \qquad \mathbf{144^2} + 2590^2 := 2594^2$$
$$\mathbf{144^2} + 130^2 := 194^2 \qquad \mathbf{144^2} + 567^2 := 585^2 \qquad \mathbf{144^2} + 5183^2 := 5185^2$$
$$\mathbf{144^2} + 165^2 := 219^2 \qquad \mathbf{144^2} + 640^2 := 656^2$$

Remark 12.1. *The above 17* ***Pythagorean triples*** *are with number 144. These numbers are due to Example 12.1. According to Result 2, there are eight triples generating magic square with* ***perfect square entries sum****:*

$$\begin{aligned}
(\mathbf{144, 108, 180}) &\Rightarrow 180 - 144 = 36 = 6^2\\
(\mathbf{144, 130, 194}) &\Rightarrow 194 - 130 = 64 = 8^2\\
(\mathbf{144, 270, 306}) &\Rightarrow 306 - 270 = 36 = 6^2\\
(\mathbf{144, 308, 340}) &\Rightarrow 340 - 144 = 196 = 14^2\\
(\mathbf{144, 567, 585}) &\Rightarrow 585 - 144 = 441 = 21^2\\
(\mathbf{144, 640, 656}) &\Rightarrow 656 - 640 = 16 = 4^2\\
(\mathbf{144, 1292, 1300}) &\Rightarrow 1300 - 144 = 1156 = 34^2\\
(\mathbf{144, 5183, 5185}) &\Rightarrow 5185 - 144 = 5041 = 71^2.
\end{aligned}$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(144, 108, 180)$ *generating a* ***perfect square entries sum*** *magic square of order 6 with the* ***odd number entries****,* $\{\{289, 291, ..., 357, 359\}\}$ *with magic sum,* $\boldsymbol{S}_{6\times 6} := 1944$*. The entries total sum is* $\boldsymbol{T}_{36} := 6 \times 1944 = 11664 = 108^2$.

2. *The triple* $(130, 144, 194)$ *generating a* ***perfect square entries sum*** *magic square of order 8 with the* ***odd number entries****,* $\{\{261, 263, ..., 385, 387\}\}$ *with magic sum,* $\boldsymbol{S}_{8\times 8} := 2592$*. The entries total sum is* $\boldsymbol{T}_{64} := 8 \times 2592 = 20736 = 144^2$.

3. *The triple* $(144,270,306)$ *generating a* ***perfect square entries sum*** *magic square of order 6 with the* ***odd number entries***, $\{\{541,543,....,539,541\}\}$ *with magic sum,* $S_{6\times6} := 3456$. *The entries total sum is* $T_{36} := 6\times3456 = 20736 = 144^2$.

4. *The triple* $(144,308,340)$ *generating a* ***perfect square entries sum*** *magic square of order 14 with the* ***odd number entries***, $\{\{289,291,....,677,679\}\}$ *with magic sum,* $S_{14\times14} := 6776$. *The entries total sum is* $T_{196} := 14\times6776 = 94864 = 308^2$.

5. *The triple* $(144,567,585)$ *generating a* ***perfect square entries sum*** *magic square of order 21 with the* ***odd number entries***, $\{\{289,291,....,1167,1169\}\}$ *with magic sum,* $S_{21\times21} := 15309$. *The entries total sum is* $T_{441} := 21\times15309 = 321489 = 567^2$.

6. *The triple* $(144,640,656)$ *generating a* ***perfect square entries sum*** *magic square of order 4 with the* ***odd number entries***, $\{\{1281,1283,....,1309,1311\}\}$ *with magic sum,* $S_{4\times4} := 5184$. *The entries total sum is* $T_{16} := 4\times5184 = 20736 = 144^2$.

7. *The triple* $(144,1292,1300)$ *generating a* ***perfect square entries sum*** *magic square of order 34 with the* ***odd number entries***, $\{\{289,291,....,2597,2599\}\}$ *with magic sum,* $S_{34\times34} := 49096$. *The entries total sum is* $T_{1156} := 34\times49096 = 1669264 = 1292^2$.

8. *The triple* $(144,5183,5185)$ *generating a* ***perfect square entries sum*** *magic square of order 71 with the* ***odd number entries***, $\{\{289,291,....,10367,10369\}\}$ *with magic sum,* $S_{71\times71} := 378359$. *The entries total sum is* $T_{5041} := 71\times378359 = 26863489 = 5183^2$.

13 Magic Squares of Order 13

According to equations (3) and (8), one can obtain **perfect square entries sum magic squares** of order 13 in two ways: first using **consecutive odd numbers**, and the second using **consecutive natural numbers**. For the **consecutive natural numbers** there are again two ways: the first one is of **uniformity**, and the second is of **minimum perfect square entries sum**.

13.1 Consecutive Odd Numbers

Taking $k = 13$ in equation (3), we get

$$1 + 3 + 5 + \ldots + (2 \times 13^2 - 1) = 13^4$$
$$\Rightarrow 1 + 3 + 5 + \ldots + 335 + 337 = 28561 = 169^2 = 13^4$$

Example 13.1. *For the* ***consecutive odd number*** *entries* $\{1, 3, 5, \ldots, 335, 337\}$, *a* ***block-bordered*** *magic square of order 13 is given by*

311	293	297	301	305	309	313	17	13	9	5	1	23
3	71	87	83	79	75	273	275	279	283	287	67	335
7	289	131	229	147	141	215	151	127	225	155	49	331
11	285	157	129	221	143	133	231	153	137	217	53	327
15	281	219	149	139	223	159	125	227	145	135	57	323
19	277	167	103	237	177	89	241	163	99	245	61	319
21	69	247	165	95	233	169	105	243	173	91	269	317
307	73	93	239	175	97	249	161	101	235	171	265	31
303	77	203	193	111	213	179	115	199	189	119	261	35
299	81	121	201	185	107	205	195	117	209	181	257	39
295	85	183	113	211	187	123	197	191	109	207	253	43
291	271	251	255	259	263	65	63	59	55	51	267	47
315	45	41	37	33	29	25	321	325	329	333	337	27

13.2 Consecutive Natural Numbers

According to equation (8),

$$G := \frac{n(n+1)}{2} - \frac{(n-169)(n-168)}{2} = 169(n-84)$$

Taking $n = 253$, we get a perfect square, i.e.,

$$G := T(253) - T(84) = \frac{253 \times 254}{2} - \frac{84 \times 85}{2} = 253 \times 127 - 42 \times 85$$
$$= 28561 = 169^2 = 13^4.$$

Simplifying, we get

$$85 + 86 + 87 + \ldots + 252 + 253 = 28561 = 169^2 = 13^4.$$

This gives a **perfect square entries sum** for the 169 **consecutive natural numbers** from to 85 to 253.

Example 13.2. *For the **consecutive natural number** entries* $\{85, 86, 87, \ldots, 252, 253\}$, *a **block-bordered** magic square of order 13 is given by*

240	231	233	235	237	239	241	93	91	89	87	85	96
86	120	128	126	124	122	221	222	224	226	228	118	252
88	229	150	199	158	155	192	160	148	197	162	109	250
90	227	163	149	195	156	151	200	161	153	193	111	248
92	225	194	159	154	196	164	147	198	157	152	113	246
94	223	168	136	203	173	129	205	166	134	207	115	244
95	119	208	167	132	201	169	137	206	171	130	219	243
238	121	131	204	172	133	209	165	135	202	170	217	100
236	123	186	181	140	191	174	142	184	179	144	215	102
234	125	145	185	177	138	187	182	143	189	175	213	104
232	127	176	141	190	178	146	183	180	139	188	211	106
230	220	210	212	214	216	117	116	114	112	110	218	108
242	107	105	103	101	99	97	245	247	249	251	253	98

In both the Examples 13.1 and 13.2, the magic sum is $S_{13\times 13} = 2197 = 13^3$, and the sum of all entries is $T_{169} := 28561 = 169^2 = 13^4$. Both the Examples 13.1 and 13.2 satisfy the uniformity property (11), i.e., $\left\langle 13, 13^2, 13^3, 13^4 \right\rangle$.

13.3 Minimum Perfect Square Entries Sum

The examples given in 13.1 and 13.2 satisfy the uniformity property (11), i.e., $\left\langle 13, 13^2, 13^3, 13^4 \right\rangle$. However, the sum of the magic square entries of 13.2 is not a **minimum perfect square entries sum**. Choosing $m = 10, 11, 12$ and 13, and $p = 13$ in equation (13), we get

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1. $L\left(10^2+\dfrac{13^2-1}{2}, 13^2\right) \to (13, 16, 184, 1300, 16900)$
2. $L\left(11^2+\dfrac{13^2-1}{2}, 13^2\right) \to (13, 37, 205, 1573, 20449)$
3. $L\left(12^2+\dfrac{13^2-1}{2}, 13^2\right) \to (13, 60, 228, 1872, 24336)$
4. $L\left(13^2+\dfrac{13^2-1}{2}, 13^2\right) \to (13, 85, 253, 2197, 28561) \Rightarrow \left\langle 13, 13^2, 13^3, 13^4 \right\rangle$

The values written above are for $m = 10, 11, 12$ and 13. The forth value for $m = 13$ satisfy the uniformity property (11). This case is already studied in Example 13.1. The first value for $m = 10$ give **minimum perfect square** magic square. In this case, we have a magic square of order 13 with the sum of all entries $16900 = 130^2$ which is a perfect square, but it does not satisfy the uniformity property (11). The magic square given below has a **minimum perfect square entries sum**.

Example 13.3. *For the* ***consecutive natural number*** *entries* $\{16, 17, 18, \ldots, 183, 184\}$, *a* ***block-bordered*** *magic square of order 13 is given by*

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171	162	164	166	168	170	172	24	22	20	18	16	27
17	51	59	57	55	53	152	153	155	157	159	49	183
19	160	81	130	89	86	123	91	79	128	93	40	181
21	158	94	80	126	87	82	131	92	84	124	42	179
23	156	125	90	85	127	95	78	129	88	83	44	177
25	154	99	67	134	104	60	136	97	65	138	46	175
26	50	139	98	63	132	100	68	137	102	61	150	174
169	52	62	135	103	64	140	96	66	133	101	148	31
167	54	117	112	71	122	105	73	115	110	75	146	33
165	56	76	116	108	69	118	113	74	120	106	144	35
163	58	107	72	121	109	77	114	111	70	119	142	37
161	151	141	143	145	147	48	47	45	43	41	149	39
173	38	36	34	32	30	28	176	178	180	182	184	29

The above example 13.3 has a magic sum $S_{13\times 13} = 1300$, and the sum of all entries is $T_{169} := 13 \times 1300 = 16900 = 130^2$. Moreover, it is **pandiagonal**.

13.4 Pythagorean Triples

According to Examples 13.1, 13.2 and 13.3, we have two perfect square entries sums, i.e., $T_{169} := 28561 = 169^2$ and $T_{169} := 13 \times 1300 = 16900 = 130^2$. Below are **Pythagorean triples** for the numbers 169 and 130:

$$\mathbf{169^2} + 1092^2 := 1105^2 \qquad \mathbf{130^2} + 312^2 := 338^2$$
$$\mathbf{169^2} + 14280^2 := 14281^2 \qquad \mathbf{130^2} + 840^2 := 850^2$$
$$\mathbf{130^2} + 144^2 := 194^2 \qquad \mathbf{130^2} + 4224^2 := 4226^2$$

Remark 13.1. *The above six* ***Pythagorean triples*** *are with numbers 169 and 130. This numbers are due to Examples 13.1 and 13.3. According to Result 2, there are only two triples generating magic square with* ***perfect square entries sum****:*

$$(\mathbf{130, 144, 194}) \Rightarrow 194 - 130 = 64 = 8^2$$
$$(\mathbf{130, 4224, 4226}) \Rightarrow 4226 - 130 = 4096 = 64^2.$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(130, 144, 194)$ *generating a* ***perfect square entries sum*** *magic square of order 8 with the* ***odd number entries****,* $\{\{261, 263, ..., 385, 387\}\}$ *with magic sum,* $S_{8\times 8} := 2592$. *The entries total sum is* $T_{64} := 8 \times 2592 = 20736 = 144^2$.

2. *The triple* $(130, 4224, 4226)$ *generating a* ***perfect square entries sum*** *magic square of order 64 with the* ***odd number entries****,* $\{\{261, 263, ..., 8449, 8451\}\}$ *with magic sum,* $S_{64\times 64} := 278784$. *The entries total sum is* $T_{4096} := 64 \times 278784 = 17842176 = 4224^2$.

14 Magic Squares of Order 14

According to equation (7), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (3), we can only obtain a magic square of order 14 with this property if we use odd number entries. Taking $k = 14$ in equation (3), we get

$$1 + 3 + 5 + \ldots + (2 \times 14^2 - 1) = 14^4$$
$$\Rightarrow 1 + 3 + 5 + \ldots + 389 + 391 = 38416 = 196^2 = 14^4.$$

Example 14.1. *For the* ***consecutive odd number*** *entries* $\{1, 3, 5, \ldots, 389, 391\}$, *a* ***block-bordered*** *magic square of order 14 is given by*

27	15	375	19	371	23	391	13	363	31	359	35	355	367
389	161	267	53	303	163	265	55	301	165	263	57	299	3
5	87	269	195	233	85	271	193	235	83	273	191	237	387
385	339	89	231	125	337	91	229	127	335	93	227	129	7
9	197	159	305	123	199	157	307	121	201	155	309	119	383
381	167	261	59	297	169	259	61	295	171	257	63	293	11
353	81	275	189	239	79	277	187	241	77	279	185	243	39
341	333	95	225	131	331	97	223	133	329	99	221	135	51
49	203	153	311	117	205	151	313	115	207	149	315	113	343
345	173	255	65	291	175	253	67	289	177	251	69	287	47
45	75	281	183	245	73	283	181	247	71	285	179	249	347
349	327	101	219	137	325	103	217	139	323	105	215	141	43
41	209	147	317	111	211	145	319	109	213	143	321	107	351
25	377	17	373	21	369	1	379	29	361	33	357	37	365

In the Example 14.1 given above, the magic sum is $S_{14\times14} = 2744 = 14^3$, and the sum of all entries is $T_{196} := 38416 = 196^2 = 14^4$. Thus , the Example 14.1 satisfies the uniformity property (11), i.e., $\left\langle 14, 14^2, 14^3, 14^4 \right\rangle$. It is **block-bordered** magic square of order 14, where inner part is **block-wise pandiagonal** magic square of order 12 with magic sum, $S_{12\times12} = 2352$. The blocks of order 4 are **pandiagonal** magic squares with equal magic sums, $S_{4\times4} = 784.$

14.1 Pythagorean Triples

According to Example 14.1, we have perfect square entries sum, i.e., $T_{196} := 38416 = 196^2$. Below are **Pythagorean triples** for the number 196:

$$\mathbf{196^2} + 147^2 := 245^2$$
$$\mathbf{196^2} + 315^2 := 371^2$$
$$\mathbf{196^2} + 672^2 := 700^2$$
$$\mathbf{196^2} + 1365^2 := 1379^2$$
$$\mathbf{196^2} + 2397^2 := 2405^2$$
$$\mathbf{196^2} + 4800^2 := 4804^2$$
$$\mathbf{196^2} + 9603^2 := 9605^2$$

Remark 14.1. *The above seven* ***Pythagorean triples*** *are with number 196. This number is due to Example 14.1. According to Result 2, there are only three triples generating magic square with* ***perfect square entries sum****:*

$$(\mathbf{196, 147, 245}) \quad \Rightarrow 245 - 196 = 49 = 7^2$$
$$(\mathbf{196, 2397, 2405}) \Rightarrow 2405 - 196 = 2209 = 47^2$$
$$(\mathbf{196, 9603, 9605}) \Rightarrow 9605 - 196 = 9409 = 97^2.$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(147, 196, 245)$ *generating a* ***perfect square entries sum*** *magic square of order 7 with the* ***odd number entries****,* $\{393, 395, ..., 487, 489\}$ *with magic sum,* $S_{7\times7} := 3087$*. The entries total sum is* $T_{49} := 7 \times 3087 = 21609 = 147^2$.

2. *The triple* $(196, 2397, 2405)$ *generating a* ***perfect square entries sum*** *magic square of order 47 with the* ***odd number entries****,* $\{393, 395, ..., 4807, 4809\}$ *with magic sum,* $S_{47\times47} := 122247$*. The entries total sum is* $T_{2209} := 47 \times 122247 = 5745609 = 2397^2$.

(iiI) *The triple* $(196, 9603, 9605)$ *generating a* ***perfect square entries sum*** *magic square of order 97 with the* ***odd number entries****,* $\{393, 395, ..., 19207, 19209\}$ *with magic sum,* $S_{97\times97} := 950697$*. The entries total sum is* $T_{9409} := 97 \times 92217609 = 9603^2$

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15 Magic Squares of Order 15

According to equations (3) and (8), one can obtain **perfect square entries sum magic squares** of order 15 in two ways: first using **consecutive odd numbers**, and the second using **consecutive natural numbers**. For the **consecutive natural numbers** there are again two ways: the first one is of **uniformity**, and the second is of **minimum perfect square entries sum**.

15.1 Consecutive Odd Numbers

Taking $k = 15$ in equation (3), we get

$$\begin{aligned} & 1+3+5+\ldots+(2\times 15^2-1) = 15^4 \\ \Rightarrow & 1+3+5+\ldots+447+449 = 50625 = 225^2 = 15^4 \end{aligned}$$

Example 15.1. *For the* ***consecutive odd number*** *entries* $\{1,3,5,\ldots,447,449\}$, *a* ***pandiagonal*** *magic square of order 15 is given by*

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		3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375
	1	173	221	355	375	5	171	219	357	373	3	169	217	359	377	3375
3375	341	385	15	151	233	339	387	13	155	231	337	389	17	153	229	3375
3375	165	211	353	371	25	163	215	351	369	27	167	213	349	367	29	3375
3375	383	11	175	225	331	381	9	177	223	335	379	7	179	227	333	3375
3375	235	345	361	23	161	237	343	365	21	159	239	347	363	19	157	3375
3375	61	143	191	325	405	65	141	189	327	403	63	139	187	329	407	3375
3375	311	415	75	121	203	309	417	73	125	201	307	419	77	123	199	3375
3375	135	181	323	401	85	133	185	321	399	87	137	183	319	397	89	3375
3375	413	71	145	195	301	411	69	147	193	305	409	67	149	197	303	3375
3375	205	315	391	83	131	207	313	395	81	129	209	317	393	79	127	3375
3375	31	113	251	295	435	35	111	249	297	433	33	109	247	299	437	3375
3375	281	445	45	91	263	279	447	43	95	261	277	449	47	93	259	3375
3375	105	241	293	431	55	103	245	291	429	57	107	243	289	427	59	3375
3375	443	41	115	255	271	441	39	117	253	275	439	37	119	257	273	3375
3375	265	285	421	53	101	267	283	425	51	99	269	287	423	49	97	3375
	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375

In this case the magic sum is $\boldsymbol{S}_{15\times 15} = 3375$, and the sum of all entries is $\boldsymbol{T}_{225} := 15 \times 3375 = 50625 = 225^2 = 15^4$. Moreover, each block of order 5 is a **pandiagonal**magic square with equal magic sums, $\boldsymbol{S}_{5\times 5} = 1125$

15.2 Consecutive Natural Numbers

According to equation (8),

$$G := \frac{n(n+1)}{2} - \frac{(n-225)(n-224)}{2} = 225(n-112)$$

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Taking $n = 337$, we get a perfect square. See below:

$$G := T(337) - T(112) = \frac{337 \times 338}{2} - \frac{112 \times 113}{2} = 50625 = 225^2 = 15^4.$$

$$\Rightarrow 113 + 114 + 115 + \ldots + 336 + 337 = 50625 = 225^2 = 15^4$$

This gives a **perfect square entries sum** for the 225 entries of **consecutive natural numbers** from to 113 to 337.

Example 15.2. *For the* ***consecutive natural number*** *entries* $\{113, 114, 115, \ldots, 336, 337\}$, *a* ***pandiagonal*** *magic square of order 15 is given by*

		3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375
	113	199	223	290	300	115	198	222	291	299	114	197	221	292	301	3375
3375	283	305	120	188	229	282	306	119	190	228	281	307	121	189	227	3375
3375	195	218	289	298	125	194	220	288	297	126	196	219	287	296	127	3375
3375	304	118	200	225	278	303	117	201	224	280	302	116	202	226	279	3375
3375	230	285	293	124	193	231	284	295	123	192	232	286	294	122	191	3375
3375	143	184	208	275	315	145	183	207	276	314	144	182	206	277	316	3375
3375	268	320	150	173	214	267	321	149	175	213	266	322	151	174	212	3375
3375	180	203	274	313	155	179	205	273	312	156	181	204	272	311	157	3375
3375	319	148	185	210	263	318	147	186	209	265	317	146	187	211	264	3375
3375	215	270	308	154	178	216	269	310	153	177	217	271	309	152	176	3375
3375	128	169	238	260	330	130	168	237	261	329	129	167	236	262	331	3375
3375	253	335	135	158	244	252	336	134	160	243	251	337	136	159	242	3375
3375	165	233	259	328	140	164	235	258	327	141	166	234	257	326	142	3375
3375	334	133	170	240	248	333	132	171	239	250	332	131	172	241	249	3375
3375	245	255	323	139	163	246	254	325	138	162	247	256	324	137	161	3375
	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375

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In the Examples 15.1 and 15.2, the magic sum is $S_{15\times 15} = 3375 = 15^3$, and the sum of all entries $T_{225} := 50625 = 225^2 = 15^4$. Moreover, the blocks of order 5 are **pandiagonal** magic squares of equal magic sums, $S_{5\times 5} = 1125$. Also, the Examples 15.1 and 15.2 satisfy the uniformity property (11), i.e., $\left\langle 15, 15^2, 15^3, 15^4 \right\rangle$.

15.3 Minimum Perfect Square Entries Sum

The examples given in 15.1 and 15.2 satisfy the property (11) of uniformity, i.e., $\left\langle 15, 15^2, 15^3, 15^4 \right\rangle$. However, the total entries of the magic square example 15.2 do not yield a **minimum perfect square entries sum**. **minimum perfect square entries sum**. Choosing $m = 11, 12, \ldots, 15$ and $p = 15$ in equation (13), we get

$$1.\quad L\left(11^2 + \frac{15^2-1}{2}, 15^2\right) \to (15, 9, 233, 1815, 27225)$$

$$2.\quad L\left(12^2 + \frac{15^2-1}{2}, 15^2\right) \to (15, 57, 281, 2535, 38025)$$

$$\ldots \quad \ldots \qquad \ldots \quad \ldots$$

$$5.\quad L\left(15^2 + \frac{15^2-1}{2}, 15^2\right) \to (15, 113, 337, 3375, 50625) \Rightarrow \left\langle 15, 15^2, 15^3, 15^4 \right\rangle$$

The values written above are for $m = 11, 12, \ldots, 15$. The fifth value for $m = 15$ satisfy the uniformity property (11). This case is already studied in Example 15.1. The first value for $m = 11$ give **minimum perfect square** magic square. In this case, we have a magic square of order 15 with **perfect square entries sum** as $27225 = 165^2$, but it does not satisfy the uniformity property (11). The magic square given below is with **minimum perfect square entries sum** for the entries from 9 to 233.

Example 15.3. *For the **consecutive natural number** entries* $\{9, 10, 11, \ldots, 232, 233\}$*, a **pandiagonal** magic square of order 15 is given by*

		1815	1815	1815	1815	1815	1815	1815	1815	1815	1815	1815	1815	1815	1815	1815
	9	95	119	186	196	11	94	118	187	195	10	93	117	188	197	1815
1815	179	201	16	84	125	178	202	15	86	124	177	203	17	85	123	1815
1815	91	114	185	194	21	90	116	184	193	22	92	115	183	192	23	1815
1815	200	14	96	121	174	199	13	97	120	176	198	12	98	122	175	1815
1815	126	181	189	20	89	127	180	191	19	88	128	182	190	18	87	1815
1815	39	80	104	171	211	41	79	103	172	210	40	78	102	173	212	1815
1815	164	216	46	69	110	163	217	45	71	109	162	218	47	70	108	1815
1815	76	99	170	209	51	75	101	169	208	52	77	100	168	207	53	1815
1815	215	44	81	106	159	214	43	82	105	161	213	42	83	107	160	1815
1815	111	166	204	50	74	112	165	206	49	73	113	167	205	48	72	1815
1815	24	65	134	156	226	26	64	133	157	225	25	63	132	158	227	1815
1815	149	231	31	54	140	148	232	30	56	139	147	233	32	55	138	1815
1815	61	129	155	224	36	60	131	154	223	37	62	130	153	222	38	1815
1815	230	29	66	136	144	229	28	67	135	146	228	27	68	137	145	1815
1815	141	151	219	35	59	142	150	221	34	58	143	152	220	33	57	1815
	1815	1815	1815	1815	1815	1815	1815	1815	1815	1815	1815	1815	1815	1815	1815	1815

The above Example 15.3 is with magic sum $S_{15\times 15} = 1815$, and the sum of all entries is $T_{225} := 15 \times 1815 = 27225 = 165^2$. This magic square of order 15 is a **minimum perfect square entries sum**. Moreover, the blocks of order 5 are **pandiagonal** magic squares of equal magic sums, $S_{5\times 5} = 605$.

15.4 Pythagorean Triples

According to Examples 15.1, 15.2 and 15.3, we have two perfect square entries sums, i.e., $T_{225} := 50625 = 225^2$ and $T_{225} := 15 \times 1815 = 27225 = 165^2$. Below are **Pythagorean triples** for the numbers 225 and 165:

$$
\begin{array}{lll}
\mathbf{225^2} + 120^2 := 255^2 & \mathbf{225^2} + 2808^2 := 2817^2 & \mathbf{165^2} + 280^2 := 325^2 \\
\mathbf{225^2} + 140^2 := 265^2 & \mathbf{225^2} + 5060^2 := 5065^2 & \mathbf{165^2} + 396^2 := 429^2 \\
\mathbf{225^2} + 272^2 := 353^2 & \mathbf{225^2} + 8436^2 := 8439^2 & \mathbf{165^2} + 532^2 := 557^2 \\
\mathbf{225^2} + 300^2 := 375^2 & \mathbf{225^2} + 25312^2 := 25313^2 & \mathbf{165^2} + 900^2 := 915^2 \\
\mathbf{225^2} + 540^2 := 585^2 & \mathbf{165^2} + 52^2 := 173^2 & \mathbf{165^2} + 1232^2 := 1243^2 \\
\mathbf{225^2} + 924^2 := 951^2 & \mathbf{165^2} + 88^2 := 187^2 & \mathbf{165^2} + 1508^2 := 1517^2 \\
\mathbf{225^2} + 1000^2 := 1025^2 & \mathbf{165^2} + 144^2 := 219^2 & \mathbf{165^2} + 2720^2 := 2725^2 \\
\mathbf{225^2} + 1680^2 := 1695^2 & \mathbf{165^2} + 220^2 := 275^2 & \mathbf{165^2} + 4536^2 := 4539^2 \\
 & & \mathbf{165^2} + 13612^2 := 13613^2
\end{array}
$$

Remark 15.1. *The above 25* ***Pythagorean triples*** *are with numbers 225 and 165. These numbers are due to Examples 15.1 and 15.3. According to Result 2, there are only five triples generating magic square with* ***perfect square entries sum****:*

$$
\begin{aligned}
(\mathbf{225, 272, 353}) &\Rightarrow 353 - 272 = 81 = 9^2 \\
(\mathbf{225, 1000, 1025}) &\Rightarrow 1025 - 1000 = 25 = 5^2 \\
(\mathbf{165, 52, 173}) &\Rightarrow 173 - 52 = 121 = 11^2 \\
(\mathbf{165, 532, 557}) &\Rightarrow 557 - 532 = 25 = 5^2 \\
(\mathbf{165, 1508, 1517}) &\Rightarrow 1517 - 1508 = 9 = 3^2.
\end{aligned}
$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(225, 272, 353)$ *generating a* ***perfect square entries sum*** *magic square of order 9 with the* ***odd number entries****,* $\{545, 547, ..., 703, 705\}$ *with magic sum,* $\boldsymbol{S}_{9\times 9} := 5625$*. The entries total sum is* $\boldsymbol{T}_{81} := 9 \times 5625 = 50625 = 225^2$.

2. *The triple* $(225, 1000, 1025)$ *generating a* ***perfect square entries sum*** *magic square of order 5 with the* ***odd***

number entries, $\{2001, 2003, ..., 2047, 2049\}$ *with magic sum,* $\boldsymbol{S}_{5\times5} := 10125$*. The entries total sum is* $\boldsymbol{T}_{25} := 5 \times 10125 = 50625 = 225^2$.

3. *The triple* $(165, 52, 173)$ *generating a* ***perfect square entries sum*** *magic square of order 11 with the* ***odd number entries***, $\{105, 107, ..., 343, 345\}$ *with magic sum,* $\boldsymbol{S}_{11\times11} := 2475$*. The entries total sum is* $\boldsymbol{T}_{121} := 11 \times 2475 = 27225 = 165^2$.

4. *The triple* $(165, 532, 557)$ *generating a* ***perfect square entries sum*** *magic square of order 5 with the* ***odd number entries***, $\{1065, 1067, ..., 1111, 1113\}$ *with magic sum,* $\boldsymbol{S}_{5\times5} := 5445$*. The entries total sum is* $\boldsymbol{T}_{25} := 5 \times 5445 = 27225 = 165^2$.

5. *The triple* $(165, 1508, 1517)$ *generating a* ***perfect square entries sum*** *magic square of order 3 with the* ***odd number entries***, $\{3017, 3019, ..., 3031, 3033\}$ *with magic sum,* $\boldsymbol{S}_{3\times3} := 9075$*. The entries total sum is* $\boldsymbol{T}_{9} := 3 \times 9075 = 27225 = 165^2$.

16 Magic Square of Order 16

According to equation (7), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (3), we can only obtain a magic square of order 16 with this property if we use odd number entries. Taking $k = 16$ in equation (3), we get

$$\begin{aligned} & 1+3+5+\ldots+(2\times 16^2-1) = 16^4 \\ \Rightarrow & 1+3+5+\ldots+509+511 \qquad = 65536 = 256^2 = 16^4. \end{aligned}$$

Example 16.1. *A* ***block-wise pandiagonal*** *magic square of order 16 for* ***consecutive odd numbers*** *entries* $1, 3, 5, ..., 509, 511$ *is given by*

		4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096
	213	315	37	459	183	345	71	425	241	287	1	495	147	381	99	397	4096
4096	43	453	219	309	73	423	185	343	15	481	255	273	109	387	157	371	4096
4096	475	53	299	197	441	87	329	167	511	17	271	225	413	115	365	131	4096
4096	293	203	469	59	327	169	439	89	257	239	497	31	355	141	403	125	4096
4096	243	285	3	493	145	383	97	399	215	313	39	457	181	347	69	427	4096
4096	13	483	253	275	111	385	159	369	41	455	217	311	75	421	187	341	4096
4096	509	19	269	227	415	113	367	129	473	55	297	199	443	85	331	165	4096
4096	259	237	499	29	353	143	401	127	295	201	471	57	325	171	437	91	4096
4096	151	377	103	393	245	283	5	491	179	349	67	429	209	319	33	463	4096
4096	105	391	153	375	11	485	251	277	77	419	189	339	47	449	223	305	4096
4096	409	119	361	135	507	21	267	229	445	83	333	163	479	49	303	193	4096
4096	359	137	407	121	261	235	501	27	323	173	435	93	289	207	465	63	4096
4096	177	351	65	431	211	317	35	461	149	379	101	395	247	281	7	489	4096
4096	79	417	191	337	45	451	221	307	107	389	155	373	9	487	249	279	4096
4096	447	81	335	161	477	51	301	195	411	117	363	133	505	23	265	231	4096
4096	321	175	433	95	291	205	467	61	357	139	405	123	263	233	503	25	4096
	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096

The magic sum is $S_{16\times16} = 4096 = 16^3$, and the sum of all entries is $\boldsymbol{T}_{256} := 65536 = 256^2 = 16^4$. Thus , the example 16.1 satisfies the uniformity property (11), i.e., $\left\langle 16, 16^2, 16^3, 16^4 \right\rangle$. Moreover, this is a **block-wise** magic square, i.e., each 4×4 block is a **pandiagonal** magic square with equal magic sums, $S_{4\times4} = 1024$.

16.1 Pythagorean Triples

According to Example 16.1, we have perfect square entries sum, i.e., $T_{256} := 65536 = 256^2$. Below are **Pythagorean triples** for the number 256:

$$\begin{aligned}
\mathbf{256^2} + 192^2 &:= 320^2 \\
\mathbf{256^2} + 480^2 &:= 544^2 \\
\mathbf{256^2} + 1008^2 &:= 1040^2 \\
\mathbf{256^2} + 2040^2 &:= 2056^2
\end{aligned}$$

$$\begin{aligned}
\mathbf{256^2} + 4092^2 &:= 4100^2 \\
\mathbf{256^2} + 8190^2 &:= 8194^2 \\
\mathbf{256^2} + 16383^2 &:= 16385^2
\end{aligned}$$

Remark 16.1. *The above 7* ***Pythagorean triples*** *are with number 256. This number is due to Example 16.1. According to Result 2, there are six triples generating magic square with* ***perfect square entries sum****:*

$$\begin{aligned}
(\mathbf{256, 192, 320}) &\Rightarrow 320 - 256 = 64 = 8^2 \\
(\mathbf{256, 480, 544}) &\Rightarrow 544 - 480 = 64 = 8^2 \\
(\mathbf{256, 1008, 1040}) &\Rightarrow 1040 - 256 = 784 = 28^2 \\
(\mathbf{256, 2040, 2056}) &\Rightarrow 2056 - 2040 = 16 = 4^2 \\
(\mathbf{256, 4092, 4100}) &\Rightarrow 4100 - 256 = 3844 = 62^2 \\
(\mathbf{256, 16383, 16385}) &\Rightarrow 16385 - 256 = 16129 = 127^2.
\end{aligned}$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(256, 192, 320)$ *generating a* ***perfect square entries sum*** *magic square of order 8 with the* ***odd number entries****,* $\{513, 515, \ldots, 637, 639\}$ *with magic sum,* $S_{8\times 8} := 4608$*. The entries total sum is* $T_{64} := 8 \times 4608 = 36864 = 192^2$.

2. *The triple* $(256, 480, 544)$ *generating a* ***perfect square entries sum*** *magic square of order 8 with the* ***odd number entries****,* $\{961, 963, \ldots, 1085, 1087\}$ *with magic sum,* $S_{8\times 8} := 8192$*. The entries total sum is* $T_{64} := 8 \times 8192 = 65536 = 256^2$.

3. *The triple* $(256, 1008, 1040)$ *generating a* ***perfect square entries sum*** *magic square of order 28 with the* ***odd number entries***, $\{513, 515, \ldots, 2077, 2079\}$ *with magic sum,* $S_{28\times 28} := 36288$*. The entries total sum is* $T_{784} := 28 \times 36288 = 1016064 = 1008^2$.

4. *The triple* $(256, 2040, 2056)$ *generating a* ***perfect square entries sum*** *magic square of order 4 with the* ***odd number entries***, $\{4081, 4083, \ldots, 4109, 4111\}$ *with magic sum,* $S_{4\times 4} := 16384$*. The entries total sum is* $T_{16} := 4 \times 16384 = 65536 = 256^2$.

5. *The triple* $(256, 4092, 4100)$ *generating a* ***perfect square entries sum*** *magic square of order 62 with the* ***odd number entries***, $\{513, 515, \ldots, 8197, 8199\}$ *with magic sum,* $S_{62\times 62} := 270072$*. The entries total sum is* $T_{3844} := 62 \times 270072 = 16744464 = 4092^2$.

6. *The triple* $(256, 16383, 16385)$ *generating a* ***perfect square entries sum*** *magic square of order 127 with the* ***odd number entries***, $\{513, 515, \ldots, 32767, 32769\}$ *with magic sum,* $S_{127\times 127} := 2113407$*. The entries total sum is* $T_{16129} := 127 \times 2113407 = 268402689 = 16383^2$.

17 Magic Squares of Order 17

According to equations (3) and (8), one can obtain **perfect square entries sum magic squares** of order 17 in two ways: first using **consecutive odd numbers**, and the second using **consecutive natural numbers**. For the **consecutive natural numbers** there are again two ways: the first one is of **uniformity**, and the second is of **minimum perfect square entries sum**.

17.1 Consecutive Odd Numbers

Taking $k = 11$ in equation (3), we get

$$\begin{aligned} & 1 + 3 + 5 + \ldots + (2 \times 17^2 - 1) = 17^4 \\ \Rightarrow & 1 + 3 + 5 + \ldots + 575 + 577 \qquad = 83521 = 289^2 = 17^4 \end{aligned}$$

Example 17.1. *The* ***block-bordered*** *magic square of order 17 with inner part as* ***block-wise*** *magic square of order 15 for the* ***consecutive odd number*** *entries* $\{1, 3, 5, ..., 575, 577\}$ *is given by*

31	575	571	567	563	559	555	551	549	39	43	47	51	55	59	63	35
1	435	193	239	437	211	219	439	213	215	441	191	235	443	187	237	577
5	223	449	195	241	429	197	243	425	199	221	445	201	217	447	203	573
9	209	225	433	189	227	451	185	229	453	205	231	431	207	233	427	569
13	135	463	269	137	481	249	139	483	245	141	461	265	143	457	267	565
17	253	149	465	271	129	467	273	125	469	251	145	471	247	147	473	561
21	479	255	133	459	257	151	455	259	153	475	261	131	477	263	127	557
25	75	493	299	77	511	279	79	513	275	81	491	295	83	487	297	553
545	283	89	495	301	69	497	303	65	499	281	85	501	277	87	503	33
541	509	285	73	489	287	91	485	289	93	505	291	71	507	293	67	37
537	375	163	329	377	181	309	379	183	305	381	161	325	383	157	327	41
533	313	389	165	331	369	167	333	365	169	311	385	171	307	387	173	45
529	179	315	373	159	317	391	155	319	393	175	321	371	177	323	367	49
525	405	103	359	407	121	339	409	123	335	411	101	355	413	97	357	53
521	343	419	105	361	399	107	363	395	109	341	415	111	337	417	113	57
517	119	345	403	99	347	421	95	349	423	115	351	401	117	353	397	61
543	3	7	11	15	19	23	27	29	539	535	531	527	523	519	515	547

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17.2 Consecutive Natural Numbers

According to equation (8),

$$G := \frac{n(n+1)}{2} - \frac{(n-289)(n-288)}{2} = 289(n-144)$$

Taking $n = 433$, we get a perfect square, i.e.,

$$G := T(433) - T(144) = \frac{433 \times 434}{2} - \frac{144 \times 145}{2} = 433 \times 217 - 72 \times 145$$
$$= 83521 = 289^2 = 17^4.$$

Simplifying, we get

$$145 + 146 + 147 + \ldots + 432 + 433 = 83521 = 289^2 = 17^4.$$

This gives a perfect square entries sum for the 289 consecutive natural numbers from to 145 to 433.

Example 17.2. *The **block-bordered** magic square of order 17 with inner part as **block-wise** magic square of order 15 for the **consecutive natural number** entries* $\{145, 146, 147, ..., 432, 433\}$ *is given by*

160	432	430	428	426	424	422	420	419	164	166	168	170	172	174	176	162
145	362	241	264	363	250	254	364	251	252	365	240	262	366	238	263	433
147	256	369	242	265	359	243	266	357	244	255	367	245	253	368	246	431
149	249	257	361	239	258	370	237	259	371	247	260	360	248	261	358	429
151	212	376	279	213	385	269	214	386	267	215	375	277	216	373	278	427
153	271	219	377	280	209	378	281	207	379	270	217	380	268	218	381	425
155	384	272	211	374	273	220	372	274	221	382	275	210	383	276	208	423
157	182	391	294	183	400	284	184	401	282	185	390	292	186	388	293	421
417	286	189	392	295	179	393	296	177	394	285	187	395	283	188	396	161
415	399	287	181	389	288	190	387	289	191	397	290	180	398	291	178	163
413	332	226	309	333	235	299	334	236	297	335	225	307	336	223	308	165
411	301	339	227	310	329	228	311	327	229	300	337	230	298	338	231	167
409	234	302	331	224	303	340	222	304	341	232	305	330	233	306	328	169
407	347	196	324	348	205	314	349	206	312	350	195	322	351	193	323	171
405	316	354	197	325	344	198	326	342	199	315	352	200	313	353	201	173
403	204	317	346	194	318	355	192	319	356	202	320	345	203	321	343	175
416	146	148	150	152	154	156	158	159	414	412	410	408	406	404	402	418

In both the Examples 17.1 and 17.2 given above the magic sum is $S_{17\times17} = 4913 = 17^3$, and the sum of all numbers is $T_{289} := 83521 = 289^2 = 17^4$. Both the Examples 17.1 and 17.2 satisfy the uniformity property (11), i.e., $\left\langle 17, 17^2, 17^3, 17^4 \right\rangle$. Moreover, the inner magic square of order 15 is **block-wise pan-diagonal** with magic sum $S_{15\times15} = 4335$. The blocks of order 3 are **semi-magic** squares with equal **semi-magic** sums, $S_{3\times3} = 867$.

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17.3 Minimum Perfect Square Entries Sum

The examples given in 17.1 and 17.2 satisfy the uniformity property (11), i.e., $\left\langle 17, 17^2, 17^3, 17^4 \right\rangle$. However, the sum of the magic square entries of Example 17.2 is not a **minimum perfect square entries sum**. Choosing $m = 13, 14, \ldots, 17$ and $p = 17$ in equation (13), we get

1. $L\left(13^2 + \frac{17^2 - 1}{2}, 17^2\right) \to (17, 25, 313, 2873, 48841)$
2. $L\left(14^2 + \frac{17^2 - 1}{2}, 17^2\right) \to (17, 52, 340, 3332, 56644)$

$\ldots \qquad \ldots \qquad \ldots$

5. $L\left(17^2 + \frac{17^2 - 1}{2}, 17^2\right) \to (145,\ 433,\ 289, 83521) \Rightarrow \left\langle 17, 17^2, 17^3, 17^4 \right\rangle$

The values written above are for $m = 13, 14, \ldots, 17$. The fifth value for $m = 17$ satisfy the uniformity property (11). This case is already studied in Example 17.1. The first value for $m = 13$ give **minimum perfect square** magic square. In this case, we have magic square of order 17 with sum of all entries $48841 = 221^2$ which is a perfect square, but it does not satisfy the uniformity property (11). The magic square given below has a **minimum perfect square entries sum**.

Example 17.3. *The **block-bordered** magic square of order 17 with inner part as **block-wise** magic square of order 15 for the **consecutive natural number** entries* $\{25, 26, 27, ..., 312, 313\}$ *is given by*

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40	312	310	308	306	304	302	300	299	44	46	48	50	52	54	56	42
25	242	121	144	243	130	134	244	131	132	245	120	142	246	118	143	313
27	136	249	122	145	239	123	146	237	124	135	247	125	133	248	126	311
29	129	137	241	119	138	250	117	139	251	127	140	240	128	141	238	309
31	92	256	159	93	265	149	94	266	147	95	255	157	96	253	158	307
33	151	99	257	160	89	258	161	87	259	150	97	260	148	98	261	305
35	264	152	91	254	153	100	252	154	101	262	155	90	263	156	88	303
37	62	271	174	63	280	164	64	281	162	65	270	172	66	268	173	301
297	166	69	272	175	59	273	176	57	274	165	67	275	163	68	276	41
295	279	167	61	269	168	70	267	169	71	277	170	60	278	171	58	43
293	212	106	189	213	115	179	214	116	177	215	105	187	216	103	188	45
291	181	219	107	190	209	108	191	207	109	180	217	110	178	218	111	47
289	114	182	211	104	183	220	102	184	221	112	185	210	113	186	208	49
287	227	76	204	228	85	194	229	86	192	230	75	202	231	73	203	51
285	196	234	77	205	224	78	206	222	79	195	232	80	193	233	81	53
283	84	197	226	74	198	235	72	199	236	82	200	225	83	201	223	55
296	26	28	30	32	34	36	38	39	294	292	290	288	286	284	282	298

The above example 17.3 has a magic sum $S_{17\times 17} = 2873$, and the sum of all entries is $T_{289} := 17 \times 2873 = 48841 = 221^2$. It is **minimum perfect square entries sum** magic square of order 17. Moreover, the inner magic square of order 15 is **block-wise pandiagonal** with magic sum $S_{15\times 15} = 2535$. The blocks of order 3 are **semi-magic** squares with equal **semi-magic** sums, $S_{3\times 3} = 507$.

17.4 Pythagorean Triples

According to Examples 17.1, 17.2 and 17.3, we have two perfect square entries sums, i.e., $\boldsymbol{T}_{289} := 83521 = 289^2$ and $\boldsymbol{T}_{289} := 17 \times 2873 = 48841 = 221^2$. Below are **Pythagorean triples** for the numbers 289 and 221:

$$\begin{aligned} \mathbf{289^2} + 2448^2 &:= 2465^2 \\ \mathbf{289^2} + 41760^2 &:= 41761^2 \\ \mathbf{221^2} + 60^2 &:= 229^2 \end{aligned} \qquad \begin{aligned} \mathbf{221^2} + 1428^2 &:= 1445^2 \\ \mathbf{221^2} + 1872^2 &:= 1885^2 \\ \mathbf{221^2} + 24420^2 &:= 24421^2 \end{aligned}$$

Remark 17.1. *The above 6* ***Pythagorean triples*** *are with numbers 289 and 221. These numbers are due to Examples 17.1 and 17.3. According to Result 2, there is only one triple generating magic square with* ***perfect square entries sum****:*

$$(221, 60, 229) \quad \Rightarrow \quad 229 - 60 = 169 = 13^2.$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(221, 60, 229)$ *generating a* ***perfect square entries sum*** *magic square of order 13 with the* ***odd number entries****,* $\{121, 123, \ldots, 455, 457\}$ *with magic sum,* $\boldsymbol{S}_{13\times 13} := 3757$*. The entries total sum is* $\boldsymbol{T}_{169} := 13 \times 3757 = 48841 = 221^2$.

18 Magic Squares of Order 18

According to equation (7), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (3), we can only obtain a magic square of order 18 with this property if we use odd number entries. Taking $k = 18$ in equation (3), we get

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$$1+3+5+\ldots+(2\times 18^2-1) = 18^4$$

$$\Rightarrow 1+3+5+\ldots+645+647 = 104976 = 324^2 = 18^4.$$

Example 18.1. *A* ***block-wise*** *magic square of order 18 for* ***consecutive odd numbers*** *entries* 1, 3, 5, ..., 645, 647 *is given by*

																		5832
1	613	611	577	35	107	3	615	609	579	33	105	5	617	607	581	31	103	5832
539	143	503	145	181	433	537	141	501	147	183	435	535	139	499	149	185	437	5832
431	397	253	287	359	217	429	399	255	285	357	219	427	401	257	283	355	221	5832
323	251	361	395	289	325	321	249	363	393	291	327	319	247	365	391	293	329	5832
109	467	179	469	505	215	111	465	177	471	507	213	113	463	175	473	509	211	5832
541	73	37	71	575	647	543	75	39	69	573	645	545	77	41	67	571	643	5832
7	619	605	583	29	101	9	621	603	585	27	99	11	623	601	587	25	97	5832
533	137	497	151	187	439	531	135	495	153	189	441	529	133	493	155	191	443	5832
425	403	259	281	353	223	423	405	261	279	351	225	421	407	263	277	349	227	5832
317	245	367	389	295	331	315	243	369	387	297	333	313	241	371	385	299	335	5832
115	461	173	475	511	209	117	459	171	477	513	207	119	457	169	479	515	205	5832
547	79	43	65	569	641	549	81	45	63	567	639	551	83	47	61	565	637	5832
13	625	599	589	23	95	15	627	597	591	21	93	17	629	595	593	19	91	5832
527	131	491	157	193	445	525	129	489	159	195	447	523	127	487	161	197	449	5832
419	409	265	275	347	229	417	411	267	273	345	231	415	413	269	271	343	233	5832
311	239	373	383	301	337	309	237	375	381	303	339	307	235	377	379	305	341	5832
121	455	167	481	517	203	123	453	165	483	519	201	125	451	163	485	521	199	5832
553	85	49	59	563	635	555	87	51	57	561	633	557	89	53	55	559	631	5832
5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832

In this case also the magic sum is $S_{18\times 18} = 5832 = 18^3$, and the sum of all entries is $T_{324} := 18 \times 5832 = 104976 = 324^2 = 18^4$.. It satisfies the uniformity property (11), i.e., $\left\langle 18, 18^2, 18^3, 18^4 \right\rangle$. Blocks of order 6 are magic squares with equal magic sums, $S_{6\times 6} = 1944$.

18.1 Pythagorean Triples

According to Example 18.1, we have perfect square entries sum, i.e., $T_{324} := 18 \times 5832 = 104976 = 324^2$. Below are **Pythagorean triples** for the number 324:

$$324^2 + 135^2 := 351^2 \qquad 324^2 + 945^2 := 999^2 \qquad 324^2 + 4368^2 := 4380^2$$
$$324^2 + 243^2 := 405^2 \qquad 324^2 + 1440^2 := 1476^2 \qquad 324^2 + 6557^2 := 6565^2$$
$$324^2 + 432^2 := 540^2 \qquad 324^2 + 2175^2 := 2199^2 \qquad 324^2 + 8745^2 := 8751^2$$
$$324^2 + 693^2 := 765^2 \qquad 324^2 + 2907^2 := 2925^2 \qquad 324^2 + 13120^2 := 13124^2$$
$$324^2 + 26243^2 := 26245^2$$

Remark 18.1. *The above 13* ***Pythagorean triples*** *are with number 324. This number is due to Example 18.1. According to Result 2, there are 6 triples generating magic square with* ***perfect square entries sum:***

$$(324, 243, 405) \Rightarrow 405 - 324 = 81 = 9^2$$
$$(324, 693, 765) \Rightarrow 765 - 324 = 441 = 21^2$$
$$(324, 1440, 1476) \Rightarrow 1476 - 1440 = 36 = 6^2$$
$$(324, 2907, 2925) \Rightarrow 2925 - 324 = 2601 = 51^2$$
$$(324, 6557, 6565) \Rightarrow 6565 - 324 = 6241 = 79^2$$
$$(324, 26243, 26245) \Rightarrow 26245 - 324 = 25921 = 161^2.$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(324, 243, 405)$ *generating a* ***perfect square entries sum*** *magic square of order 9 with the* ***odd number entries***, $\{649, 651, \ldots, 807, 809\}$ *with magic sum,* $S_{9\times 9} := 6561$. *The entries total sum is* $T_{81} :=$

$9 \times 6561 = 59049 = 243^2$.

2. *The triple* $(324, 693, 765)$ *generating a* ***perfect square entries sum*** *magic square of order 21 with the* ***odd number entries***, $\{649, 651, \ldots, 1527, 1529\}$ *with magic sum,* $\boldsymbol{S}_{21\times 21} := 22869$. *The entries total sum is* $\boldsymbol{T}_{441} := 21 \times 22869 = 480249 = 693^2$.

3. *The triple* $(324, 1440, 1476)$ *generating a* ***perfect square entries sum*** *magic square of order 6 with the* ***odd number entries***, $\{2881, 2883, \ldots, 2949, 2951\}$ *with magic sum,* $\boldsymbol{S}_{6\times 6} := 17496$. *The entries total sum is* $\boldsymbol{T}_{36} := 6 \times 17496 = 104976 = 324^2$.

4. *The triple* $(324, 2907, 2925)$ *generating a* ***perfect square entries sum*** *magic square of order 51 with the* ***odd number entries***, $\{649, 651, \ldots, 5847, 5849\}$ *with magic sum,* $\boldsymbol{S}_{51\times 51} := 165699$. *The entries total sum is* $\boldsymbol{T}_{2601} := 51 \times 165699 = 8450649 = 2907^2$.

5. *The triple* $(324, 6557, 6565)$ *generating a* ***perfect square entries sum*** *magic square of order 79 with the* ***odd number entries***, $\{649, 651, \ldots, 13127, 13129\}$ *with magic sum,* $\boldsymbol{S}_{79\times 79} := 544231$. *The entries total sum is* $\boldsymbol{T}_{6241} := 79 \times 544231 = 42994249 = 6557^2$.

6. *The triple* $(324, 26243, 26245)$ *generating a* ***perfect square entries sum*** *magic square of order 161 with the* ***odd number entries***, $\{649, 651, \ldots, 52487, 52489\}$ *with magic sum,* $\boldsymbol{S}_{161\times 161} := 4277609$. *The entries total sum is* $\boldsymbol{T}_{25921} := 161 \times 4277609 = 688695049 = 26243^2$.

19 Magic Squares of Order 19

According to equations (3) and (8), one can obtain **perfect square entries sum magic squares** of order 19 in two ways: first using **consecutive odd numbers**, and the second using **consecutive natural numbers**. For the **consecutive natural numbers** there are again two ways: the first one is of **uniformity**, and the second is of **minimum perfect square entries sum**.

19.1 Consecutive Odd Numbers

Taking $k = 19$ in equation (3), we get

$$1+3+5+\ldots+(2\times 19^2-1)=19^4$$
$$\Rightarrow 1+3+5+\ldots+719+721 \qquad =130321=361^2=19^4$$

Example 19.1. *For the* ***consecutive odd number*** *entries* $\{1,3,5,\ldots,719,731\}$ *a* ***block-bordered*** *magic square of order 19 is given by*

39	71	67	63	59	55	51	47	43	689	691	695	699	703	707	711	715	719	35
721	103	647	643	639	635	631	627	623	621	111	115	119	123	127	131	135	107	1
717	73	137	571	195	373	529	197	511	285	343	469	227	451	315	403	409	649	5
713	77	375	553	139	557	181	345	493	199	497	271	405	433	229	437	301	645	9
709	81	559	167	361	555	163	499	257	331	495	223	439	287	391	435	253	641	13
705	85	541	165	583	169	347	481	225	523	259	317	421	255	463	289	377	637	17
701	89	193	349	527	151	585	283	319	467	211	525	313	379	407	241	465	633	21
697	93	141	569	191	369	535	201	509	281	339	475	231	449	311	399	415	629	25
693	97	371	549	145	561	179	341	489	205	501	269	401	429	235	441	299	625	29
37	617	565	171	359	551	159	505	261	329	491	219	445	291	389	431	249	105	685
41	613	539	161	579	175	351	479	221	519	265	321	419	251	459	295	381	109	681
45	609	189	355	531	149	581	279	325	471	209	521	309	385	411	239	461	113	677
49	605	143	573	187	365	537	203	513	277	335	477	233	453	307	395	417	117	673
53	601	367	545	147	563	183	337	485	207	503	273	397	425	237	443	303	121	669
57	597	567	173	363	547	155	507	263	333	487	215	447	293	393	427	245	125	665
61	593	543	157	575	177	353	483	217	515	267	323	423	247	455	297	383	129	661
65	589	185	357	533	153	577	275	327	473	213	517	305	387	413	243	457	133	657
69	615	75	79	83	87	91	95	99	101	611	607	603	599	595	591	587	619	653
687	651	655	659	663	667	671	675	679	33	31	27	23	19	15	11	7	3	683

19.2 Consecutive Natural Numbers

According to equation (8),

$$G := \frac{n(n+1)}{2} - \frac{(n-361)(n-360)}{2} = 361(n-180)$$

Taking $n = 541$, we get a perfect square, i.e.,

$$\begin{aligned} G := T(541) - T(180) = \frac{541 \times 542}{2} - \frac{180 \times 181}{2} &= 541 \times 271 - 90 \times 181 \\ &= 130321 = 361^2 = 19^4. \end{aligned}$$

Simplifying, we get

$$181 + 182 + 183 + \ldots + 540 + 541 = 130321 = 361^2 = 19^4.$$

This gives a perfect square entries sum for the 361 **consecutive natural numbers** from to 181 to 541.

Example 19.2. *For the* ***consecutive natural number*** *entries* $\{181, 182, 183, \ldots, 540, 541\}$ *a* ***block-bordered*** *magic square of order 19 is given by*

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200	216	214	212	210	208	206	204	202	525	526	528	530	532	534	536	538	540	198
541	232	504	502	500	498	496	494	492	491	236	238	240	242	244	246	248	234	181
539	217	249	466	278	367	445	279	436	323	352	415	294	406	338	382	385	505	183
537	219	368	457	250	459	271	353	427	280	429	316	383	397	295	399	331	503	185
535	221	460	264	361	458	262	430	309	346	428	292	400	324	376	398	307	501	187
533	223	451	263	472	265	354	421	293	442	310	339	391	308	412	325	369	499	189
531	225	277	355	444	256	473	322	340	414	286	443	337	370	384	301	413	497	191
529	227	251	465	276	365	448	281	435	321	350	418	296	405	336	380	388	495	193
527	229	366	455	253	461	270	351	425	283	431	315	381	395	298	401	330	493	195
199	489	463	266	360	456	260	433	311	345	426	290	403	326	375	396	305	233	523
201	487	450	261	470	268	356	420	291	440	313	341	390	306	410	328	371	235	521
203	485	275	358	446	255	471	320	343	416	285	441	335	373	386	300	411	237	519
205	483	252	467	274	363	449	282	437	319	348	419	297	407	334	378	389	239	517
207	481	364	453	254	462	272	349	423	284	432	317	379	393	299	402	332	241	515
209	479	464	267	362	454	258	434	312	347	424	288	404	327	377	394	303	243	513
211	477	452	259	468	269	357	422	289	438	314	342	392	304	408	329	372	245	511
213	475	273	359	447	257	469	318	344	417	287	439	333	374	387	302	409	247	509
215	488	218	220	222	224	226	228	230	231	486	484	482	480	478	476	474	490	507
524	506	508	510	512	514	516	518	520	197	196	194	192	190	188	186	184	182	522

In both the Examples 19.1 and 19.2, the magic sum is $S_{19\times 19} = 6859 = 19^3$, and the sum of all entries is $T_{361} := 130321 = 361^2 = 19^4$. Both the Examples 19.1 and 19.2 satisfy the uniformity property (11), i.e., $\left\langle 19, 19^2, 19^3, 19^4 \right\rangle$.

The Examples 19.1 and 19.2 are **block-bordered** magic squares of equal sums, where inner **block-wise** magic square of order 15 is **pandiagonal**. Moreover, blocks of order 5 are equal sum magic squares, with magic sums: $S_{17\times 17} = 6137$, $S_{15\times 15} = 5415$ and $S_{5\times 5} = 1805$

19.3 Minimum Perfect Square Entries Sum

The examples given in 19.1 and 19.2 satisfy the uniformity property (11), i.e., $\left\langle 19, 19^2, 19^3, 19^4 \right\rangle$. However, the sum of the magic square entries of 19.2 is not a **minimum perfect square entries sum**. Choosing $m = 14, 15, \ldots, 19$ and $p = 19$ in equation (13), we get

$$\text{1.}\quad L\left(14^2 + \frac{19^2-1}{2}, 19^2\right) \rightarrow (19, 16, 376, 3724, 70756)$$

$$\text{2.}\quad L\left(15^2 + \frac{19^2-1}{2}, 19^2\right) \rightarrow (19, 45, 405, 4275, 81225)$$

$$\ldots \qquad\qquad \ldots \qquad \ldots$$

$$\text{6.}\quad L\left(19^2 + \frac{19^2-1}{2}, 19^2\right) \rightarrow (19, 181, 541, 6859, 130321) \Rightarrow \left\langle 19, 19^2, 19^3, 19^4 \right\rangle$$

The values written above are for $m = 14, 15, \ldots, 19$. The sixth value for $m = 19$ satisfy the uniformity property (11). This case is already studied in Example 19.1. The first value for $m = 14$ give **minimum perfect square** magic square. In this case, we have magic square of order 19 with sum of all entries $70756 = 266^2$ which is a perfect square, but it does not satisfy the uniformity property (11). The magic square given below has a **minimum perfect square entries sum**.

Example 19.3. *For the **consecutive natural number** entries* $\{16, 17, 18, \ldots, 375, 376\}$, *a **block-bordered** magic square of order 19 is given by*

35	51	49	47	45	43	41	39	37	360	361	363	365	367	369	371	373	375	33
376	67	339	337	335	333	331	329	327	326	71	73	75	77	79	81	83	69	16
374	52	84	301	113	202	280	114	271	158	187	250	129	241	173	217	220	340	18
372	54	203	292	85	294	106	188	262	115	264	151	218	232	130	234	166	338	20
370	56	295	99	196	293	97	265	144	181	263	127	235	159	211	233	142	336	22
368	58	286	98	307	100	189	256	128	277	145	174	226	143	247	160	204	334	24
366	60	112	190	279	91	308	157	175	249	121	278	172	205	219	136	248	332	26
364	62	86	300	111	200	283	116	270	156	185	253	131	240	171	215	223	330	28
362	64	201	290	88	296	105	186	260	118	266	150	216	230	133	236	165	328	30
34	324	298	101	195	291	95	268	146	180	261	125	238	161	210	231	140	68	358
36	322	285	96	305	103	191	255	126	275	148	176	225	141	245	163	206	70	356
38	320	110	193	281	90	306	155	178	251	120	276	170	208	221	135	246	72	354
40	318	87	302	109	198	284	117	272	154	183	254	132	242	169	213	224	74	352
42	316	199	288	89	297	107	184	258	119	267	152	214	228	134	237	167	76	350
44	314	299	102	197	289	93	269	147	182	259	123	239	162	212	229	138	78	348
46	312	287	94	303	104	192	257	124	273	149	177	227	139	243	164	207	80	346
48	310	108	194	282	92	304	153	179	252	122	274	168	209	222	137	244	82	344
50	323	53	55	57	59	61	63	65	66	321	319	317	315	313	311	309	325	342
359	341	343	345	347	349	351	353	355	32	31	29	27	25	23	21	19	17	357

The above example 19.3 has a magic sum $S_{19\times 19} = 3724$, and the sum of all entries is $T_{361} := 19 \times 3724 = 70756 = 266^2$. The Example 19.3 is **block-bordered** magic squares, where inner **block-wise** magic square of order 15 is **pandiagonal**. Moreover, blocks of order 5 are equal sum magic squares, with magic sums: $S_{17\times 17} = 3332$, $S_{15\times 15} = 2940$ and $S_{5\times 5} = 980$

19.4 Pythagorean Triples

According to Examples 19.1, 19.2 and 19.3, we have two perfect square entries sums, i.e., $T_{289} := 130321 = 361^2$ and $T_{361} := 19 \times 3724 = 70756 = 266^2$. Below are **Pythagorean triples** for the numbers 361 and 266:

$$\begin{aligned} 361^2 + 3420^2 &:= 3439^2 \\ 361^2 + 65160^2 &:= 65161^2 \\ 266^2 + 312^2 &:= 410^2 \end{aligned} \qquad \begin{aligned} 266^2 + 912^2 &:= 950^2 \\ 266^2 + 2520^2 &:= 2534^2 \\ 266^2 + 17688^2 &:= 17690^2 \end{aligned}$$

Remark 19.1. *The above 6* ***Pythagorean triples*** *are with numbers 361 and 266. These numbers are due to Examples 19.1 and 19.3. According to Result 2, there are only 2 triples generating magic square with* ***perfect square entries sum****:*

$$\begin{aligned} (266, 312, 410) &\Rightarrow 410 - 266 = 144 = 12^2 \\ (266, 17688, 17690) &\Rightarrow 17690 - 266 = 17424 = 132^2. \end{aligned}$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(266, 312, 410)$ *generating a* ***perfect square entries sum*** *magic square of order 12 with the* ***odd number entries****,* $\{533, 535, \ldots, 817, 819\}$ *with magic sum,* $S_{12\times 12} := 8112$*. The entries total sum is* $T_{144} := 12 \times 6561 = 97344 = 312^2$*.*

2. *The triple* $(266, 17688, 17690)$ *generating a* ***perfect square entries sum*** *magic square of order 132 with the* ***odd number entries****,* $\{533, 535, \ldots, 35377, 35379\}$ *with magic sum,* $S_{132\times 132} := 2370192$*. The entries total sum is* $T_{441} := 132 \times 2370192 = 312865344 = 17688^2$*.*

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20 Magic Squares of Order 20

According to equation (7), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (3), we can only obtain a magic square of order 20 with this property if we use odd number entries. Taking $k = 20$ in equation (3), we get

$$\begin{aligned} & 1+3+5+\ldots+(2\times 20^2-1) &&= 20^4 \\ \Rightarrow & 1+3+5+\ldots+797+799 &&= 160000 = 400^2 = 20^4. \end{aligned}$$

Example 20.1. *A* ***block-wise pandiagonal*** *magic square of order 16 for* ***consecutive odd numbers*** *entries* $1, 3, 5, ..., 797, 799$ *is given by*

		8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000
	379	401	39	781	333	447	73	747	297	483	117	703	251	529	151	669	215	565	195	625	8000
8000	21	799	361	419	67	753	327	453	103	717	283	497	149	671	249	531	185	635	205	575	8000
8000	761	19	421	399	727	53	467	353	683	97	503	317	649	131	549	271	605	175	585	235	8000
8000	439	381	779	1	473	347	733	47	517	303	697	83	551	269	651	129	595	225	615	165	8000
8000	257	523	157	663	211	569	191	629	375	405	35	785	339	441	79	741	293	487	113	707	8000
8000	143	677	243	537	189	631	209	571	25	795	365	415	61	759	321	459	107	713	287	493	8000
8000	643	137	543	277	609	171	589	231	765	15	425	395	721	59	461	359	687	93	507	313	8000
8000	557	263	657	123	591	229	611	169	435	385	775	5	479	341	739	41	513	307	693	87	8000
8000	335	445	75	745	299	481	119	701	253	527	153	667	217	563	197	623	371	409	31	789	8000
8000	65	755	325	455	101	719	281	499	147	673	247	533	183	637	203	577	29	791	369	411	8000
8000	725	55	465	355	681	99	501	319	647	133	547	273	603	177	583	237	769	11	429	391	8000
8000	475	345	735	45	519	301	699	81	553	267	653	127	597	223	617	163	431	389	771	9	8000
8000	213	567	193	627	377	403	37	783	331	449	71	749	295	485	115	705	259	521	159	661	8000
8000	187	633	207	573	23	797	363	417	69	751	329	451	105	715	285	495	141	679	241	539	8000
8000	607	173	587	233	763	17	423	397	729	51	469	351	685	95	505	315	641	139	541	279	8000
8000	593	227	613	167	437	383	777	3	471	349	731	49	515	305	695	85	559	261	659	121	8000
8000	291	489	111	709	255	525	155	665	219	561	199	621	373	407	33	787	337	443	77	743	8000
8000	109	711	289	491	145	675	245	535	181	639	201	579	27	793	367	413	63	757	323	457	8000
8000	689	91	509	311	645	135	545	275	601	179	581	239	767	13	427	393	723	57	463	357	8000
8000	511	309	691	89	555	265	655	125	599	221	619	161	433	387	773	7	477	343	737	43	8000
	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000

In this case the magic sum is $S_{20\times 20} = 8000 = 20^3$, and the sum of all entries is $T_{400} := 20 \times 8000 = 160000 = 400^2 = 20^4$. Each 4×4 block is a **pandiagonal** magic square of order 4 with the same magic sum $S_{4\times 4} = 1600$. These blocks also have the perfect square entries sum property, i.e., the sum of all entries in each block is $T_{16} := 4 \times 1600 = 6400 = 80^2$.

20.1 Pythagorean Triples

According to Example 20.1, we have perfect square entries sum, i.e., $T_{400} := 20 \times 8000 = 160000 = 400^2$. Below are **Pythagorean triples** for the number 400:

$$\begin{array}{lll}
400^2 + 90^2 := 410^2 & 400^2 + 960^2 := 1040^2 & 400^2 + 3990^2 := 4010^2 \\
400^2 + 195^2 := 445^2 & 400^2 + 1218^2 := 1282^2 & 400^2 + 4992^2 := 5008^2 \\
400^2 + 300^2 := 500^2 & 400^2 + 1575^2 := 1625^2 & 400^2 + 7995^2 := 8005^2 \\
400^2 + 420^2 := 580^2 & 400^2 + 1980^2 := 2020^2 & 400^2 + 9996^2 := 10004^2 \\
400^2 + 561^2 := 689^2 & 400^2 + 2484^2 := 2516^2 & 400^2 + 19998^2 := 20002^2 \\
400^2 + 750^2 := 850^2 & & 400^2 + 39999^2 := 40001^2
\end{array}$$

Remark 20.1. *The above 17* ***Pythagorean triples*** *are with number 400. This number is due to Example 20.1. According to Result 2, there are only 9 triples generating magic square with* ***perfect square entries sum****:*

$$\begin{array}{ll}
(400, 300, 500) & \Rightarrow 500 - 400 = 100 = 10^2 \\
(400, 561, 689) & \Rightarrow 689 - 400 = 289 = 17^2 \\
(400, 750, 850) & \Rightarrow 850 - 750 = 100 = 10^2 \\
(400, 1218, 1282) & \Rightarrow 1282 - 1218 = 64 = 8^2 \\
(400, 1575, 1625) & \Rightarrow 1625 - 400 = 1225 = 35^2 \\
(400, 2484, 2516) & \Rightarrow 2516 - 400 = 2116 = 46^2 \\
(400, 4992, 5008) & \Rightarrow 5008 - 4992 = 16 = 4^2 \\
(400, 9996, 10004) & \Rightarrow 10004 - 400 = 9604 = 98^2 \\
(400, 39999, 40001) & \Rightarrow 40001 - 400 = 39601 = 199^2.
\end{array}$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(300, 400, 500)$ *generating a* ***perfect square entries sum*** *magic square of order 10 with the* ***odd number entries***, $\{801, 803, \ldots, 997, 999\}$ *with magic sum*, $S_{10\times 10} := 9000$. *The entries total sum is* $T_{100} := 10 \times 9000 = 90000 = 300^2$.

2. *The triple* $(400, 561, 689)$ *generating a* ***perfect square entries sum*** *magic square of order 17 with the* ***odd number entries***, $\{801, 803, \ldots, 1375, 1377\}$ *with magic sum*, $S_{17\times 17} := 18513$. *The entries total sum is* $T_{289} := 17 \times 18513 = 314721 = 561^2$.

3. *The triple* $(400, 750, 850)$ *generating a* ***perfect square entries sum*** *magic square of order 10 with the* ***odd number entries***, $\{1501, 1503, \ldots, 1697, 1699\}$ *with magic sum*, $S_{10\times 10} := 16000$. *The entries total sum is* $T_{100} := 10 \times 16000 = 160000 = 400^2$.

4. *The triple* $(400, 1218, 1282)$ *generating a* ***perfect square entries sum*** *magic square of order 8 with the* ***odd number entries***, $\{2437, 2439, \ldots, 2561, 2563\}$ *with magic sum*, $S_{8\times 8} := 20000$. *The entries total sum is* $T_{64} := 8 \times 20000 = 160000 = 400^2$.

5. *The triple* $(400, 1575, 1625)$ *generating a* ***perfect square entries sum*** *magic square of order 35 with the* ***odd number entries***, $\{801, 803, \ldots, 3247, 3249\}$ *with magic sum*, $S_{35\times 35} := 134136$. *The entries total sum is* $T_{1225} := 35 \times 134136 = 2480625 = 1575^2$.

6. *The triple* $(400, 2484, 2516)$ *generating a* ***perfect square entries sum*** *magic square of order 46 with the* ***odd number entries***, $\{801, 803, \ldots, 5029, 5031\}$ *with magic sum*, $S_{46\times 46} := 4277609$. *The entries total sum is* $T_{2116} := 46 \times 4277609 = 688695049 = 26243^2$.

7. *The triple* $(400, 4992, 5008)$ *generating a* ***perfect square entries sum*** *magic square of order 4 with the* ***odd number entries***, $\{9985, 9987, \ldots, 10013, 10015\}$ *with magic sum*, $S_{4\times 4} := 40000$. *The entries total sum is* $T_{16} := 4 \times 40000 = 160000 = 400^2$.

8. *The triple* $(400, 9996, 10004)$ *generating a* ***perfect square entries sum*** *magic square of order 98 with the* ***odd number entries***, $\{801, 803, \ldots, 20005, 20007\}$ *with magic sum*, $S_{98\times 98} := 1019592$. *The entries total sum is* $T_{9604} := 98 \times 1019592 = 99920016 = 9996^2$.

9. *The triple* $(400, 39999, 40001)$ *generating a* ***perfect square entries sum*** *magic square of order 199 with the* ***odd number entries***, $\{801, 803, \ldots, 79999, 80001\}$ *with magic sum,* $\boldsymbol{S}_{199\times 199} := 8039799$. *The entries total sum is* $\boldsymbol{T}_{39601} := 199 \times 8039799 = 1599920001 = 39999^2$.

21 Magic Square of Order 21

According to equations (3) and (8), one can obtain **perfect square entries sum magic squares** of order 21 in two ways: first using **consecutive odd numbers**, and the second using **consecutive natural numbers**. For the **consecutive natural numbers** there are again two ways: the first one is of **uniformity**, and the second is of **minimum perfect square entries sum**.

21.1 Consecutive Odd Number Entries

Taking $k = 21$ in equation (3), we get

$$\begin{aligned} & 1+3+5+\ldots+(2\times 21^2-1) = 21^4 \\ \Rightarrow & 1+3+5+\ldots+879+881 \qquad = 194481 = 441^2 = 21^4 \end{aligned}$$

Example 21.1. *For the* ***consecutive odd number*** *entries* $\{1, 3, 5, \ldots, 879, 881\}$ *a* ***block-wise*** *magic square of order 21 is given by*

		9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261
	1	221	309	485	529	749	793	5	219	307	483	531	747	795	3	217	311	481	533	745	797	9261
9261	739	791	37	211	305	477	527	741	789	39	215	303	475	525	743	787	41	213	301	479	523	9261
9261	473	519	737	781	35	247	295	471	517	735	783	33	249	299	469	521	733	785	31	251	297	9261
9261	245	331	463	515	729	779	25	243	333	467	513	727	777	27	241	335	465	511	731	775	29	9261
9261	771	23	235	329	499	505	725	769	21	237	327	501	509	723	773	19	239	325	503	507	721	9261
9261	541	715	767	15	233	319	497	543	719	765	13	231	321	495	545	717	763	17	229	323	493	9261
9261	317	487	539	751	757	11	225	315	489	537	753	761	9	223	313	491	535	755	759	7	227	9261
9261	85	179	267	443	571	707	835	89	177	265	441	573	705	837	87	175	269	439	575	703	839	9261
9261	697	833	121	169	263	435	569	699	831	123	173	261	433	567	701	829	125	171	259	437	565	9261
9261	431	561	695	823	119	205	253	429	559	693	825	117	207	257	427	563	691	827	115	209	255	9261
9261	203	289	421	557	687	821	109	201	291	425	555	685	819	111	199	293	423	553	689	817	113	9261
9261	813	107	193	287	457	547	683	811	105	195	285	459	551	681	815	103	197	283	461	549	679	9261
9261	583	673	809	99	191	277	455	585	677	807	97	189	279	453	587	675	805	101	187	281	451	9261
9261	275	445	581	709	799	95	183	273	447	579	711	803	93	181	271	449	577	713	801	91	185	9261
9261	43	137	351	401	613	665	877	47	135	349	399	615	663	879	45	133	353	397	617	661	881	9261
9261	655	875	79	127	347	393	611	657	873	81	131	345	391	609	659	871	83	129	343	395	607	9261
9261	389	603	653	865	77	163	337	387	601	651	867	75	165	341	385	605	649	869	73	167	339	9261
9261	161	373	379	599	645	863	67	159	375	383	597	643	861	69	157	377	381	595	647	859	71	9261
9261	855	65	151	371	415	589	641	853	63	153	369	417	593	639	857	61	155	367	419	591	637	9261
9261	625	631	851	57	149	361	413	627	635	849	55	147	363	411	629	633	847	59	145	365	409	9261
9261	359	403	623	667	841	53	141	357	405	621	669	845	51	139	355	407	619	671	843	49	143	9261
	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261

The above magic square is **pandiagonal** with magic sum $S_{21\times 21} = 9261$. The blocks of order 7 are **pandiagonal** magic squares with equal magic sums $S_{7\times 7} := 3087$. The total entries sum is $T_{441} := 9261 \times 21 = 194481 = 441^2$. It satisfies the uniformity property (11), i.e., $\left\langle 21, 21^2, 21^3, 21^4 \right\rangle$.

21.2 Consecutive Natural Number Entries

According to equation (8),

$$G := \frac{n(n+1)}{2} - \frac{(n-441)(n-440)}{2} = 441(n-220)$$

Taking $n = 661$, we get a perfect square, i.e.,

$$G := T(661) - T(220) = \frac{661 \times 662}{2} - \frac{220 \times 221}{2} = 661 \times 331 - 110 \times 221$$
$$= 194481 = 441^2 = 21^4.$$

Simplifying, we get

$$221 + 222 + 223 + \ldots + 660 + 661 = 194481 = 441^2 = 21^4.$$

This gives a perfect square entries sum for the 441 **consecutive natural numbers** from to 221 to 661.

Example 21.2. *For the* ***consecutive natural number*** *entries* $\{221, 222, 223, \ldots, 660, 661\}$ *a* ***block-wise pan-diagonal*** *magic square of order 21 is given by*

Generating Pythagorean Triples and Magic Squares: Orders 3 to 31,
May 28, 2021, pp. 1-153, https://doi.org/10.5281/zenodo.4837491

		9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261
	221	331	375	463	485	595	617	223	330	374	462	486	594	618	222	329	376	461	487	593	619	9261
9261	590	616	239	326	373	459	484	591	615	240	328	372	458	483	592	614	241	327	371	460	482	9261
9261	457	480	589	611	238	344	368	456	479	588	612	237	345	370	455	481	587	613	236	346	369	9261
9261	343	386	452	478	585	610	233	342	387	454	477	584	609	234	341	388	453	476	586	608	235	9261
9261	606	232	338	385	470	473	583	605	231	339	384	471	475	582	607	230	340	383	472	474	581	9261
9261	491	578	604	228	337	380	469	492	580	603	227	336	381	468	493	579	602	229	335	382	467	9261
9261	379	464	490	596	599	226	333	378	465	489	597	601	225	332	377	466	488	598	600	224	334	9261
9261	263	310	354	442	506	574	638	265	309	353	441	507	573	639	264	308	355	440	508	572	640	9261
9261	569	637	281	305	352	438	505	570	636	282	307	351	437	504	571	635	283	306	350	439	503	9261
9261	436	501	568	632	280	323	347	435	500	567	633	279	324	349	434	502	566	634	278	325	348	9261
9261	322	365	431	499	564	631	275	321	366	433	498	563	630	276	320	367	432	497	565	629	277	9261
9261	627	274	317	364	449	494	562	626	273	318	363	450	496	561	628	272	319	362	451	495	560	9261
9261	512	557	625	270	316	359	448	513	559	624	269	315	360	447	514	558	623	271	314	361	446	9261
9261	358	443	511	575	620	268	312	357	444	510	576	622	267	311	356	445	509	577	621	266	313	9261
9261	242	289	396	421	527	553	659	244	288	395	420	528	552	660	243	287	397	419	529	551	661	9261
9261	548	658	260	284	394	417	526	549	657	261	286	393	416	525	550	656	262	285	392	418	524	9261
9261	415	522	547	653	259	302	389	414	521	546	654	258	303	391	413	523	545	655	257	304	390	9261
9261	301	407	410	520	543	652	254	300	408	412	519	542	651	255	299	409	411	518	544	650	256	9261
9261	648	253	296	406	428	515	541	647	252	297	405	429	517	540	649	251	298	404	430	516	539	9261
9261	533	536	646	249	295	401	427	534	538	645	248	294	402	426	535	537	644	250	293	403	425	9261
9261	400	422	532	554	641	247	291	399	423	531	555	643	246	290	398	424	530	556	642	245	292	9261
	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261

The magic squares of order 21 given in Examples 21.1 and 21.2 are with equal magic sums, i.e., $S_{21\times 21} := 9261$. The blocks of order 7 are **pandiagonal** magic squares with equal magic sums $S_{7\times 7} := 3087$. The total entries sum is $T_{441} := 9261 \times 21 = 194481 = 441^2$. It satisfies the uniformity property (11), i.e., $\left\langle 21, 21^2, 21^3, 21^4 \right\rangle$.

21.3 Minimum Perfect Square Entries Sum

The examples given in 21.1 and 21.2 satisfy the uniformity property (11), i.e., $\left\langle 21, 21^2, 21^3, 21^4 \right\rangle$. However, the sum of the magic square entries of 21.2 is not a **minimum perfect square entries sum**. Choosing $m = 15, 16, \ldots, 21$ and $p = 21$ in equation (13), we get

$$
\begin{array}{lll}
1. & L\left(15^2 + \dfrac{21^2-1}{2}, 21^2\right) & \rightarrow (21, 5, 445, 4725, 99225) \\
2. & L\left(16^2 + \dfrac{21^2-1}{2}, 21^2\right) & \rightarrow (21, 36, 476, 5376, 112896) \\
\ldots & \ldots & \ldots \quad \ldots \\
7. & L\left(21^2 + \dfrac{21^2-1}{2}, 21^2\right) & \rightarrow (21, 221, 661, 9261, 194481) \Rightarrow \left\langle 21, 21^2, 21^3, 21^4 \right\rangle
\end{array}
$$

The values written above are for $m = 15, 16, \ldots, 21$. The seventh value for $m = 21$ satisfy the uniformity property (11). This case is already studied in Example 21.1. The first value for $m = 15$ give **minimum perfect square** magic square. In this case, we have magic square of order 21 with sum of all entries $99225 = 315^2$ which is a perfect square, but it does not satisfy the uniformity property (11). The magic square given below has a **minimum perfect square entries sum**.

Example 21.3. *For the* ***consecutive natural number*** *entries* $\{5, 6, 7, \ldots, 444, 445\}$*, a* ***block-bordered*** *magic square of order 21 is given by*

		4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725
	5	115	159	247	269	379	401	7	114	158	246	270	378	402	6	113	160	245	271	377	403	4725
4725	374	400	23	110	157	243	268	375	399	24	112	156	242	267	376	398	25	111	155	244	266	4725
4725	241	264	373	395	22	128	152	240	263	372	396	21	129	154	239	265	371	397	20	130	153	4725
4725	127	170	236	262	369	394	17	126	171	238	261	368	393	18	125	172	237	260	370	392	19	4725
4725	390	16	122	169	254	257	367	389	15	123	168	255	259	366	391	14	124	167	256	258	365	4725
4725	275	362	388	12	121	164	253	276	364	387	11	120	165	252	277	363	386	13	119	166	251	4725
4725	163	248	274	380	383	10	117	162	249	273	381	385	9	116	161	250	272	382	384	8	118	4725
4725	47	94	138	226	290	358	422	49	93	137	225	291	357	423	48	92	139	224	292	356	424	4725
4725	353	421	65	89	136	222	289	354	420	66	91	135	221	288	355	419	67	90	134	223	287	4725
4725	220	285	352	416	64	107	131	219	284	351	417	63	108	133	218	286	350	418	62	109	132	4725
4725	106	149	215	283	348	415	59	105	150	217	282	347	414	60	104	151	216	281	349	413	61	4725
4725	411	58	101	148	233	278	346	410	57	102	147	234	280	345	412	56	103	146	235	279	344	4725
4725	296	341	409	54	100	143	232	297	343	408	53	99	144	231	298	342	407	55	98	145	230	4725
4725	142	227	295	359	404	52	96	141	228	294	360	406	51	95	140	229	293	361	405	50	97	4725
4725	26	73	180	205	311	337	443	28	72	179	204	312	336	444	27	71	181	203	313	335	445	4725
4725	332	442	44	68	178	201	310	333	441	45	70	177	200	309	334	440	46	69	176	202	308	4725
4725	199	306	331	437	43	86	173	198	305	330	438	42	87	175	197	307	329	439	41	88	174	4725
4725	85	191	194	304	327	436	38	84	192	196	303	326	435	39	83	193	195	302	328	434	40	4725
4725	432	37	80	190	212	299	325	431	36	81	189	213	301	324	433	35	82	188	214	300	323	4725
4725	317	320	430	33	79	185	211	318	322	429	32	78	186	210	319	321	428	34	77	187	209	4725
4725	184	206	316	338	425	31	75	183	207	315	339	427	30	74	182	208	314	340	426	29	76	4725
	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725

The above Example 21.3 is with magic sum $S_{21\times 21} = 4275$, and the sum of all entries is $T_{441} := 21 \times 4725 = 99225 = 315^2$. This magic square of order 21 is a **minimum perfect square entries sum**. Moreover, the blocks of order 3 are **semi-magic** squares of equal **semi-magic** sums, $S_{3\times 3} = 945$.

21.4 Pythagorean Triples

According to Examples 21.1, 21.2 and 21.3, we have two perfect square entries sums, i.e., $T_{441} := 9261 \times 21 = 194481 = 441^2$ and $T_{441} := 21 \times 4725 = 99225 = 315^2$. Below are **Pythagorean triples** for the numbers 441 and 315:

$$\begin{array}{lll}
\mathbf{441^2}+112^2 := 455^2 & \mathbf{315^2}+80^2 := 325^2 & \mathbf{315^2}+1080^2 := 1125^2 \\
\mathbf{441^2}+420^2 := 609^2 & \mathbf{315^2}+108^2 := 333^2 & \mathbf{315^2}+1400^2 := 1435^2 \\
\mathbf{441^2}+588^2 := 735^2 & \mathbf{315^2}+168^2 := 357^2 & \mathbf{315^2}+1824^2 := 1851^2 \\
\mathbf{441^2}+1160^2 := 1241^2 & \mathbf{315^2}+196^2 := 371^2 & \mathbf{315^2}+1972^2 := 1997^2 \\
\mathbf{441^2}+1512^2 := 1575^2 & \mathbf{315^2}+264^2 := 411^2 & \mathbf{315^2}+2352^2 := 2373^2 \\
\mathbf{441^2}+1960^2 := 2009^2 & \mathbf{315^2}+300^2 := 435^2 & \mathbf{315^2}+3300^2 := 3315^2 \\
\mathbf{441^2}+3588^2 := 3615^2 & \mathbf{315^2}+420^2 := 525^2 & \mathbf{315^2}+5508^2 := 5517^2 \\
\mathbf{441^2}+4620^2 := 4641^2 & \mathbf{315^2}+572^2 := 653^2 & \mathbf{315^2}+7084^2 := 7091^2 \\
\mathbf{441^2}+10800^2 := 10809^2 & \mathbf{315^2}+624^2 := 699^2 & \mathbf{315^2}+9920^2 := 9925^2 \\
\mathbf{441^2}+13888^2 := 13895^2 & \mathbf{315^2}+756^2 := 819^2 & \mathbf{315^2}+16536^2 := 16539^2 \\
\mathbf{441^2}+32412^2 := 32415^2 & \mathbf{315^2}+988^2 := 1037^2 & \mathbf{315^2}+49612^2 := 49613^2 \\
\mathbf{441^2}+97240^2 := 97241^2 & &
\end{array}$$

Remark 21.1. *The above 34* ***Pythagorean triples*** *are with numbers 441 and 315. These numbers are due to Examples 21.1 and 21.3. According to Result 2, there are only 8 triples generating magic square with* ***perfect square entries sum****:*

$$\begin{aligned}
(\mathbf{441, 1160, 1241}) &\Rightarrow 1241 - 1160 &= 81 = 9^2 \\
(\mathbf{441, 1960, 2009}) &\Rightarrow 2009 - 1960 &= 49 = 7^2 \\
(\mathbf{441, 10800, 10809}) &\Rightarrow 10809 - 10800 &= 9 = 3^2 \\
(\mathbf{315, 108, 333}) &\Rightarrow 333 - 108 &= 225 = 15^2 \\
(\mathbf{315, 572, 653}) &\Rightarrow 653 - 572 &= 81 = 9^2 \\
(\mathbf{315, 988, 1037}) &\Rightarrow 1037 - 988 &= 49 = 7^2 \\
(\mathbf{315, 1972, 1997}) &\Rightarrow 1997 - 1972 &= 25 = 5^2 \\
(\mathbf{315, 5508, 5517}) &\Rightarrow 5517 - 5508 &= 9 = 3^2.
\end{aligned}$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(441, 1160, 1241)$ *generating a* ***perfect square entries sum*** *magic square of order 9 with the* ***odd number entries***, $\{2321, 2323, \ldots, 2479, 2481\}$ *with magic sum,* $\boldsymbol{S}_{9\times 9} := 21609$. *The entries total sum is* $\boldsymbol{T}_{81} := 9 \times 21609 = 194481 = 441^2$.

2. *The triple* $(441, 1960, 2009)$ *generating a* ***perfect square entries sum*** *magic square of order 7 with the* ***odd number entries***, $\{3921, 3923, \ldots, 4015, 4017\}$ *with magic sum,* $\boldsymbol{S}_{7\times 7} := 27783$. *The entries total sum is* $\boldsymbol{T}_{49} := 7 \times 27783 = 194481 = 441^2$.

3. *The triple* $(441, 10800, 10809)$ *generating a* ***perfect square entries sum*** *magic square of order 3 with the* ***odd number entries***, $\{21601, 21603, \ldots, 21615, 21617\}$ *with magic sum,* $\boldsymbol{S}_{3\times 3} := 64827$. *The entries total sum is* $\boldsymbol{T}_{9} := 3 \times 64827 = 194481 = 441^2$.

4. *The triple* $(315, 108, 333)$ *generating a* ***perfect square entries sum*** *magic square of order 15 with the* ***odd number entries***, $\{217, 219, \ldots, 663, 665\}$ *with magic sum,* $\boldsymbol{S}_{15\times 15} := 6615$. *The entries total sum is* $\boldsymbol{T}_{225} := 15 \times 6615 = 99225 = 315^2$.

5. *The triple* $(315, 572, 653)$ *generating a* ***perfect square entries sum*** *magic square of order 9 with the* ***odd number entries***, $\{1145, 1147, \ldots, 1303, 1305\}$ *with magic sum,* $\boldsymbol{S}_{9\times 9} := 11025$. *The entries total sum is* $\boldsymbol{T}_{81} := 9 \times 11025 = 99225 = 315^2$.

6. *The triple* $(315, 988, 1037)$ *generating a* ***perfect square entries sum*** *magic square of order 7 with the* ***odd number entries***, $\{1977, 1979, \ldots, 2071, 2073\}$ *with magic sum,* $\boldsymbol{S}_{7\times7} := 14175$*. The entries total sum is* $\boldsymbol{T}_{49} := 7 \times 14175 = 99225 = 315^2$.

7. *The triple* $(315, 1972, 1997)$ *generating a* ***perfect square entries sum*** *magic square of order 5 with the* ***odd number entries***, $\{3945, 3947, \ldots, 3991, 3993\}$ *with magic sum,* $\boldsymbol{S}_{5\times5} := 19845$*. The entries total sum is* $\boldsymbol{T}_{25} := 5 \times 19845 = 99225 = 315^2$.

8. *The triple* $(315, 5508, 5517)$ *generating a* ***perfect square entries sum*** *magic square of order 3 with the* ***odd number entries***, $\{11017, 11019, \ldots, 11031, 11033\}$ *with magic sum,* $\boldsymbol{S}_{3\times3} := 33075$*. The entries total sum is* $\boldsymbol{T}_9 := 3 \times 33075 = 99225 = 315^2$.

22 Magic Square of Order 22

According to equation (7), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (3), we can only obtain a magic square of order 22 with this property if we use odd number entries. Taking $k = 22$ in equation (3), we get

$$\begin{aligned} &1 + 3 + 5 + \ldots + (2 \times 22^2 - 1) = 22^4 \\ \Rightarrow &1 + 3 + 5 + \ldots + 965 + 967 \quad = 234256 = 484^2 = 22^4. \end{aligned}$$

Example 22.1. *For the* ***consecutive odd number*** *entries* $\{1, 3, 5, \ldots, 965, 967\}$*, a* ***block-bordered*** *magic square of order 22 is given by*

41	945	25	941	29	937	33	933	37	929	1	947	45	921	49	917	53	913	57	909	61	925
65	97	353	449	705	801	107	363	459	715	811	85	341	437	693	789	111	367	463	719	815	903
901	609	865	161	257	513	619	875	171	267	523	597	853	149	245	501	623	879	175	271	527	67
69	321	417	673	769	225	331	427	683	779	235	309	405	661	757	213	335	431	687	783	239	899
897	833	129	385	481	577	843	139	395	491	587	821	117	373	469	565	847	143	399	495	591	71
73	545	641	737	193	289	555	651	747	203	299	533	629	725	181	277	559	655	751	207	303	895
893	87	343	439	695	791	109	365	461	717	813	99	355	451	707	803	105	361	457	713	809	75
77	599	855	151	247	503	621	877	173	269	525	611	867	163	259	515	617	873	169	265	521	891
889	311	407	663	759	215	333	429	685	781	237	323	419	675	771	227	329	425	681	777	233	79
81	823	119	375	471	567	845	141	397	493	589	835	131	387	483	579	841	137	393	489	585	887
885	535	631	727	183	279	557	653	749	205	301	547	643	739	195	291	553	649	745	201	297	83
905	115	371	467	723	819	89	345	441	697	793	103	359	455	711	807	93	349	445	701	797	63
949	627	883	179	275	531	601	857	153	249	505	615	871	167	263	519	605	861	157	253	509	19
17	339	435	691	787	243	313	409	665	761	217	327	423	679	775	231	317	413	669	765	221	951
953	851	147	403	499	595	825	121	377	473	569	839	135	391	487	583	829	125	381	477	573	15
13	563	659	755	211	307	537	633	729	185	281	551	647	743	199	295	541	637	733	189	285	955
957	101	357	453	709	805	95	351	447	703	799	113	369	465	721	817	91	347	443	699	795	11
9	613	869	165	261	517	607	863	159	255	511	625	881	177	273	529	603	859	155	251	507	959
961	325	421	677	773	229	319	415	671	767	223	337	433	689	785	241	315	411	667	763	219	7
5	837	133	389	485	581	831	127	383	479	575	849	145	401	497	593	827	123	379	475	571	963
965	549	645	741	197	293	543	639	735	191	287	561	657	753	209	305	539	635	731	187	283	3
43	23	943	27	939	31	935	35	931	39	967	21	923	47	919	51	915	55	911	59	907	927

The above **block-bordered** magic square of order 22 is with inner part as **block-wise pandiagonal** magic square of order 20. The blocks of order 4 are **pandiagonal** magic squares with equal magic sums.

Below are respective magic sums:

$$S_{22\times 22} = 10648$$
$$S_{20\times 20} = 9680$$
$$S_{4\times 4} = 1936$$

The total entries sum is $T_{484} := 10648 \times 22 = 234256 = 484^2$. It satisfies the uniformity property (11), i.e., $\left\langle 22, 22^2, 22^3, 22^4 \right\rangle$.

22.1 Pythagorean Triples

According to Example 22.1, we have perfect square entries sum, i.e., $T_{484} := 10648 \times 22 = 234256 = 484^2$. Below are **Pythagorean triples** for the number 484:

$$\mathbf{484^2} + 363^2 := 605^2$$
$$\mathbf{484^2} + 1287^2 := 1375^2$$
$$\mathbf{484^2} + 2640^2 := 2684^2$$
$$\mathbf{484^2} + 5313^2 := 5335^2$$
$$\mathbf{484^2} + 14637^2 := 14645^2$$
$$\mathbf{484^2} + 29280^2 := 29284^2$$
$$\mathbf{484^2} + 58563^2 := 58565^2$$

Remark 22.1. *The above 7* ***Pythagorean triples*** *are with number 484. This number is due to Example 22.1. According to Result 2, there are only 3 triples generating magic square with* ***perfect square entries sum****:*

$$(\mathbf{484, 363, 605}) \Rightarrow 605 - 484 = 121 = 11^2$$
$$(\mathbf{484, 14637, 14645}) \Rightarrow 14645 - 484 = 14161 = 119^2$$
$$(\mathbf{484, 58563, 58565}) \Rightarrow 58565 - 484 = 58081 = 241^2.$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(363, 484, 605)$ *generating a* ***perfect square entries sum*** *magic square of order 11 with the* ***odd number entries****,* $\{969, 971, \ldots, 1207, 1209\}$ *with magic sum,* $S_{11\times 11} := 11979$*. The entries total sum is*

$T_{121} := 11 \times 11979 = 131769 = 363^2$.

2. *The triple* $(484, 14637, 14645)$ *generating a* ***perfect square entries sum*** *magic square of order 119 with the* ***odd number entries***, $\{969, 971, \ldots, 29287, 29289\}$ *with magic sum,* $S_{119\times119} := 27783$. *The entries total sum is* $T_{1800351} := 119 \times 1800351 = 214241769 = 14637^2$.

3. *The triple* $(484, 58563, 58565)$ *generating a* ***perfect square entries sum*** *magic square of order 241 with the* ***odd number entries***, $\{969, 971, \ldots, 117127, 117129\}$ *with magic sum,* $S_{241\times241} := 14230809$. *The entries total sum is* $T_{58081} := 241 \times 14230809 = 3429624969 = 58563^2$.

23 Magic Square of Order 23

According to equations (3) and (8), one can obtain **perfect square entries sum magic squares** of order 23 in two ways: first using **consecutive odd numbers**, and the second using **consecutive natural numbers**. For the **consecutive natural numbers** there are again two ways: the first one is of **uniformity**, and the second is of **minimum perfect square entries sum**.

23.1 Consecutive Odd Number Entries

According to equation (7), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (3), we can only obtain a magic square of order 14 with this property if we use odd number entries. Taking $k = 23$ in equation (3), we get

$$\begin{aligned} & 1+3+5+\ldots+(2\times 23^2-1) = 23^4 \\ \Rightarrow & 1+3+5+\ldots+1055+1057 \quad = 279841 = 529^2 = 234^4. \end{aligned}$$

Example 23.1. *For the* ***consecutive odd number*** *entries* $\{1, 3, 5, \ldots, 1055, 1057\}$, *a* ***block-bordered*** *magic square of order 23 is given by*

1015	85	81	77	73	69	65	61	57	53	49	45	1021	1025	1029	1033	1037	1041	1045	1049	1053	1057	47
971	579	309	699	635	837	115	461	265	861	511	925	151	423	221	943	597	793	197	497	353	737	87
975	687	573	327	123	661	803	853	441	293	169	529	889	935	397	255	205	617	765	731	485	371	83
979	321	705	561	829	89	669	273	881	433	907	133	547	229	969	389	785	177	625	359	749	479	79
983	633	767	187	471	373	743	581	311	695	663	813	111	425	291	871	545	895	147	385	253	949	75
987	179	607	801	751	491	345	689	569	329	99	657	831	879	451	257	139	525	923	967	403	217	71
991	775	213	599	365	723	499	317	707	563	825	117	645	283	845	459	903	167	517	235	931	421	67
995	509	921	157	419	223	945	595	799	193	507	347	733	555	331	701	665	815	107	453	267	867	63
999	165	535	887	937	399	251	211	613	763	725	481	381	709	575	303	101	653	833	855	447	285	59
1003	913	131	543	231	965	391	781	175	631	355	759	473	323	681	583	821	119	647	279	873	435	55
1007	639	835	113	455	269	863	537	897	153	383	249	955	629	769	189	469	379	739	591	305	691	51
41	121	659	807	857	443	287	141	531	915	963	409	215	181	609	797	757	487	343	683	565	339	1017
39	827	93	667	275	875	437	909	159	519	241	929	417	777	209	601	361	721	505	313	717	557	1019
35	503	349	735	553	337	697	675	809	103	429	289	869	539	899	149	411	225	951	593	795	199	1023
31	727	483	377	715	571	301	95	649	843	877	449	261	143	527	917	939	405	243	207	619	761	1027
27	357	755	475	319	679	589	817	129	641	281	849	457	905	161	521	237	957	393	787	173	627	1031
23	413	227	947	621	771	195	467	375	745	587	307	693	637	841	109	465	263	859	513	919	155	1035
19	941	401	245	183	615	789	753	493	341	685	567	335	127	655	805	851	439	297	163	533	891	1039
15	233	959	395	783	201	603	367	719	501	315	713	559	823	91	673	271	885	431	911	135	541	1043
11	427	295	865	549	893	145	387	247	953	623	773	191	495	351	741	551	333	703	671	811	105	1047
7	883	445	259	137	523	927	961	407	219	185	611	791	729	489	369	711	577	299	97	651	839	1051
3	277	847	463	901	171	515	239	933	415	779	203	605	363	747	477	325	677	585	819	125	643	1055
1011	973	977	981	985	989	993	997	1001	1005	1009	1013	37	33	29	25	21	17	13	9	5	1	43

The above **block-bordered** magic square of order 23 is with inner part as **block-wise pandiagonal** magic square of order 21 with blocks of order 3. The blocks of order 3 are **semi-magic** squares with equal

semi-magic sums. Below are respective sums

$$S_{23\times 23} = 12167$$
$$S_{21\times 21} = 11109$$
$$S_{3\times 3} = 1587$$

The total entries sum is $T_{529} := 12167 \times 23 = 279841 = 529^2$. It satisfies the uniformity property (11), i.e., $\left\langle 23, 23^2, 23^3, 23^4 \right\rangle$.

23.2 Consecutive Natural Number Entries

According to equation (8),

$$G := \frac{n(n+1)}{2} - \frac{(n-529)(n-528)}{2} = 529(n-264)$$

Taking $n = 793$, we get a perfect square, i.e.,

$$\begin{aligned} G := T(793) - T(264) = \frac{793 \times 794}{2} - \frac{264 \times 265}{2} &= 793 \times 397 - 132 \times 265 \\ &= 279841 = 529^2 = 23^4. \end{aligned}$$

Simplifying, we get

$$265 + 266 + 267 + \ldots + 792 + 793 = 279841 = 529^2 = 23^4.$$

This gives a perfect square entries sum for the 441 **consecutive natural numbers** from to 265 to 793.

Example 23.2. *For the **consecutive natural number** entries* $\{265, 266, 267, \ldots, 792, 793\}$ *a **block-wise pan-diagonal** magic square of order 21 is given by*

772	307	305	303	301	299	297	295	293	291	289	287	775	777	779	781	783	785	787	789	791	793	288
750	554	419	614	582	683	322	495	397	695	520	727	340	476	375	736	563	661	363	513	441	633	308
752	608	551	428	326	595	666	691	485	411	349	529	709	732	463	392	367	573	647	630	507	450	306
754	425	617	545	679	309	599	401	705	481	718	331	538	379	749	459	657	353	577	444	639	504	304
756	581	648	358	500	451	636	555	420	612	596	671	320	477	410	700	537	712	338	457	391	739	302
758	354	568	665	640	510	437	609	549	429	314	593	680	704	490	393	334	527	726	748	466	373	300
760	652	371	564	447	626	514	423	618	546	677	323	587	406	687	494	716	348	523	382	730	475	298
762	519	725	343	474	376	737	562	664	361	518	438	631	542	430	615	597	672	318	491	398	698	296
764	347	532	708	733	464	390	370	571	646	627	505	455	619	552	416	315	591	681	692	488	407	294
766	721	330	536	380	747	460	655	352	580	442	644	501	426	605	556	675	324	588	404	701	482	292
768	584	682	321	492	399	696	533	713	341	456	389	742	579	649	359	499	454	634	560	417	610	290
285	325	594	668	693	486	408	335	530	722	746	469	372	355	569	663	643	508	436	606	547	434	773
284	678	311	598	402	702	483	719	344	524	385	729	473	653	369	565	445	625	517	421	623	543	774
282	516	439	632	541	433	613	602	669	316	479	409	699	534	714	339	470	377	740	561	662	364	776
280	628	506	453	622	550	415	312	589	686	703	489	395	336	528	723	734	467	386	368	574	645	778
278	443	642	502	424	604	559	673	329	585	405	689	493	717	345	525	383	743	461	658	351	578	780
276	471	378	738	575	650	362	498	452	637	558	418	611	583	685	319	497	396	694	521	724	342	782
274	735	465	387	356	572	659	641	511	435	607	548	432	328	592	667	690	484	413	346	531	710	784
272	381	744	462	656	365	566	448	624	515	422	621	544	676	310	601	400	707	480	720	332	535	786
270	478	412	697	539	711	337	458	388	741	576	651	360	512	440	635	540	431	616	600	670	317	788
268	706	487	394	333	526	728	745	468	374	357	570	660	629	509	449	620	553	414	313	590	684	790
266	403	688	496	715	350	522	384	731	472	654	366	567	446	638	503	427	603	557	674	327	586	792
770	751	753	755	757	759	761	763	765	767	769	771	283	281	279	277	275	273	271	269	267	265	286

The **block-bordered** magic squares of order 23 given in Examples 23.1 and 23.2 are of equal magic sums, i.e., $S_{23\times 23} := 12167$.

23.3 Minimum Perfect Square Entries Sum

The examples given in 23.1 and 23.2 satisfy the uniformity property (11), i.e., $\left\langle 23, 23^2, 23^3, 23^4 \right\rangle$. However, the sum of the magic square entries of 23.2 is not a **minimum perfect square entries sum**. Choosing $m = 17, 18, \ldots, 23$ and $p = 23$ in equation (13), we get

$$1.\quad L\left(17^2 + \frac{23^2-1}{2}, 23^2\right) \rightarrow (23, 25, 553, 6647, 152881)$$

$$2.\quad L\left(18^2 + \frac{23^2-1}{2}, 23^2\right) \rightarrow (23, 60, 588, 7452, 171396)$$

$$\ldots \quad \ldots \qquad\qquad \ldots \quad \ldots$$

$$7.\quad L\left(23^2 + \frac{23^2-1}{2}, 23^2\right) \rightarrow (23, 265, 793, 12167, 279841) \Rightarrow \left\langle 23, 23^2, 23^3, 23^4 \right\rangle$$

The values written above are for $m = 17, 18, \ldots, 23$. The seventh value for $m = 23$ satisfy the uniformity property (11). This case is already studied in Example 23.1. The first value for $m = 17$ give **minimum perfect square** magic square. In this case, we have magic square of order 23 with sum of all entries $152881 = 391^2$ which is a perfect square, but it does not satisfy the uniformity property (11). The magic square given below has a **minimum perfect square entries sum**.

Example 23.3. *For the **consecutive natural number** entries $\{25, 26, 27, \ldots, 552, 553\}$ a **block-bordered** magic square of order 23 is given by*

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532	67	65	63	61	59	57	55	53	51	49	47	535	537	539	541	543	545	547	549	551	553	48
510	314	179	374	342	443	82	255	157	455	280	487	100	236	135	496	323	421	123	273	201	393	68
512	368	311	188	86	355	426	451	245	171	109	289	469	492	223	152	127	333	407	390	267	210	66
514	185	377	305	439	69	359	161	465	241	478	91	298	139	509	219	417	113	337	204	399	264	64
516	341	408	118	260	211	396	315	180	372	356	431	80	237	170	460	297	472	98	217	151	499	62
518	114	328	425	400	270	197	369	309	189	74	353	440	464	250	153	94	287	486	508	226	133	60
520	412	131	324	207	386	274	183	378	306	437	83	347	166	447	254	476	108	283	142	490	235	58
522	279	485	103	234	136	497	322	424	121	278	198	391	302	190	375	357	432	78	251	158	458	56
524	107	292	468	493	224	150	130	331	406	387	265	215	379	312	176	75	351	441	452	248	167	54
526	481	90	296	140	507	220	415	112	340	202	404	261	186	365	316	435	84	348	164	461	242	52
528	344	442	81	252	159	456	293	473	101	216	149	502	339	409	119	259	214	394	320	177	370	50
45	85	354	428	453	246	168	95	290	482	506	229	132	115	329	423	403	268	196	366	307	194	533
44	438	71	358	162	462	243	479	104	284	145	489	233	413	129	325	205	385	277	181	383	303	534
42	276	199	392	301	193	373	362	429	76	239	169	459	294	474	99	230	137	500	321	422	124	536
40	388	266	213	382	310	175	72	349	446	463	249	155	96	288	483	494	227	146	128	334	405	538
38	203	402	262	184	364	319	433	89	345	165	449	253	477	105	285	143	503	221	418	111	338	540
36	231	138	498	335	410	122	258	212	397	318	178	371	343	445	79	257	156	454	281	484	102	542
34	495	225	147	116	332	419	401	271	195	367	308	192	88	352	427	450	244	173	106	291	470	544
32	141	504	222	416	125	326	208	384	275	182	381	304	436	70	361	160	467	240	480	92	295	546
30	238	172	457	299	471	97	218	148	501	336	411	120	272	200	395	300	191	376	360	430	77	548
28	466	247	154	93	286	488	505	228	134	117	330	420	389	269	209	380	313	174	73	350	444	550
26	163	448	256	475	110	282	144	491	232	414	126	327	206	398	263	187	363	317	434	87	346	552
530	511	513	515	517	519	521	523	525	527	529	531	43	41	39	37	35	33	31	29	27	25	46

The above **block-bordered** magic square of order 23 is with inner part as **block-wise pandiagonal** magic square of order 21 with blocks of order 3. The blocks of order 3 are **semi-magic** squares with equal

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semi-magic sums. Below are respective magic sums

$$S_{23\times 23} = 6647$$
$$S_{21\times 21} = 6069$$
$$S_{3\times 3} = 867$$

The total entries sum is $T_{529} := 23 \times 6647 = 152881 = 361^2$. It is **minimum perfect square entries sum** magic square of order 23.

23.4 Pythagorean Triples

According to Examples 23.1, 23.2 and 23.3, we have two perfect square entries sums, i.e., $T_{529} := 12167 \times 23 = 279841 = 529^2$ and $T_{529} := 23 \times 6647 = 152881 = 361^2$. Below are **Pythagorean triples** for the numbers 529 and 361:

$$529^2 + 6072^2 := 6095^2 \qquad 361^2 + 3420^2 := 3439^2$$
$$529^2 + 139920^2 := 139921^2 \qquad 361^2 + 65160^2 := 65161^2$$

Remark 23.1. *The above 4* ***Pythagorean triples*** *are with numbers 529 and 361 These numbers are due to Examples 23.1 and 23.3. According to Result 2, there is not even a single triple generating magic square with* ***perfect square entries sum****. The same already happened with magic square of order 3.*

24 Magic Square of Order 24

According to equation (7), we cannot obtain a **consecutive natural number** magic square where the sum of all the entries is a perfect square. According to equation (3), we can only obtain a magic square of order 24 with this property if we use odd numbers entries. Taking $k = 24$ in equation (3), we get

$$1+3+5+\ldots+(2\times 24^2-1)=24^4$$
$$\Rightarrow 1+3+5+\ldots+1149+1151 \quad =331776=576^2=24^4.$$

Example 24.1. *A* ***block-wise pandiagonal*** *magic square of order 24 for* ***consecutive odd numbers*** *entries* $1,3,5,...,1149,1151$ *is given by*

		13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824
	433	863	1	1007	435	861	3	1005	437	859	5	1003	439	857	7	1001	441	855	9	999	443	853	11	997	13824
13824	143	865	575	721	141	867	573	723	139	869	571	725	137	871	569	727	135	873	567	729	133	875	565	731	13824
13824	1151	145	719	289	1149	147	717	291	1147	149	715	293	1145	151	713	295	1143	153	711	297	1141	155	709	299	13824
13824	577	431	1009	287	579	429	1011	285	581	427	1013	283	583	425	1015	281	585	423	1017	279	587	421	1019	277	13824
13824	445	851	13	995	447	849	15	993	449	847	17	991	451	845	19	989	453	843	21	987	455	841	23	985	13824
13824	131	877	563	733	129	879	561	735	127	881	559	737	125	883	557	739	123	885	555	741	121	887	553	743	13824
13824	1139	157	707	301	1137	159	705	303	1135	161	703	305	1133	163	701	307	1131	165	699	309	1129	167	697	311	13824
13824	589	419	1021	275	591	417	1023	273	593	415	1025	271	595	413	1027	269	597	411	1029	267	599	409	1031	265	13824
13824	457	839	25	983	459	837	27	981	461	835	29	979	463	833	31	977	465	831	33	975	467	829	35	973	13824
13824	119	889	551	745	117	891	549	747	115	893	547	749	113	895	545	751	111	897	543	753	109	899	541	755	13824
13824	1127	169	695	313	1125	171	693	315	1123	173	691	317	1121	175	689	319	1119	177	687	321	1117	179	685	323	13824
13824	601	407	1033	263	603	405	1035	261	605	403	1037	259	607	401	1039	257	609	399	1041	255	611	397	1043	253	13824
13824	469	827	37	971	471	825	39	969	473	823	41	967	475	821	43	965	477	819	45	963	479	817	47	961	13824
13824	107	901	539	757	105	903	537	759	103	905	535	761	101	907	533	763	99	909	531	765	97	911	529	767	13824
13824	1115	181	683	325	1113	183	681	327	1111	185	679	329	1109	187	677	331	1107	189	675	333	1105	191	673	335	13824
13824	613	395	1045	251	615	393	1047	249	617	391	1049	247	619	389	1051	245	621	387	1053	243	623	385	1055	241	13824
13824	481	815	49	959	483	813	51	957	485	811	53	955	487	809	55	953	489	807	57	951	491	805	59	949	13824
13824	95	913	527	769	93	915	525	771	91	917	523	773	89	919	521	775	87	921	519	777	85	923	517	779	13824
13824	1103	193	671	337	1101	195	669	339	1099	197	667	341	1097	199	665	343	1095	201	663	345	1093	203	661	347	13824
13824	625	383	1057	239	627	381	1059	237	629	379	1061	235	631	377	1063	233	633	375	1065	231	635	373	1067	229	13824
13824	493	803	61	947	495	801	63	945	497	799	65	943	499	797	67	941	501	795	69	939	503	793	71	937	13824
13824	83	925	515	781	81	927	513	783	79	929	511	785	77	931	509	787	75	933	507	789	73	935	505	791	13824
13824	1091	205	659	349	1089	207	657	351	1087	209	655	353	1085	211	653	355	1083	213	651	357	1081	215	649	359	13824
13824	637	371	1069	227	639	369	1071	225	641	367	1073	223	643	365	1075	221	645	363	1077	219	647	361	1079	217	13824
	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824

It is **pandiagonal** magic square with magic sum, $S_{24\times 24} = 13824 = 24^3$. The sum of all entries is $T_{576} := 24 \times 13824 = 331776 = 576^2 = 24^4$. Each 4×4 block is a **pandiagonal** magic square of order 4 with the same magic sum $S_{4\times 4} = 2304$. These blocks also have the perfect square entries sum property, i.e., the sum of all entries in each block is $T_{16} := 4 \times 2304 = 9216 = 96^2$. It also satisfy the uniformity property (11), i.e., $\left\langle 24, 24^2, 24^3, 24^4 \right\rangle$.

24.1 Pythagorean Triples

According to Example 24.1, we have perfect square entries sum, i.e., $T_{576} := 24 \times 13824 = 331776 = 576^2$. Below are **Pythagorean triples** for the number 576:

$$576^2 + 68^2 := 580^2$$
$$576^2 + 168^2 := 600^2$$
$$576^2 + 240^2 := 624^2$$
$$576^2 + 350^2 := 674^2$$
$$576^2 + 432^2 := 720^2$$
$$576^2 + 520^2 := 776^2$$
$$576^2 + 660^2 := 876^2$$
$$576^2 + 768^2 := 960^2$$
$$576^2 + 943^2 := 1105^2$$
$$576^2 + 1080^2 := 1224^2$$
$$576^2 + 1232^2 := 1360^2$$
$$576^2 + 1482^2 := 1590^2$$
$$576^2 + 1680^2 := 1776^2$$
$$576^2 + 2268^2 := 2340^2$$
$$576^2 + 2560^2 := 2624^2$$
$$576^2 + 3045^2 := 3099^2$$
$$576^2 + 3432^2 := 3480^2$$
$$576^2 + 4590^2 := 4626^2$$
$$576^2 + 5168^2 := 5200^2$$
$$576^2 + 6900^2 := 6924^2$$
$$576^2 + 9207^2 := 9225^2$$
$$576^2 + 10360^2 := 10376^2$$
$$576^2 + 13818^2 := 13830^2$$
$$576^2 + 20732^2 := 20740^2$$
$$576^2 + 27645^2 := 27651^2$$
$$576^2 + 41470^2 := 41474^2$$

Remark 24.1. *The above 26* ***Pythagorean triples*** *are with number 576. This number is due to Example 24.1. According to Result 2, there are 13 triples generating magic square with* ***perfect square entries sum****:*

$$\begin{aligned}
&(\mathbf{576, 350, 674}) &&\Rightarrow 674 - 350 &&= 324 = 18^2\\
&(\mathbf{576, 432, 720}) &&\Rightarrow 720 - 576 &&= 144 = 12^2\\
&(\mathbf{576, 520, 776}) &&\Rightarrow 776 - 520 &&= 256 = 16^2\\
&(\mathbf{576, 943, 1105}) &&\Rightarrow 1105 - 576 &&= 529 = 23^2\\
&(\mathbf{576, 1080, 1224}) &&\Rightarrow 1224 - 1080 &&= 144 = 12^2\\
&(\mathbf{576, 1232, 1360}) &&\Rightarrow 1360 - 576 &&= 784 = 28^2\\
&(\mathbf{576, 2268, 2340}) &&\Rightarrow 2340 - 576 &&= 1764 = 42^2\\
&(\mathbf{576, 2560, 2624}) &&\Rightarrow 2624 - 2560 &&= 64 = 8^2\\
&(\mathbf{576, 4590, 4626}) &&\Rightarrow 4626 - 4590 &&= 36 = 6^2\\
&(\mathbf{576, 5168, 5200}) &&\Rightarrow 5200 - 576 &&= 4624 = 68^2\\
&(\mathbf{576, 9207, 9225}) &&\Rightarrow 9225 - 576 &&= 8649 = 93^2\\
&(\mathbf{576, 10360, 10376}) &&\Rightarrow 10376 - 10360 &&= 16 = 4^2\\
&(\mathbf{576, 20732, 20740}) &&\Rightarrow 20740 - 576 &&= 20164 = 142^2.
\end{aligned}$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(576, 350, 674)$ *generating a* ***perfect square entries sum*** *magic square of order 18 with the* ***odd number entries***, $\{701, 703, \ldots, 1345, 1347\}$ *with magic sum,* $\boldsymbol{S}_{18\times 18} := 18432$. *The entries total sum is* $\boldsymbol{T}_{324} := 18 \times 18432 = 331776 = 576^2$.

2. *The triple* $(576, 432, 720)$ *generating a* ***perfect square entries sum*** *magic square of order 12 with the* ***odd number entries***, $\{1153, 1155, \ldots, 1437, 1439\}$ *with magic sum,* $\boldsymbol{S}_{12\times 12} := 15552$. *The entries total sum is* $\boldsymbol{T}_{144} := 12 \times 15552 = 186624 = 432^2$.

3. *The triple* $(576, 520, 776)$ *generating a* ***perfect square entries sum*** *magic square of order 16 with the* ***odd number entries***, $\{1041, 1043, \ldots, 1549, 1551\}$ *with magic sum,* $\boldsymbol{S}_{16\times 16} := 20736$. *The entries total sum is* $\boldsymbol{T}_{256} := 16 \times 20736 = 331776 = 576^2$.

4. *The triple* $(576, 943, 1105)$ *generating a* ***perfect square entries sum*** *magic square of order 23 with the* ***odd number entries***, $\{1153, 1155, \ldots, 2207, 2209\}$ *with magic sum,* $\boldsymbol{S}_{23\times 23} := 38663$. *The entries total sum is* $\boldsymbol{T}_{529} := 23 \times 38663 = 889249 = 943^2$.

5. *The triple* $(576, 1080, 1224)$ *generating a* ***perfect square entries sum*** *magic square of order 12 with the* ***odd number entries***, $\{2161, 2163, \ldots, 2445, 2447\}$ *with magic sum,* $\boldsymbol{S}_{12\times 12} := 27648$. *The entries total sum is* $\boldsymbol{T}_{144} := 12 \times 27648 = 331776 = 576^2$.

6. *The triple* $(576, 1232, 1360)$ *generating a* ***perfect square entries sum*** *magic square of order 28 with the* ***odd number entries***, $\{1153, 1155, \ldots, 2717, 2719\}$ *with magic sum,* $\boldsymbol{S}_{28\times 28} := 54208$. *The entries total sum is* $\boldsymbol{T}_{784} := 28 \times 54208 = 1517824 = 1232^2$.

7. *The triple* $(576, 2268, 2340)$ *generating a* ***perfect square entries sum*** *magic square of order 42 with the* ***odd number entries***, $\{1153, 1155, \ldots, 4677, 4679\}$ *with magic sum,* $\boldsymbol{S}_{42\times 42} := 122472$. *The entries total sum is* $\boldsymbol{T}_{1764} := 42 \times 122472 = 5143824 = 2268^2$.

8. *The triple* $(576, 2560, 2624)$ *generating a* ***perfect square entries sum*** *magic square of order 8 with the* ***odd number entries***, $\{5121, 5123, \ldots, 5245, 5247\}$ *with magic sum,* $\boldsymbol{S}_{8\times 8} := 41472$. *The entries total sum is* $\boldsymbol{T}_{64} := 8 \times 41472 = 331776 = 576^2$.

9. *The triple* $(576, 4590, 4626)$ *generating a* ***perfect square entries sum*** *magic square of order 6 with the* ***odd number entries***, $\{9181, 9183, \ldots, 9249, 9251\}$ *with magic sum,* $\boldsymbol{S}_{6\times 6} := 55296$. *The entries total sum is* $\boldsymbol{T}_{36} := 6 \times 55296 = 331776 = 576^2$.

10. *The triple* $(576, 5168, 5200)$ *generating a* ***perfect square entries sum*** *magic square of order 68 with the* ***odd number entries***, $\{1153, 1155, \ldots, 10397, 10399\}$ *with magic sum,* $\boldsymbol{S}_{68\times 68} := 392768$. *The entries total sum is* $\boldsymbol{T}_{4624} := 68 \times 392768 = 26708224 = 5168^2$.

11. *The triple* $(576, 9207, 9225)$ *generating a* ***perfect square entries sum*** *magic square of order 93 with the* ***odd number entries***, $\{1153, 1155, \ldots, 18447, 18449\}$ *with magic sum,* $\boldsymbol{S}_{93\times 93} := 911493$. *The entries total sum is* $\boldsymbol{T}_{8949} := 241 \times 911493 = 84768849 = 9207^2$.

12. *The triple* $(576, 10360, 10376)$ *generating a* ***perfect square entries sum*** *magic square of order 4 with the* ***odd number entries***, $\{20721, 20723, \ldots, 20749, 20751\}$ *with magic sum,* $\boldsymbol{S}_{4\times4} := 82944$. *The entries total sum is* $\boldsymbol{T}_{16} := 4 \times 82944 = 331776 = 576^2$.

13. *The triple* $(576, 20732, 20740)$ *generating a* ***perfect square entries sum*** *magic square of order 142 with the* ***odd number entries***, $\{1153, 1155, \ldots, 41477, 41479\}$ *with magic sum,* $\boldsymbol{S}_{142\times142} = 3026872$. *The entries total sum is* $\boldsymbol{T}_{20164} := 142 \times 3026872 = 429815824 = 20732^2$.

25 Magic Square of Order 25

According to equations (3) and (8), one can obtain **perfect square entries sum magic squares** of order 25 in two ways: first using **consecutive odd numbers**, and the second using **consecutive natural numbers**. For the **consecutive natural numbers** there are again two ways: the first one is of **uniformity**, and the second is of **minimum perfect square entries sum**.

25.1 Consecutive Odd Number Entries

Taking $k = 25$ in equation (3), we get

$$\begin{aligned} 1+3+5+\ldots+(2\times25^2-1) &= 25^4 \\ \Rightarrow 1+3+5+\ldots+1247+1249 &= 390625 = 625^2 = 25^4 \end{aligned}$$

Example 25.1. *For the* ***consecutive odd number*** *entries* $\{1, 3, 5, \ldots, 1247, 1249\}$ *a* ***block-wise*** *magic square of order 21 is given by*

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		15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625
	1	337	613	949	1225	885	1161	247	273	559	469	545	821	1107	183	1093	129	405	731	767	677	953	1039	65	391	15625
15625	913	1249	25	301	637	297	573	859	1185	211	1121	157	483	519	845	705	781	1067	143	429	89	365	691	977	1003	15625
15625	325	601	937	1213	49	1159	235	261	597	873	533	819	1145	171	457	117	443	729	755	1081	991	1027	53	389	665	15625
15625	1237	13	349	625	901	561	897	1173	209	285	195	471	507	833	1119	779	1055	131	417	743	353	689	965	1041	77	15625
15625	649	925	1201	37	313	223	259	585	861	1197	807	1133	169	495	521	431	717	793	1079	105	1015	91	377	653	989	15625
15625	1069	145	421	707	783	693	979	1005	81	367	27	303	639	915	1241	851	1187	213	299	575	485	511	847	1123	159	15625
15625	721	757	1083	119	445	55	381	667	993	1029	939	1215	41	327	603	263	599	875	1151	237	1147	173	459	535	811	15625
15625	133	419	745	771	1057	967	1043	79	355	681	341	627	903	1239	15	1175	201	287	563	899	509	835	1111	197	473	15625
15625	795	1071	107	433	719	379	655	981	1017	93	1203	39	315	641	927	587	863	1199	225	251	161	497	523	809	1135	15625
15625	407	733	769	1095	121	1031	67	393	679	955	615	941	1227	3	339	249	275	551	887	1163	823	1109	185	461	547	15625
15625	877	1153	239	265	591	451	537	813	1149	175	1085	111	447	723	759	669	995	1021	57	383	43	329	605	931	1217	15625
15625	289	565	891	1177	203	1113	199	475	501	837	747	773	1059	135	411	71	357	683	969	1045	905	1231	17	343	629	15625
15625	1191	227	253	589	865	525	801	1137	163	499	109	435	711	797	1073	983	1019	95	371	657	317	643	929	1205	31	15625
15625	553	889	1165	241	277	187	463	549	825	1101	761	1097	123	409	735	395	671	957	1033	69	1229	5	331	617	943	15625
15625	215	291	577	853	1189	849	1125	151	487	513	423	709	785	1061	147	1007	83	369	695	971	631	917	1243	29	305	15625
15625	685	961	1047	73	359	19	345	621	907	1233	893	1179	205	281	567	477	503	839	1115	191	1051	137	413	749	775	15625
15625	97	373	659	985	1011	921	1207	33	319	645	255	581	867	1193	229	1139	165	491	527	803	713	799	1075	101	437	15625
15625	959	1035	61	397	673	333	619	945	1221	7	1167	243	279	555	881	541	827	1103	189	465	125	401	737	763	1099	15625
15625	361	697	973	1009	85	1245	21	307	633	919	579	855	1181	217	293	153	489	515	841	1127	787	1063	149	425	701	15625
15625	1023	59	385	661	997	607	933	1219	45	321	231	267	593	879	1155	815	1141	177	453	539	449	725	751	1087	113	15625
15625	493	529	805	1131	167	1077	103	439	715	791	651	987	1013	99	375	35	311	647	923	1209	869	1195	221	257	583	15625
15625	1105	181	467	543	829	739	765	1091	127	403	63	399	675	951	1037	947	1223	9	335	611	271	557	883	1169	245	15625
15625	517	843	1129	155	481	141	427	703	789	1065	975	1001	87	363	699	309	635	911	1247	23	1183	219	295	571	857	15625
15625	179	455	531	817	1143	753	1089	115	441	727	387	663	999	1025	51	1211	47	323	609	935	595	871	1157	233	269	15625
15625	831	1117	193	479	505	415	741	777	1053	139	1049	75	351	687	963	623	909	1235	11	347	207	283	569	895	1171	15625
	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625

The above magic square is **bimagic** and **pandiagonal** with magic sums: $S_{25\times25} = 15625$ and $Sb_{25\times25} = 13020825$. The blocks of order 5 are **pandiagonal** magic squares with equal magic sums, $S_{5\times5} := 3125$. The total entries sum is $\boldsymbol{T}_{625} := 15625 \times 25 = 390625 = 625^2$. It satisfies the uniformity property (11), i.e., $\left\langle 25, 25^2, 25^3, 25^4 \right\rangle$.

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25.2 Consecutive Natural Number Entries

According to equation (8),

$$G := \frac{n(n+1)}{2} - \frac{(n-625)(n-624)}{2} = 625(n-312)$$

Taking $n = 937$, we get a perfect square, i.e.,

$$G := T(937) - T(312) = \frac{937 \times 938}{2} - \frac{312 \times 313}{2} = 937 \times 469 - 156 \times 313$$
$$= 390625 = 625^2 = 25^4.$$

Simplifying, we get

$$313 + 314 + 315 + \ldots + 936 + 937 = 390625 = 625^2 = 25^4.$$

This gives a perfect square entries sum for the 625 **consecutive natural numbers** from to 313 to 937.

Example 25.2. *For the* ***consecutive natural number*** *entries* $\{313, 314, 315, \ldots, 936, 937\}$ *a* ***block-wise pan-diagonal*** *magic square of order 25 is given by*

		15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625
	313	481	619	787	925	755	893	436	449	592	547	585	723	866	404	859	377	515	678	696	651	789	832	345	508	15625
15625	769	937	325	463	631	461	599	742	905	418	873	391	554	572	735	665	703	846	384	527	357	495	658	801	814	15625
15625	475	613	781	919	337	892	430	443	611	749	579	722	885	398	541	371	534	677	690	853	808	826	339	507	645	15625
15625	931	319	487	625	763	593	761	899	417	455	410	548	566	729	872	702	840	378	521	684	489	657	795	833	351	15625
15625	637	775	913	331	469	424	442	605	743	911	716	879	397	560	573	528	671	709	852	365	820	358	501	639	807	15625
15625	847	385	523	666	704	659	802	815	353	496	326	464	632	770	933	738	906	419	462	600	555	568	736	874	392	15625
15625	673	691	854	372	535	340	503	646	809	827	782	920	333	476	614	444	612	750	888	431	886	399	542	580	718	15625
15625	379	522	685	698	841	796	834	352	490	653	483	626	764	932	320	900	413	456	594	762	567	730	868	411	549	15625
15625	710	848	366	529	672	502	640	803	821	359	914	332	470	633	776	606	744	912	425	438	393	561	574	717	880	15625
15625	516	679	697	860	373	828	346	509	652	790	620	783	926	314	482	437	450	588	756	894	724	867	405	543	586	15625
15625	751	889	432	445	608	538	581	719	887	400	855	368	536	674	692	647	810	823	341	504	334	477	615	778	921	15625
15625	457	595	758	901	414	869	412	550	563	731	686	699	842	380	518	348	491	654	797	835	765	928	321	484	627	15625
15625	908	426	439	607	745	575	713	881	394	562	367	530	668	711	849	804	822	360	498	641	471	634	777	915	328	15625
15625	589	757	895	433	451	406	544	587	725	863	693	861	374	517	680	510	648	791	829	347	927	315	478	621	784	15625
15625	420	458	601	739	907	737	875	388	556	569	524	667	705	843	386	816	354	497	660	798	628	771	934	327	465	15625
15625	655	793	836	349	492	322	485	623	766	929	759	902	415	453	596	551	564	732	870	408	838	381	519	687	700	15625
15625	361	499	642	805	818	773	916	329	472	635	440	603	746	909	427	882	395	558	576	714	669	712	850	363	531	15625
15625	792	830	343	511	649	479	622	785	923	316	896	434	452	590	753	583	726	864	407	545	375	513	681	694	862	15625
15625	493	661	799	817	355	935	323	466	629	772	602	740	903	421	459	389	557	570	733	876	706	844	387	525	663	15625
15625	824	342	505	643	811	616	779	922	335	473	428	446	609	752	890	720	883	401	539	582	537	675	688	856	369	15625
15625	559	577	715	878	396	851	364	532	670	708	638	806	819	362	500	330	468	636	774	917	747	910	423	441	604	15625
15625	865	403	546	584	727	682	695	858	376	514	344	512	650	788	831	786	924	317	480	618	448	591	754	897	435	15625
15625	571	734	877	390	553	383	526	664	707	845	800	813	356	494	662	467	630	768	936	324	904	422	460	598	741	15625
15625	402	540	578	721	884	689	857	370	533	676	506	644	812	825	338	918	336	474	617	780	610	748	891	429	447	15625
15625	728	871	409	552	565	520	683	701	839	382	837	350	488	656	794	624	767	930	318	486	416	454	597	760	898	15625
	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625	15625

The magic squares of order 25 given in Examples 25.1 and 25.2 are **bimagic** and **pandiagonal** with equal magic sums, i.e., $S_{25\times 25} := 15625$, but the **bimgic sums** are different, $Sb^1_{25\times 25} = 13020825$ and $Sb^2_{25\times 25} = 10563175$. The blocks of order 5 are **pandiagonal** magic squares with equal magic sums $S_{5\times 5} := 3125$. The total entries sum is $\boldsymbol{T}_{625} := 15625 \times 25 = 390625 = 625^2$. It satisfies the uniformity property (11), i.e., $\left\langle 25, 25^2, 25^3, 25^4 \right\rangle$.

25.3 Minimum Perfect Square Entries Sum

The examples given in 25.1 and 25.2 satisfy the uniformity property (11), i.e., $\left\langle 25, 25^2, 25^3, 25^4 \right\rangle$. However, the sum of the magic square entries of 25.2 is not a **minimum perfect square entries sum**. Choosing $m = 18, 19, \ldots, 25$ and $p = 25$ in equation (13), we get

1. $L\left(18^2 + \dfrac{25^2-1}{2}, 25^2\right) \rightarrow (25, 12, 636, 8100, 202500)$
2. $L\left(19^2 + \dfrac{25^2-1}{2}, 25^2\right) \rightarrow (25, 49, 673, 9025, 225625)$

$\ldots \qquad \ldots \qquad \ldots$

8. $L\left(25^2 + \dfrac{25^2-1}{2}, 25^2\right) \rightarrow (25, 313, 937, 15625, 390625) \Rightarrow \left\langle 25, 25^2, 25^3, 25^4 \right\rangle$

The values written above are for $m = 18, 19, \ldots, 25$. The 8^{th} value for $m = 25$ satisfy the uniformity property (11). This case is already studied in Example 25.1. The first value for $m = 17$ give **minimum perfect square** magic square. In this case, we have magic square of order 25 with sum of all entries $202500 = 450^2$ which is a **perfect square**, but it does not satisfy the uniformity property (11). The magic square given below has a **minimum perfect square entries sum**.

Example 25.3. *For the* ***consecutive natural number*** *entries* $\{12, 13, 14, \ldots, 635, 636\}$*, a* ***pandiagonal*** *and* ***bimagic*** *magic square of order 25 is given by*

		8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100
	12	180	318	486	624	454	592	135	148	291	246	284	422	565	103	558	76	214	377	395	350	488	531	44	207	8100
8100	468	636	24	162	330	160	298	441	604	117	572	90	253	271	434	364	402	545	83	226	56	194	357	500	513	8100
8100	174	312	480	618	36	591	129	142	310	448	278	421	584	97	240	70	233	376	389	552	507	525	38	206	344	8100
8100	630	18	186	324	462	292	460	598	116	154	109	247	265	428	571	401	539	77	220	383	188	356	494	532	50	8100
8100	336	474	612	30	168	123	141	304	442	610	415	578	96	259	272	227	370	408	551	64	519	57	200	338	506	8100
8100	546	84	222	365	403	358	501	514	52	195	25	163	331	469	632	437	605	118	161	299	254	267	435	573	91	8100
8100	372	390	553	71	234	39	202	345	508	526	481	619	32	175	313	143	311	449	587	130	585	98	241	279	417	8100
8100	78	221	384	397	540	495	533	51	189	352	182	325	463	631	19	599	112	155	293	461	266	429	567	110	248	8100
8100	409	547	65	228	371	201	339	502	520	58	613	31	169	332	475	305	443	611	124	137	92	260	273	416	579	8100
8100	215	378	396	559	72	527	45	208	351	489	319	482	625	13	181	136	149	287	455	593	423	566	104	242	285	8100
8100	450	588	131	144	307	237	280	418	586	99	554	67	235	373	391	346	509	522	40	203	33	176	314	477	620	8100
8100	156	294	457	600	113	568	111	249	262	430	385	398	541	79	217	47	190	353	496	534	464	627	20	183	326	8100
8100	607	125	138	306	444	274	412	580	93	261	66	229	367	410	548	503	521	59	197	340	170	333	476	614	27	8100
8100	288	456	594	132	150	105	243	286	424	562	392	560	73	216	379	209	347	490	528	46	626	14	177	320	483	8100
8100	119	157	300	438	606	436	574	87	255	268	223	366	404	542	85	515	53	196	359	497	327	470	633	26	164	8100
8100	354	492	535	48	191	21	184	322	465	628	458	601	114	152	295	250	263	431	569	107	537	80	218	386	399	8100
8100	60	198	341	504	517	472	615	28	171	334	139	302	445	608	126	581	94	257	275	413	368	411	549	62	230	8100
8100	491	529	42	210	348	178	321	484	622	15	595	133	151	289	452	282	425	563	106	244	74	212	380	393	561	8100
8100	192	360	498	516	54	634	22	165	328	471	301	439	602	120	158	88	256	269	432	575	405	543	86	224	362	8100
8100	523	41	204	342	510	315	478	621	34	172	127	145	308	451	589	419	582	100	238	281	236	374	387	555	68	8100
8100	258	276	414	577	95	550	63	231	369	407	337	505	518	61	199	29	167	335	473	616	446	609	122	140	303	8100
8100	564	102	245	283	426	381	394	557	75	213	43	211	349	487	530	485	623	16	179	317	147	290	453	596	134	8100
8100	270	433	576	89	252	82	225	363	406	544	499	512	55	193	361	166	329	467	635	23	603	121	159	297	440	8100
8100	101	239	277	420	583	388	556	69	232	375	205	343	511	524	37	617	35	173	316	479	309	447	590	128	146	8100
8100	427	570	108	251	264	219	382	400	538	81	536	49	187	355	493	323	466	629	17	185	115	153	296	459	597	8100
	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100	8100

The above Example 25.3 is with magic sum $S_{25\times 25} = 8100$, and the sum of all entries is $T_{625} := 25 \times 8100 = 202500 = 450^2$. This magic square of order 25 is a **minimum perfect square entries sum**. Moreover, the blocks of order 5 are **pandiagonal** magic squares of equal sums, $S_{5\times 5} = 1620$. It is also a **bimagic** square with **bimagic** $Sb_{25\times 25} = 3438200$.

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25.4 Pythagorean Triples

According to Examples 25.1, 25.2 and 25.3, we have two perfect square entries sums, i.e., $T_{625} := 15625 \times 25 = 390625 = 625^2$ and $T_{625} := 25 \times 8100 = 202500 = 450^2$. Below are **Pythagorean triples** for the numbers 625 and 450:

$$
\begin{array}{lll}
\mathbf{625^2} + 1500^2 := 1625^2 & \mathbf{450^2} + 544^2 := 706^2 & \mathbf{450^2} + 3360^2 := 3390^2 \\
\mathbf{625^2} + 7800^2 := 7825^2 & \mathbf{450^2} + 600^2 := 750^2 & \mathbf{450^2} + 5616^2 := 5634^2 \\
\mathbf{625^2} + 39060^2 := 39065^2 & \mathbf{450^2} + 1080^2 := 1170^2 & \mathbf{450^2} + 10120^2 := 10130^2 \\
\mathbf{450^2} + 240^2 := 510^2 & \mathbf{450^2} + 1848^2 := 1902^2 & \mathbf{450^2} + 16872^2 := 16878^2 \\
\mathbf{450^2} + 280^2 := 530^2 & \mathbf{450^2} + 2000^2 := 2050^2 & \mathbf{450^2} + 50624^2 := 50626^2
\end{array}
$$

Remark 25.1. *The above 15* ***Pythagorean triples*** *are with numbers 625 and 450. These numbers are due to Examples 25.1 and 25.3. According to Result 2, there are 5 triples generating magic square with* ***perfect square entries sum****:*

$$
\begin{array}{ll}
(\mathbf{625, 7800, 7825}) & \Rightarrow 7825 - 7800 = 25 = 5^2 \\
(\mathbf{450, 544, 706}) & \Rightarrow 706 - 450 = 256 = 16^2 \\
(\mathbf{450, 2000, 2050}) & \Rightarrow 2050 - 450 = 1600 = 40^2 \\
(\mathbf{450, 5616, 5634}) & \Rightarrow 5634 - 450 = 5184 = 72^2 \\
(\mathbf{450, 50624, 50626}) & \Rightarrow 50626 - 450 = 50176 = 224^2.
\end{array}
$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(625, 7800, 7825)$ *generating a* ***perfect square entries sum*** *magic square of order 5 with the* ***odd number entries****,* $\{15601, 15603, \ldots, 15647, 15649\}$ *with magic sum,* $S_{5\times5} := 78125$*. The entries total sum is* $T_{25} := 5 \times 78125 = 390625 = 625^2$.
2. *The triple* $(450, 544, 706)$ *generating a* ***perfect square entries sum*** *magic square of order 16 with the* ***odd***

number entries, $\{901, 903, \ldots, 1409, 1411\}$ *with magic sum*, $S_{16\times16} := 18496$. *The entries total sum is* $T_{256} := 16 \times 18496 = 295936 = 544^2$.

3. *The triple* $(450, 2000, 2050)$ *generating a* ***perfect square entries sum*** *magic square of order 40 with the* ***odd number entries***, $\{901, 903, \ldots, 4097, 4099\}$ *with magic sum*, $S_{40\times40} := 100000$. *The entries total sum is* $T_{1600} := 40 \times 100000 = 4000000 = 2000^2$.

4. *The triple* $(450, 5616, 5634)$ *generating a* ***perfect square entries sum*** *magic square of order 72 with the* ***odd number entries***, $\{901, 903, \ldots, 11265, 11267\}$ *with magic sum*, $S_{72\times72} := 438048$. *The entries total sum is* $T_{5184} := 72 \times 438048 = 31539456 = 5616^2$.

5. *The triple* $(450, 50624, 50626)$ *generating a* ***perfect square entries sum*** *magic square of order 224 with the* ***odd number entries***, $\{901, 903, \ldots, 101249, 101251\}$ *with magic sum*, $S_{224\times224} := 27648$. *The entries total sum is* $T_{50176} := 224 \times 11441024 = 2562789376 = 50624^2$.

26 Magic Square of Order 26

According to equation (7), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (3), we can only obtain a magic square of order 26 with this property if we use odd number entries. Taking $k = 26$ in equation (3), we get

$$\begin{aligned} 1 + 3 + 5 + \ldots + (2 \times 26^2 - 1) &= 26^4 \\ \Rightarrow 1 + 3 + 5 + \ldots + 1349 + 1351 &= 456976 = 676^2 = 26^4. \end{aligned}$$

Example 26.1. *For the* ***consecutive odd number*** *entries* $\{1, 3, 5, \ldots, 1349, 1351\}$, *a* ***block-bordered*** *magic square of order 26 is given by*

Generating Pythagorean Triples and Magic Squares: Orders 3 to 31,
May 28, 2021, pp. 1-153, *https://doi.org/10.5281/zenodo.4837491*

49	1325	29	1321	33	1317	37	1313	41	1309	45	1305	1	1327	53	1297	57	1293	61	1289	65	1285	69	1281	73	1301
77	533	963	101	1107	535	961	103	1105	537	959	105	1103	539	957	107	1101	541	955	109	1099	543	953	111	1097	1275
1273	243	965	675	821	241	967	673	823	239	969	671	825	237	971	669	827	235	973	667	829	233	975	665	831	79
81	1251	245	819	389	1249	247	817	391	1247	249	815	393	1245	251	813	395	1243	253	811	397	1241	255	809	399	1271
1269	677	531	1109	387	679	529	1111	385	681	527	1113	383	683	525	1115	381	685	523	1117	379	687	521	1119	377	83
85	545	951	113	1095	547	949	115	1093	549	947	117	1091	551	945	119	1089	553	943	121	1087	555	941	123	1085	1267
1265	231	977	663	833	229	979	661	835	227	981	659	837	225	983	657	839	223	985	655	841	221	987	653	843	87
89	1239	257	807	401	1237	259	805	403	1235	261	803	405	1233	263	801	407	1231	265	799	409	1229	267	797	411	1263
1261	689	519	1121	375	691	517	1123	373	693	515	1125	371	695	513	1127	369	697	511	1129	367	699	509	1131	365	91
93	557	939	125	1083	559	937	127	1081	561	935	129	1079	563	933	131	1077	565	931	133	1075	567	929	135	1073	1259
1257	219	989	651	845	217	991	649	847	215	993	647	849	213	995	645	851	211	997	643	853	209	999	641	855	95
97	1227	269	795	413	1225	271	793	415	1223	273	791	417	1221	275	789	419	1219	277	787	421	1217	279	785	423	1255
1253	701	507	1133	363	703	505	1135	361	705	503	1137	359	707	501	1139	357	709	499	1141	355	711	497	1143	353	99
1277	569	927	137	1071	571	925	139	1069	573	923	141	1067	575	921	143	1065	577	919	145	1063	579	917	147	1061	75
1329	207	1001	639	857	205	1003	637	859	203	1005	635	861	201	1007	633	863	199	1009	631	865	197	1011	629	867	23
21	1215	281	783	425	1213	283	781	427	1211	285	779	429	1209	287	777	431	1207	289	775	433	1205	291	773	435	1331
1333	713	495	1145	351	715	493	1147	349	717	491	1149	347	719	489	1151	345	721	487	1153	343	723	485	1155	341	19
17	581	915	149	1059	583	913	151	1057	585	911	153	1055	587	909	155	1053	589	907	157	1051	591	905	159	1049	1335
1337	195	1013	627	869	193	1015	625	871	191	1017	623	873	189	1019	621	875	187	1021	619	877	185	1023	617	879	15
13	1203	293	771	437	1201	295	769	439	1199	297	767	441	1197	299	765	443	1195	301	763	445	1193	303	761	447	1339
1341	725	483	1157	339	727	481	1159	337	729	479	1161	335	731	477	1163	333	733	475	1165	331	735	473	1167	329	11
9	593	903	161	1047	595	901	163	1045	597	899	165	1043	599	897	167	1041	601	895	169	1039	603	893	171	1037	1343
1345	183	1025	615	881	181	1027	613	883	179	1029	611	885	177	1031	609	887	175	1033	607	889	173	1035	605	891	7
5	1191	305	759	449	1189	307	757	451	1187	309	755	453	1185	311	753	455	1183	313	751	457	1181	315	749	459	1347
1349	737	471	1169	327	739	469	1171	325	741	467	1173	323	743	465	1175	321	745	463	1177	319	747	461	1179	317	3
51	27	1323	31	1319	35	1315	39	1311	43	1307	47	1351	25	1299	55	1295	59	1291	63	1287	67	1283	71	1279	1303

The above **block-bordered** magic square of order 26 is with inner part as **block-wise pandiagonal** magic square of order 24. The blocks of order 4 are **pandiagonal** magic squares with equal magic sums.

Below are respective magic sums:

$$S_{26\times 26} = 17576$$
$$S_{20\times 20} = 16224$$
$$S_{4\times 4} = 2704$$

The total entries sum is $T_{676} := 17576 \times 26 = 456976 = 676^2 = 26^4$. It satisfies the uniformity property (11), i.e., $\left\langle 26, 26^2, 26^3, 26^4 \right\rangle$.

26.1 Pythagorean Triples

According to Example 26.1, we have perfect square entries sum, i.e., $T_{676} := 17576 \times 26 = 456976 = 676^2$. Below are **Pythagorean triples** for the number 676:

$$676^2 + 507^2 := 845^2 \qquad 676^2 + 8775^2 := 8801^2$$
$$676^2 + 2145^2 := 2249^2 \qquad 676^2 + 28557^2 := 28565^2$$
$$676^2 + 4368^2 := 4420^2 \qquad 676^2 + 57120^2 := 57124^2$$

Remark 26.1. *The above 6* ***Pythagorean triples*** *are with number 676. This number is due to Example 26.1. According to Result 2, there are only 2 triples generating magic square with* ***perfect square entries sum****:*

$$(676, 507, 845) \Rightarrow 845 - 676 = 169 = 13^2$$
$$(676, 28557, 28565) \Rightarrow 28565 - 676 = 27889 = 167^2.$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(676, 507, 845)$ *generating a* ***perfect square entries sum*** *magic square of order 13 with the* ***odd number entries***, $\{1353, 1355, \ldots, 1687, 1689\}$ *with magic sum,* $S_{13\times 13} := 19773$*. The entries total sum is* $T_{169} := 13 \times 19773 = 257049 = 507^2$.

2. *The triple* $(676, 28557, 28565)$ *generating a* ***perfect square entries sum*** *magic square of order 12 with the* ***odd number entries****,* $\{1353, 1355, \ldots, 57127, 57129\}$ *with magic sum,* $\boldsymbol{S}_{167\times167} := 4883247$*. The entries total sum is* $\boldsymbol{T}_{27889} := 167 \times 4883247 = 815502249 = 28557^2$.

27 Magic Square of Order 27

According to equations (3) and (8), one can obtain **perfect square entries sum magic squares** of order 27 in two ways: first using **consecutive odd numbers**, and the second using **consecutive natural numbers**. For the **consecutive natural numbers** there are again two ways: the first one is of **uniformity**, and the second is of **minimum perfect square entries sum**.

27.1 Consecutive Odd Number Entries

Taking $k = 27$ in equation (3), we get

$$1 + 3 + 5 + \ldots + (2 \times 27^2 - 1) = 27^4$$
$$\Rightarrow 1 + 3 + 5 + \ldots + 1455 + 1457 = 531441 = 729^2 = 27^4$$

Example 27.1. *For the* ***consecutive odd number*** *entries* $\{1, 3, 5, \ldots, 1455, 1457\}$ *a* ***block-wise pandigonal*** *magic square of order 27 is given by*

		19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683
	1243	897	47	1271	867	49	1251	901	35	1273	875	39	1283	877	27	1285	887	15	1259	909	19	1293	889	5	1263	917	7	19683
19683	33	1289	865	3	1291	893	37	1277	873	11	1281	895	13	1269	905	23	1257	907	45	1261	881	25	1247	915	53	1249	885	19683
19683	911	1	1275	913	29	1245	899	9	1279	903	31	1253	891	41	1255	879	43	1265	883	17	1287	869	51	1267	871	21	1295	19683
19683	1297	87	803	1325	57	805	1305	91	791	1327	65	795	1337	67	783	1339	77	771	1313	99	775	1347	79	761	1317	107	763	19683
19683	789	1343	55	759	1345	83	793	1331	63	767	1335	85	769	1323	95	779	1311	97	801	1315	71	781	1301	105	809	1303	75	19683
19683	101	757	1329	103	785	1299	89	765	1333	93	787	1307	81	797	1309	69	799	1319	73	773	1341	59	807	1321	61	777	1349	19683
19683	919	1005	263	947	975	265	927	1009	251	949	983	255	959	985	243	961	995	231	935	1017	235	969	997	221	939	1025	223	19683
19683	249	965	973	219	967	1001	253	953	981	227	957	1003	229	945	1013	239	933	1015	261	937	989	241	923	1023	269	925	993	19683
19683	1019	217	951	1021	245	921	1007	225	955	1011	247	929	999	257	931	987	259	941	991	233	963	977	267	943	979	237	971	19683
19683	1027	303	857	1055	273	859	1035	307	845	1057	281	849	1067	283	837	1069	293	825	1043	315	829	1077	295	815	1047	323	817	19683
19683	843	1073	271	813	1075	299	847	1061	279	821	1065	301	823	1053	311	833	1041	313	855	1045	287	835	1031	321	863	1033	291	19683
19683	317	811	1059	319	839	1029	305	819	1063	309	841	1037	297	851	1039	285	853	1049	289	827	1071	275	861	1051	277	831	1079	19683
19683	703	357	1127	731	327	1129	711	361	1115	733	335	1119	743	337	1107	745	347	1095	719	369	1099	753	349	1085	723	377	1087	19683
19683	1113	749	325	1083	751	353	1117	737	333	1091	741	355	1093	729	365	1103	717	367	1125	721	341	1105	707	375	1133	709	345	19683
19683	371	1081	735	373	1109	705	359	1089	739	363	1111	713	351	1121	715	339	1123	725	343	1097	747	329	1131	727	331	1101	755	19683
19683	379	627	1181	407	597	1183	387	631	1169	409	605	1173	419	607	1161	421	617	1149	395	639	1153	429	619	1139	399	647	1141	19683
19683	1167	425	595	1137	427	623	1171	413	603	1145	417	625	1147	405	635	1157	393	637	1179	397	611	1159	383	645	1187	385	615	19683
19683	641	1135	411	643	1163	381	629	1143	415	633	1165	389	621	1175	391	609	1177	401	613	1151	423	599	1185	403	601	1155	431	19683
19683	487	1221	479	515	1191	481	495	1225	467	517	1199	471	527	1201	459	529	1211	447	503	1233	451	537	1213	437	507	1241	439	19683
19683	465	533	1189	435	535	1217	469	521	1197	443	525	1219	445	513	1229	455	501	1231	477	505	1205	457	491	1239	485	493	1209	19683
19683	1235	433	519	1237	461	489	1223	441	523	1227	463	497	1215	473	499	1203	475	509	1207	449	531	1193	483	511	1195	453	539	19683
19683	109	681	1397	137	651	1399	117	685	1385	139	659	1389	149	661	1377	151	671	1365	125	693	1369	159	673	1355	129	701	1357	19683
19683	1383	155	649	1353	157	677	1387	143	657	1361	147	679	1363	135	689	1373	123	691	1395	127	665	1375	113	699	1403	115	669	19683
19683	695	1351	141	697	1379	111	683	1359	145	687	1381	119	675	1391	121	663	1393	131	667	1367	153	653	1401	133	655	1371	161	19683
19683	163	1437	587	191	1407	589	171	1441	575	193	1415	579	203	1417	567	205	1427	555	179	1449	559	213	1429	545	183	1457	547	19683
19683	573	209	1405	543	211	1433	577	197	1413	551	201	1435	553	189	1445	563	177	1447	585	181	1421	565	167	1455	593	169	1425	19683
19683	1451	541	195	1453	569	165	1439	549	199	1443	571	173	1431	581	175	1419	583	185	1423	557	207	1409	591	187	1411	561	215	19683
	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683

The above magic square is **pandiagonal** with magic sum $S_{27\times 27} = 19683$. The blocks of order 3 are **semi-magic** squares with equal sum **semi-magic** sums, $S_{3\times 3} := 2187$. The total entries sum is $T_{729} := 19683 \times 27 = 531441 = 729^2$. It satisfies the uniformity property (11), i.e., $\left\langle 27, 27^2, 27^3, 27^4 \right\rangle$.

27.2 Consecutive Natural Number Entries

According to equation (8),

$$G := \frac{n(n+1)}{2} - \frac{(n-729)(n-728)}{2} = 729(n-364)$$

Taking $n = 1093$, we get a perfect square, i.e.,

$$G := T(1093) - T(364) = \frac{1093 \times 1094}{2} - \frac{364 \times 365}{2} = 1093 \times 547 - 182 \times 365$$
$$= 531441 = 729^2 = 27^4.$$

Simplifying, we get

$$365 + 366 + 367 + \ldots + 1092 + 1093 = 531441 = 729^2 = 27^4.$$

This gives a perfect square entries sum for the 7291 **consecutive natural numbers** from to 365 to 1093.

Example 27.2. *For the* ***consecutive natural number*** *entries* $\{365, 366, 367, \ldots, 1092, 1093\}$ *a* ***block-wise pandiagonal*** *magic square of order 21 is given by*

		19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683
	986	813	388	1000	798	389	990	815	382	1001	802	384	1006	803	378	1007	808	372	994	819	374	1011	809	367	996	823	368	19683
19683	381	1009	797	366	1010	811	383	1003	801	370	1005	812	371	999	817	376	993	818	387	995	805	377	988	822	391	989	807	19683
19683	820	365	1002	821	379	987	814	369	1004	816	380	991	810	385	992	804	386	997	806	373	1008	799	390	998	800	375	1012	19683
19683	1013	408	766	1027	393	767	1017	410	760	1028	397	762	1033	398	756	1034	403	750	1021	414	752	1038	404	745	1023	418	746	19683
19683	759	1036	392	744	1037	406	761	1030	396	748	1032	407	749	1026	412	754	1020	413	765	1022	400	755	1015	417	769	1016	402	19683
19683	415	743	1029	416	757	1014	409	747	1031	411	758	1018	405	763	1019	399	764	1024	401	751	1035	394	768	1025	395	753	1039	19683
19683	824	867	496	838	852	497	828	869	490	839	856	492	844	857	486	845	862	480	832	873	482	849	863	475	834	877	476	19683
19683	489	847	851	474	848	865	491	841	855	478	843	866	479	837	871	484	831	872	495	833	859	485	826	876	499	827	861	19683
19683	874	473	840	875	487	825	868	477	842	870	488	829	864	493	830	858	494	835	860	481	846	853	498	836	854	483	850	19683
19683	878	516	793	892	501	794	882	518	787	893	505	789	898	506	783	899	511	777	886	522	779	903	512	772	888	526	773	19683
19683	786	901	500	771	902	514	788	895	504	775	897	515	776	891	520	781	885	521	792	887	508	782	880	525	796	881	510	19683
19683	523	770	894	524	784	879	517	774	896	519	785	883	513	790	884	507	791	889	509	778	900	502	795	890	503	780	904	19683
19683	716	543	928	730	528	929	720	545	922	731	532	924	736	533	918	737	538	912	724	549	914	741	539	907	726	553	908	19683
19683	921	739	527	906	740	541	923	733	531	910	735	542	911	729	547	916	723	548	927	725	535	917	718	552	931	719	537	19683
19683	550	905	732	551	919	717	544	909	734	546	920	721	540	925	722	534	926	727	536	913	738	529	930	728	530	915	742	19683
19683	554	678	955	568	663	956	558	680	949	569	667	951	574	668	945	575	673	939	562	684	941	579	674	934	564	688	935	19683
19683	948	577	662	933	578	676	950	571	666	937	573	677	938	567	682	943	561	683	954	563	670	944	556	687	958	557	672	19683
19683	685	932	570	686	946	555	679	936	572	681	947	559	675	952	560	669	953	565	671	940	576	664	957	566	665	942	580	19683
19683	608	975	604	622	960	605	612	977	598	623	964	600	628	965	594	629	970	588	616	981	590	633	971	583	618	985	584	19683
19683	597	631	959	582	632	973	599	625	963	586	627	974	587	621	979	592	615	980	603	617	967	593	610	984	607	611	969	19683
19683	982	581	624	983	595	609	976	585	626	978	596	613	972	601	614	966	602	619	968	589	630	961	606	620	962	591	634	19683
19683	419	705	1063	433	690	1064	423	707	1057	434	694	1059	439	695	1053	440	700	1047	427	711	1049	444	701	1042	429	715	1043	19683
19683	1056	442	689	1041	443	703	1058	436	693	1045	438	704	1046	432	709	1051	426	710	1062	428	697	1052	421	714	1066	422	699	19683
19683	712	1040	435	713	1054	420	706	1044	437	708	1055	424	702	1060	425	696	1061	430	698	1048	441	691	1065	431	692	1050	445	19683
19683	446	1083	658	460	1068	659	450	1085	652	461	1072	654	466	1073	648	467	1078	642	454	1089	644	471	1079	637	456	1093	638	19683
19683	651	469	1067	636	470	1081	653	463	1071	640	465	1082	641	459	1087	646	453	1088	657	455	1075	647	448	1092	661	449	1077	19683
19683	1090	635	462	1091	649	447	1084	639	464	1086	650	451	1080	655	452	1074	656	457	1076	643	468	1069	660	458	1070	645	472	19683
	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683	19683

The magic squares of order 27 given in Examples 27.1 and 27.2 are with equal magic sums, i.e., $S_{27\times 27} := 19683$. Blocks of order 3 are **semi-magic** squares with equal sum **semi-magic** sums, $S_{3\times 3} := 2187$. The total entries sum is $T_{729} := 19683 \times 27 = 531441 = 729^2$. It satisfies the uniformity property (11), i.e., $\left\langle 27, 27^2, 27^3, 27^4 \right\rangle$.

27.3 Minimum Perfect Square Entries Sum

The examples given in 27.1 and 27.2 satisfy the uniformity property (11), i.e., $\left\langle 27, 27^2, 27^3, 27^4 \right\rangle$. However, the sum of the magic square entries of 27.2 is not a **minimum perfect square entries sum**. Choosing $m = 20, 21, \ldots, 27$ and $p = 27$ in equation (13), we get

$$1. \quad L\left(20^2 + \frac{27^2 - 1}{2}, 27^2\right) \rightarrow (27, 36, 764, 10800, 291600)$$

$$2. \quad L\left(21^2 + \frac{27^2 - 1}{2}, 27^2\right) \rightarrow (27, 77, 805, 11907, 321489)$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$8. \quad L\left(27^2 + \frac{27^2 - 1}{2}, 27^2\right) \rightarrow (27, 365, 1093, 19683, 531441) \Rightarrow \left\langle 27, 27^2, 27^3, 27^4 \right\rangle$$

The values written above are for $m = 20, 21, \ldots, 27$. The 8^{th} value for $m = 27$ satisfy the uniformity property (11). This case is already studied in Example 27.1. The first value for $m = 20$ give **minimum perfect square** magic square. In this case, we have magic square of order 27 with sum of all entries $291600 = 540^2$ which is a perfect square, but it does not satisfy the uniformity property (11). The magic square given below has a **minimum perfect square entries sum**.

Example 27.3. *For the* ***consecutive natural number*** *entries* $\{36, 37, 38, \ldots, 763, 764\}$, *a* ***pandiagonal*** *magic square of order 27 is given by*

		10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800
	657	484	59	671	469	60	661	486	53	672	473	55	677	474	49	678	479	43	665	490	45	682	480	38	667	494	39	10800
10800	52	680	468	37	681	482	54	674	472	41	676	483	42	670	488	47	664	489	58	666	476	48	659	493	62	660	478	10800
10800	491	36	673	492	50	658	485	40	675	487	51	662	481	56	663	475	57	668	477	44	679	470	61	669	471	46	683	10800
10800	684	79	437	698	64	438	688	81	431	699	68	433	704	69	427	705	74	421	692	85	423	709	75	416	694	89	417	10800
10800	430	707	63	415	708	77	432	701	67	419	703	78	420	697	83	425	691	84	436	693	71	426	686	88	440	687	73	10800
10800	86	414	700	87	428	685	80	418	702	82	429	689	76	434	690	70	435	695	72	422	706	65	439	696	66	424	710	10800
10800	495	538	167	509	523	168	499	540	161	510	527	163	515	528	157	516	533	151	503	544	153	520	534	146	505	548	147	10800
10800	160	518	522	145	519	536	162	512	526	149	514	537	150	508	542	155	502	543	166	504	530	156	497	547	170	498	532	10800
10800	545	144	511	546	158	496	539	148	513	541	159	500	535	164	501	529	165	506	531	152	517	524	169	507	525	154	521	10800
10800	549	187	464	563	172	465	553	189	458	564	176	460	569	177	454	570	182	448	557	193	450	574	183	443	559	197	444	10800
10800	457	572	171	442	573	185	459	566	175	446	568	186	447	562	191	452	556	192	463	558	179	453	551	196	467	552	181	10800
10800	194	441	565	195	455	550	188	445	567	190	456	554	184	461	555	178	462	560	180	449	571	173	466	561	174	451	575	10800
10800	387	214	599	401	199	600	391	216	593	402	203	595	407	204	589	408	209	583	395	220	585	412	210	578	397	224	579	10800
10800	592	410	198	577	411	212	594	404	202	581	406	213	582	400	218	587	394	219	598	396	206	588	389	223	602	390	208	10800
10800	221	576	403	222	590	388	215	580	405	217	591	392	211	596	393	205	597	398	207	584	409	200	601	399	201	586	413	10800
10800	225	349	626	239	334	627	229	351	620	240	338	622	245	339	616	246	344	610	233	355	612	250	345	605	235	359	606	10800
10800	619	248	333	604	249	347	621	242	337	608	244	348	609	238	353	614	232	354	625	234	341	615	227	358	629	228	343	10800
10800	356	603	241	357	617	226	350	607	243	352	618	230	346	623	231	340	624	236	342	611	247	335	628	237	336	613	251	10800
10800	279	646	275	293	631	276	283	648	269	294	635	271	299	636	265	300	641	259	287	652	261	304	642	254	289	656	255	10800
10800	268	302	630	253	303	644	270	296	634	257	298	645	258	292	650	263	286	651	274	288	638	264	281	655	278	282	640	10800
10800	653	252	295	654	266	280	647	256	297	649	267	284	643	272	285	637	273	290	639	260	301	632	277	291	633	262	305	10800
10800	90	376	734	104	361	735	94	378	728	105	365	730	110	366	724	111	371	718	98	382	720	115	372	713	100	386	714	10800
10800	727	113	360	712	114	374	729	107	364	716	109	375	717	103	380	722	97	381	733	99	368	723	92	385	737	93	370	10800
10800	383	711	106	384	725	91	377	715	108	379	726	95	373	731	96	367	732	101	369	719	112	362	736	102	363	721	116	10800
10800	117	754	329	131	739	330	121	756	323	132	743	325	137	744	319	138	749	313	125	760	315	142	750	308	127	764	309	10800
10800	322	140	738	307	141	752	324	134	742	311	136	753	312	130	758	317	124	759	328	126	746	318	119	763	332	120	748	10800
10800	761	306	133	762	320	118	755	310	135	757	321	122	751	326	123	745	327	128	747	314	139	740	331	129	741	316	143	10800
	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800	10800

The above Example 27.3 is with magic sum $S_{27\times 27} = 10800$, and the sum of all entries is $T_{729} := 27 \times 10800 = 291600 = 540^2$. This magic square of order 27 is a **minimum perfect square entries sum**. The blocks of order 3 are **semi-magic** squares of order 3 with equal **semi-magic** sums, $Sm_{3\times 3} = 1200$. It is also **perfect square entries sum**, i.e. $T_9 := 3 \times 1200 = 3600 = 60^2$.

27.4 Pythagorean Triples

According to Examples 27.1, 27.2 and 27.3, we have two perfect square entries sums, i.e., $T_{729} := 19683 \times 27 = 531441 = 729^2$ and $T_{729} := 27 \times 10800 = 291600 = 540^2$. Below are **Pythagorean triples** for the numbers 729 and 540:

$$
\begin{array}{lll}
729^2 + 972^2 := 1215^2 & 540^2 + 629^2 := 829^2 & 540^2 + 2891^2 := 2941^2 \\
729^2 + 3240^2 := 3321^2 & 540^2 + 720^2 := 900^2 & 540^2 + 3625^2 := 3665^2 \\
729^2 + 9828^2 := 9855^2 & 540^2 + 819^2 := 981^2 & 540^2 + 4032^2 := 4068^2 \\
729^2 + 29520^2 := 29529^2 & 540^2 + 897^2 := 1047^2 & 540^2 + 4845^2 := 4875^2 \\
729^2 + 88572^2 := 88575^2 & 540^2 + 1155^2 := 1275^2 & 540^2 + 6063^2 := 6087^2 \\
540^2 + 57^2 := 543^2 & 540^2 + 1296^2 := 1404^2 & 540^2 + 7280^2 := 7300^2 \\
540^2 + 99^2 := 549^2 & 540^2 + 1408^2 := 1508^2 & 540^2 + 8091^2 := 8109^2 \\
540^2 + 225^2 := 585^2 & 540^2 + 1575^2 := 1665^2 & 540^2 + 12144^2 := 12156^2 \\
540^2 + 288^2 := 612^2 & 540^2 + 1989^2 := 2061^2 & 540^2 + 14575^2 := 14585^2 \\
540^2 + 336^2 := 636^2 & 540^2 + 2400^2 := 2460^2 & 540^2 + 24297^2 := 24303^2 \\
540^2 + 405^2 := 675^2 & 540^2 + 2673^2 := 2727^2 & 540^2 + 36448^2 := 36452^2 \\
540^2 + 567^2 := 783^2 & & 540^2 + 72899^2 := 72901^2
\end{array}
$$

Remark 27.1. *The above 35* ***Pythagorean triples*** *are with numbers 729 and 540. These numbers are due to Examples 27.1 and 27.3. According to Result 2, there are 13 triples generating magic square with* ***perfect square entries sum****:*

$$\begin{aligned}
&(\mathbf{729, 3240, 3321}) &&\Rightarrow 3321 - 3240 &&= 81 = 9^2\\
&(\mathbf{729, 29520, 29529}) &&\Rightarrow 29529 - 29520 &&= 9 = 3^2\\
&(\mathbf{540, 99, 549}) &&\Rightarrow 549 - 540 &&= 9 = 3^2\\
&(\mathbf{540, 288, 612}) &&\Rightarrow 612 - 288 &&= 324 = 18^2\\
&(\mathbf{540, 629, 829}) &&\Rightarrow 829 - 540 &&= 289 = 17^2\\
&(\mathbf{540, 819, 981}) &&\Rightarrow 981 - 540 &&= 441 = 21^2\\
&(\mathbf{540, 1408, 1508}) &&\Rightarrow 1508 - 1408 &&= 100 = 10^2\\
&(\mathbf{540, 1989, 2061}) &&\Rightarrow 2061 - 540 &&= 1521 = 39^2\\
&(\mathbf{540, 2891, 2941}) &&\Rightarrow 2941 - 540 &&= 2401 = 49^2\\
&(\mathbf{540, 4032, 4068}) &&\Rightarrow 4068 - 4032 &&= 36 = 6^2\\
&(\mathbf{540, 8091, 8109}) &&\Rightarrow 8109 - 540 &&= 7569 = 87^2\\
&(\mathbf{540, 18221, 18229}) &&\Rightarrow 18229 - 540 &&= 17689 = 133^2\\
&(\mathbf{540, 72899, 72901}) &&\Rightarrow 72901 - 540 &&= 72361 = 269^2.
\end{aligned}$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(729, 3240, 3321)$ *generating a* ***perfect square entries sum*** *magic square of order 9 with the* ***odd number entries***, $\{6481, 6483, \ldots, 6639, 6641\}$ *with magic sum,* $S_{9\times 9} := 59049$. *The entries total sum is* $T_{81} := 9 \times 59049 = 531441 = 729^2$.

2. *The triple* $(729, 29520, 29529)$ *generating a* ***perfect square entries sum*** *magic square of order 3 with the* ***odd number entries***, $\{59041, 59043, \ldots, 59055, 59057\}$ *with magic sum,* $S_{3\times 3} := 177147$. *The entries total sum is* $T_9 := 3 \times 177147 = 531441 = 729^2$.

3. *The triple* $(540, 99, 549)$ *generating a* ***perfect square entries sum*** *magic square of order 3 with the* ***odd number entries***, $\{1081, 1083, \ldots, 1095, 1097\}$ *with magic sum,* $S_{3\times 3} := 3267$. *The entries total sum is* $T_9 := 3 \times 3267 = 9801 = 99^2$.

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4. *The triple* $(540, 288, 612)$ *generating a* ***perfect square entries sum*** *magic square of order 18 with the* ***odd number entries***, $\{577, 579, \ldots, 1221, 1223\}$ *with magic sum,* $S_{18\times 18} := 16200$. *The entries total sum is* $T_{324} := 18 \times 16200 = 291600 = 540^2$.

5. *The triple* $(540, 629, 829)$ *generating a* ***perfect square entries sum*** *magic square of order 17 with the* ***odd number entries***, $\{1081, 1083, \ldots, 1655, 1657\}$ *with magic sum,* $S_{17\times 17} := 23273$. *The entries total sum is* $T_{289} := 17 \times 23273 = 395641 = 629^2$.

6. *The triple* $(540, 819, 981)$ *generating a* ***perfect square entries sum*** *magic square of order 21 with the* ***odd number entries***, $\{1081, 1083, \ldots, 1959, 1961\}$ *with magic sum,* $S_{21\times 21} := 31941$. *The entries total sum is* $T_{441} := 21 \times 31941 = 670761 = 819^2$.

7. *The triple* $(540, 1408, 1508)$ *generating a* ***perfect square entries sum*** *magic square of order 10 with the* ***odd number entries***, $\{2817, 2819, \ldots, 3013, 3015\}$ *with magic sum,* $S_{10\times 10} := 29160$. *The entries total sum is* $T_{100} := 10 \times 29160 = 291600 = 540^2$.

8. *The triple* $(540, 1989, 2061)$ *generating a* ***perfect square entries sum*** *magic square of order 39 with the* ***odd number entries***, $\{1081, 1083, \ldots, 4119, 4121\}$ *with magic sum,* $S_{39\times 39} := 101439$. *The entries total sum is* $T_{1521} := 39 \times 101439 = 3956121 = 1989^2$.

9. *The triple* $(540, 2891, 2941)$ *generating a* ***perfect square entries sum*** *magic square of order 49 with the* ***odd number entries***, $\{1081, 1083, \ldots, 5879, 5881\}$ *with magic sum,* $S_{49\times 49} := 170569$. *The entries total sum is* $T_{2401} := 49 \times 170569 = 8357881 = 2891^2$.

10. *The triple* $(540, 4032, 4068)$ *generating a* ***perfect square entries sum*** *magic square of order 6 with the* ***odd number entries***, $\{8065, 8067, 8133, \ldots, 8135\}$ *with magic sum,* $S_{6\times 6} := 48600$. *The entries total sum is* $T_{36} := 6 \times 48600 = 291600 = 540^2$.

11. *The triple* $(540, 8091, 8109)$ *generating a* ***perfect square entries sum*** *magic square of order 87 with the* ***odd number entries***, $\{1081, 1083, \ldots, 16215, 16217\}$ *with magic sum,* $S_{87\times 87} := 752463$. *The entries total sum is* $T_{7569} := 87 \times 752463 = 65464281 = 8091^2$.

12. *The triple* $(540, 18221, 18229)$ *generating a* ***perfect square entries sum*** *magic square of order 133 with the* ***odd number entries***, $\{1081, 1083, \ldots, 36455, 36457\}$ *with magic sum,* $\boldsymbol{S}_{133\times133} := 2496277$. *The entries total sum is* $\boldsymbol{T}_{17689} := 133 \times 2496277 = 332004841 = 18221^2$.

13. *The triple* $(540, 72899, 72901)$ *generating a* ***perfect square entries sum*** *magic square of order 269 with the* ***odd number entries***, $\{1081, 1083, \ldots, 145799, 145801\}$ *with magic sum,* $\boldsymbol{S}_{269\times269} := 19755629$. *The entries total sum is* $\boldsymbol{T}_{72361} := 269 \times 19755629 = 5314264201 = 72899^2$.

28 Magic Square of Order 28

According to equation (7), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (3), we can only obtain a magic square of order 28 with this property if we use odd number entries. Taking $k = 20$ in equation (3), we get

$$\begin{aligned} &1 + 3 + 5 + \ldots + (2 \times 28^2 - 1) = 28^4 \\ \Rightarrow &1 + 3 + 5 + \ldots + 1565 + 1567 \quad = 614656 = 784^2 = 28^4. \end{aligned}$$

Example 28.1. *A* ***block-wise pandiagonal*** *magic square of order 28 for* ***consecutive odd numbers*** *entries* $1, 3, 5, ..., 1565, 1567$ *is given by*

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		21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952
	589	1175	1	1371	591	1173	3	1369	593	1171	5	1367	595	1169	7	1365	597	1167	9	1363	599	1165	11	1361	601	1163	13	1359	21952
21952	195	1177	783	981	193	1179	781	983	191	1181	779	985	189	1183	777	987	187	1185	775	989	185	1187	773	991	183	1189	771	993	21952
21952	1567	197	979	393	1565	199	977	395	1563	201	975	397	1561	203	973	399	1559	205	971	401	1557	207	969	403	1555	209	967	405	21952
21952	785	587	1373	391	787	585	1375	389	789	583	1377	387	791	581	1379	385	793	579	1381	383	795	577	1383	381	797	575	1385	379	21952
21952	603	1161	15	1357	605	1159	17	1355	607	1157	19	1353	609	1155	21	1351	611	1153	23	1349	613	1151	25	1347	615	1149	27	1345	21952
21952	181	1191	769	995	179	1193	767	997	177	1195	765	999	175	1197	763	1001	173	1199	761	1003	171	1201	759	1005	169	1203	757	1007	21952
21952	1553	211	965	407	1551	213	963	409	1549	215	961	411	1547	217	959	413	1545	219	957	415	1543	221	955	417	1541	223	953	419	21952
21952	799	573	1387	377	801	571	1389	375	803	569	1391	373	805	567	1393	371	807	565	1395	369	809	563	1397	367	811	561	1399	365	21952
21952	617	1147	29	1343	619	1145	31	1341	621	1143	33	1339	623	1141	35	1337	625	1139	37	1335	627	1137	39	1333	629	1135	41	1331	21952
21952	167	1205	755	1009	165	1207	753	1011	163	1209	751	1013	161	1211	749	1015	159	1213	747	1017	157	1215	745	1019	155	1217	743	1021	21952
21952	1539	225	951	421	1537	227	949	423	1535	229	947	425	1533	231	945	427	1531	233	943	429	1529	235	941	431	1527	237	939	433	21952
21952	813	559	1401	363	815	557	1403	361	817	555	1405	359	819	553	1407	357	821	551	1409	355	823	549	1411	353	825	547	1413	351	21952
21952	631	1133	43	1329	633	1131	45	1327	635	1129	47	1325	637	1127	49	1323	639	1125	51	1321	641	1123	53	1319	643	1121	55	1317	21952
21952	153	1219	741	1023	151	1221	739	1025	149	1223	737	1027	147	1225	735	1029	145	1227	733	1031	143	1229	731	1033	141	1231	729	1035	21952
21952	1525	239	937	435	1523	241	935	437	1521	243	933	439	1519	245	931	441	1517	247	929	443	1515	249	927	445	1513	251	925	447	21952
21952	827	545	1415	349	829	543	1417	347	831	541	1419	345	833	539	1421	343	835	537	1423	341	837	535	1425	339	839	533	1427	337	21952
21952	645	1119	57	1315	647	1117	59	1313	649	1115	61	1311	651	1113	63	1309	653	1111	65	1307	655	1109	67	1305	657	1107	69	1303	21952
21952	139	1233	727	1037	137	1235	725	1039	135	1237	723	1041	133	1239	721	1043	131	1241	719	1045	129	1243	717	1047	127	1245	715	1049	21952
21952	1511	253	923	449	1509	255	921	451	1507	257	919	453	1505	259	917	455	1503	261	915	457	1501	263	913	459	1499	265	911	461	21952
21952	841	531	1429	335	843	529	1431	333	845	527	1433	331	847	525	1435	329	849	523	1437	327	851	521	1439	325	853	519	1441	323	21952
21952	659	1105	71	1301	661	1103	73	1299	663	1101	75	1297	665	1099	77	1295	667	1097	79	1293	669	1095	81	1291	671	1093	83	1289	21952
21952	125	1247	713	1051	123	1249	711	1053	121	1251	709	1055	119	1253	707	1057	117	1255	705	1059	115	1257	703	1061	113	1259	701	1063	21952
21952	1497	267	909	463	1495	269	907	465	1493	271	905	467	1491	273	903	469	1489	275	901	471	1487	277	899	473	1485	279	897	475	21952
21952	855	517	1443	321	857	515	1445	319	859	513	1447	317	861	511	1449	315	863	509	1451	313	865	507	1453	311	867	505	1455	309	21952
21952	673	1091	85	1287	675	1089	87	1285	677	1087	89	1283	679	1085	91	1281	681	1083	93	1279	683	1081	95	1277	685	1079	97	1275	21952
21952	111	1261	699	1065	109	1263	697	1067	107	1265	695	1069	105	1267	693	1071	103	1269	691	1073	101	1271	689	1075	99	1273	687	1077	21952
21952	1483	281	895	477	1481	283	893	479	1479	285	891	481	1477	287	889	483	1475	289	887	485	1473	291	885	487	1471	293	883	489	21952
21952	869	503	1457	307	871	501	1459	305	873	499	1461	303	875	497	1463	301	877	495	1465	299	879	493	1467	297	881	491	1469	295	21952
	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952	21952

In this case the magic sum is $S_{28\times 28} = 21952 = 28^3$, and the sum of all entries is $T_{784} := 28 \times 21952 = 614656 = 784^2 = 28^4$. Block of order 4 are **pandiagonal** magic squares equal magic sums, i.e., $S_{4\times 4} = 3136$. These blocks are also **perfect square entries sum**, i.e., the sum of all entries in each block is $T_{16} := 4 \times 3136 = 12544 = 112^2$.

28.1 Pythagorean Triples

According to Example 28.1, we have perfect square entries sum, i.e., $T_{784} := 28 \times 21952 = 614656 = 784^2$. Below are **Pythagorean triples** for the number 784:

$$\mathbf{784^2} + 105^2 := 791^2 \qquad \mathbf{784^2} + 2337^2 := 2465^2 \qquad \mathbf{784^2} + 9588^2 := 9620^2$$
$$\mathbf{784^2} + 462^2 := 910^2 \qquad \mathbf{784^2} + 2688^2 := 2800^2 \qquad \mathbf{784^2} + 10962^2 := 10990^2$$
$$\mathbf{784^2} + 588^2 := 980^2 \qquad \mathbf{784^2} + 3087^2 := 3185^2 \qquad \mathbf{784^2} + 19200^2 := 19216^2$$
$$\mathbf{784^2} + 1260^2 := 1484^2 \qquad \mathbf{784^2} + 4770^2 := 4834^2 \qquad \mathbf{784^2} + 21945^2 := 21959^2$$
$$\mathbf{784^2} + 1470^2 := 1666^2 \qquad \mathbf{784^2} + 5460^2 := 5516^2 \qquad \mathbf{784^2} + 38412^2 := 38420^2$$

Remark 28.1. *The above 15* ***Pythagorean triples*** *are with number 784. This number is due to Example 28.1. According to Result 2, there are 8 triples generating magic square with* ***perfect square entries sum****:*

$$(\mathbf{784, 588, 980}) \Rightarrow 980 - 784 = 196 = 14^2$$
$$(\mathbf{784, 1470, 1666}) \Rightarrow 1666 - 1470 = 196 = 14^2$$
$$(\mathbf{784, 2337, 2465}) \Rightarrow 2465 - 784 = 1681 = 41^2$$
$$(\mathbf{784, 3087, 3185}) \Rightarrow 3185 - 784 = 2401 = 49^2$$
$$(\mathbf{784, 4770, 4834}) \Rightarrow 4834 - 4770 = 64 = 8^2$$
$$(\mathbf{784, 9588, 9620}) \Rightarrow 9620 - 784 = 8836 = 94^2$$
$$(\mathbf{784, 19200, 19216}) \Rightarrow 19216 - 19200 = 16 = 4^2$$
$$(\mathbf{784, 38412, 38420}) \Rightarrow 38420 - 784 = 37636 = 194^2.$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(784, 588, 980)$ *generating a* ***perfect square entries sum*** *magic square of order 14 with the* ***odd number entries****,* $\{1569, 1571, \ldots, 1957, 1959\}$ *with magic sum,* $S_{14\times 14} := 24696$*. The entries total sum is* $T_{196} := 14 \times 24696 = 345744 = 588^2$.

2. *The triple* $(784, 1470, 1666)$ *generating a* ***perfect square entries sum*** *magic square of order 14 with the* ***odd number entries***, $\{2941, 2943, \ldots, 3329, 3331\}$ *with magic sum,* $\boldsymbol{S}_{14\times 14} := 43904$. *The entries total sum is* $\boldsymbol{T}_{196} := 14 \times 43904 = 614656 = 784^2$.

3. *The triple* $(784, 2337, 2465)$ *generating a* ***perfect square entries sum*** *magic square of order 41 with the* ***odd number entries***, $\{1569, 1571, \ldots, 4927, 4929\}$ *with magic sum,* $\boldsymbol{S}_{41\times 41} := 133209$. *The entries total sum is* $\boldsymbol{T}_{1681} := 41 \times 133209 = 5461569 = 2337^2$.

4. *The triple* $(784, 3087, 3185)$ *generating a* ***perfect square entries sum*** *magic square of order 49 with the* ***odd number entries***, $\{1569, 1571, \ldots, 6367, 6369\}$ *with magic sum,* $\boldsymbol{S}_{49\times 49} := 194481$. *The entries total sum is* $\boldsymbol{T}_{2401} := 49 \times 194481 = 9529569 = 3087^2$.

5. *The triple* $(784, 4770, 4834)$ *generating a* ***perfect square entries sum*** *magic square of order 8 with the* ***odd number entries***, $\{9541, 9543, \ldots, 9665, 9667\}$ *with magic sum,* $\boldsymbol{S}_{8\times 8} := 76832$. *The entries total sum is* $\boldsymbol{T}_{64} := 8 \times 76832 = 614656 = 784^2$.

6. *The triple* $(784, 9588, 9620)$ *generating a* ***perfect square entries sum*** *magic square of order 94 with the* ***odd number entries***, $\{1569, 1571, \ldots, 19237, 19239\}$ *with magic sum,* $\boldsymbol{S}_{94\times 94} := 977976$. *The entries total sum is* $\boldsymbol{T}_{8836} := 94 \times 977976 = 91929744 = 9588^2$.

7. *The triple* $(784, 19200, 19216)$ *generating a* ***perfect square entries sum*** *magic square of order 4 with the* ***odd number entries***, $\{38401, 38403, \ldots, 38429, 38431\}$ *with magic sum,* $\boldsymbol{S}_{4\times} := 153664$. *The entries total sum is* $\boldsymbol{T}_{16} := 4 \times 153664 = 614656 = 784^2$.

8. *The triple* $(784, 38412, 38420)$ *generating a* ***perfect square entries sum*** *magic square of order 194 with the* ***odd number entries***, $\{1569, 1571, \ldots, 76837, 76839\}$ *with magic sum,* $\boldsymbol{S}_{194\times 194} := 7605576$. *The entries total sum is* $\boldsymbol{T}_{37636} := 194 \times 7605576 = 1475481744 = 38412^2$.

29 Magic Square of Order 29

According to equations (3) and (8), one can obtain **perfect square entries sum magic squares** of order 29 in two ways: first using **consecutive odd numbers**, and the second using **consecutive natural numbers**.

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For the **consecutive natural numbers** there are again two ways: the first one is of **uniformity**, and the second is of **minimum perfect square entries sum**.

29.1 Consecutive Odd Number Entries

According to equation (7), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (3), we can only obtain a magic square of order 29 with this property if we use odd number entries. Taking $k = 29$ in equation (3), we get

$$1+3+5+\ldots+(2\times 29^2-1) = 29^4$$
$$\Rightarrow 1+3+5+\ldots+1679+1681 = 707281 = 841^2 = 29^4.$$

Example 29.1. *For the* ***consecutive odd number*** *entries* $\{1,3,5,\ldots,1679,1681\}$, *a* ***block-bordered*** *magic square of order 29 is given by*

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1627	1571	1575	1579	1583	1587	1591	1595	1599	1603	1607	1611	1615	1619	53	51	47	43	39	35	31	27	23	19	15	11	7	3	1623
109	1355	1009	159	1383	979	161	1363	1013	147	1385	987	151	1395	989	139	1397	999	127	1371	1021	131	1405	1001	117	1375	1029	119	1573
105	145	1401	977	115	1403	1005	149	1389	985	123	1393	1007	125	1381	1017	135	1369	1019	157	1373	993	137	1359	1027	165	1361	997	1577
101	1023	113	1387	1025	141	1357	1011	121	1391	1015	143	1365	1003	153	1367	991	155	1377	995	129	1399	981	163	1379	983	133	1407	1581
97	1409	199	915	1437	169	917	1417	203	903	1439	177	907	1449	179	895	1451	189	883	1425	211	887	1459	191	873	1429	219	875	1585
93	901	1455	167	871	1457	195	905	1443	175	879	1447	197	881	1435	207	891	1423	209	913	1427	183	893	1413	217	921	1415	187	1589
89	213	869	1441	215	897	1411	201	877	1445	205	899	1419	193	909	1421	181	911	1431	185	885	1453	171	919	1433	173	889	1461	1593
85	1031	1117	375	1059	1087	377	1039	1121	363	1061	1095	367	1071	1097	355	1073	1107	343	1047	1129	347	1081	1109	333	1051	1137	335	1597
81	361	1077	1085	331	1079	1113	365	1065	1093	339	1069	1115	341	1057	1125	351	1045	1127	373	1049	1101	353	1035	1135	381	1037	1105	1601
77	1131	329	1063	1133	357	1033	1119	337	1067	1123	359	1041	1111	369	1043	1099	371	1053	1103	345	1075	1089	379	1055	1091	349	1083	1605
73	1139	415	969	1167	385	971	1147	419	957	1169	393	961	1179	395	949	1181	405	937	1155	427	941	1189	407	927	1159	435	929	1609
69	955	1185	383	925	1187	411	959	1173	391	933	1177	413	935	1165	423	945	1153	425	967	1157	399	947	1143	433	975	1145	403	1613
65	429	923	1171	431	951	1141	417	931	1175	421	953	1149	409	963	1151	397	965	1161	401	939	1183	387	973	1163	389	943	1191	1617
61	815	469	1239	843	439	1241	823	473	1227	845	447	1231	855	449	1219	857	459	1207	831	481	1211	865	461	1197	835	489	1199	1621
57	1225	861	437	1195	863	465	1229	849	445	1203	853	467	1205	841	477	1215	829	479	1237	833	453	1217	819	487	1245	821	457	1625
1633	483	1193	847	485	1221	817	471	1201	851	475	1223	825	463	1233	827	451	1235	837	455	1209	859	441	1243	839	443	1213	867	49
1637	491	739	1293	519	709	1295	499	743	1281	521	717	1285	531	719	1273	533	729	1261	507	751	1265	541	731	1251	511	759	1253	45
1641	1279	537	707	1249	539	735	1283	525	715	1257	529	737	1259	517	747	1269	505	749	1291	509	723	1271	495	757	1299	497	727	41
1645	753	1247	523	755	1275	493	741	1255	527	745	1277	501	733	1287	503	721	1289	513	725	1263	535	711	1297	515	713	1267	543	37
1649	599	1333	591	627	1303	593	607	1337	579	629	1311	583	639	1313	571	641	1323	559	615	1345	563	649	1325	549	619	1353	551	33
1653	577	645	1301	547	647	1329	581	633	1309	555	637	1331	557	625	1341	567	613	1343	589	617	1317	569	603	1351	597	605	1321	29
1657	1347	545	631	1349	573	601	1335	553	635	1339	575	609	1327	585	611	1315	587	621	1319	561	643	1305	595	623	1307	565	651	25
1661	221	793	1509	249	763	1511	229	797	1497	251	771	1501	261	773	1489	263	783	1477	237	805	1481	271	785	1467	241	813	1469	21
1665	1495	267	761	1465	269	789	1499	255	769	1473	259	791	1475	247	801	1485	235	803	1507	239	777	1487	225	811	1515	227	781	17
1669	807	1463	253	809	1491	223	795	1471	257	799	1493	231	787	1503	233	775	1505	243	779	1479	265	765	1513	245	767	1483	273	13
1673	275	1549	699	303	1519	701	283	1553	687	305	1527	691	315	1529	679	317	1539	667	291	1561	671	325	1541	657	295	1569	659	9
1677	685	321	1517	655	323	1545	689	309	1525	663	313	1547	665	301	1557	675	289	1559	697	293	1533	677	279	1567	705	281	1537	5
1681	1563	653	307	1565	681	277	1551	661	311	1555	683	285	1543	693	287	1531	695	297	1535	669	319	1521	703	299	1523	673	327	1
59	111	107	103	99	95	91	87	83	79	75	71	67	63	1629	1631	1635	1639	1643	1647	1651	1655	1659	1663	1667	1671	1675	1679	55

The above **block-bordered** magic square of order 29 is with inner part as **block-wise pandiagonal** magic square of order 27 with blocks of orders 3 as **semi-magic** squares with equal magic sums. Below are respective magic sums:

$$S_{29\times 29} = 24389$$
$$S_{27\times 27} = 22707$$
$$Sm_{3\times 3} = 2523$$

The total entries sum is $T_{841} := 24389 \times 29 = 707281 = 841^2 = 29^4$. It satisfies the uniformity property

(11), i.e., $\left\langle 29, 29^2, 29^3, 29^4 \right\rangle$. Magic squares of orders 27 and 3 are also satisfy uniformity property (11), i.e., $\left\langle 27, 27^2, 27^3, 27^4 \right\rangle$ and $\left\langle 3, 3^2, 3^3, 3^4 \right\rangle$. In case of order 3, we are talking about **semi-magic** squares.

29.2 Consecutive Natural Number Entries

According to equation (8),

$$G := \frac{n(n+1)}{2} - \frac{(n-841)(n-840)}{2} = 841(n-420)$$

Taking $n = 1261$, we get a perfect square, i.e.,

$$G := T(1261) - T(420) = \frac{1261 \times 1262}{2} - \frac{420 \times 421}{2} = 1261 \times 631 - 210 \times 421$$
$$= 707281 = 841^2 = 29^4.$$

Simplifying, we get

$$421 + 422 + 423 + \ldots + 1260 + 1261 = 279841 = 529^2 = 23^4.$$

This gives a perfect square entries sum for the 441 **consecutive natural numbers** from to 265 to 793.

Example 29.2. *For the* ***consecutive natural number*** *entries* $\{421, 422, 423, \ldots, 1260, 1261\}$ *a* ***block-wise pandiagonal*** *magic square of order 27 is given by*

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1234	1206	1208	1210	1212	1214	1216	1218	1220	1222	1224	1226	1228	1230	447	446	444	442	440	438	436	434	432	430	428	426	424	422	1232
475	1098	925	500	1112	910	501	1102	927	494	1113	914	496	1118	915	490	1119	920	484	1106	931	486	1123	921	479	1108	935	480	1207
473	493	1121	909	478	1122	923	495	1115	913	482	1117	924	483	1111	929	488	1105	930	499	1107	917	489	1100	934	503	1101	919	1209
471	932	477	1114	933	491	1099	926	481	1116	928	492	1103	922	497	1104	916	498	1109	918	485	1120	911	502	1110	912	487	1124	1211
469	1125	520	878	1139	505	879	1129	522	872	1140	509	874	1145	510	868	1146	515	862	1133	526	864	1150	516	857	1135	530	858	1213
467	871	1148	504	856	1149	518	873	1142	508	860	1144	519	861	1138	524	866	1132	525	877	1134	512	867	1127	529	881	1128	514	1215
465	527	855	1141	528	869	1126	521	859	1143	523	870	1130	517	875	1131	511	876	1136	513	863	1147	506	880	1137	507	865	1151	1217
463	936	979	608	950	964	609	940	981	602	951	968	604	956	969	598	957	974	592	944	985	594	961	975	587	946	989	588	1219
461	601	959	963	586	960	977	603	953	967	590	955	978	591	949	983	596	943	984	607	945	971	597	938	988	611	939	973	1221
459	986	585	952	987	599	937	980	589	954	982	600	941	976	605	942	970	606	947	972	593	958	965	610	948	966	595	962	1223
457	990	628	905	1004	613	906	994	630	899	1005	617	901	1010	618	895	1011	623	889	998	634	891	1015	624	884	1000	638	885	1225
455	898	1013	612	883	1014	626	900	1007	616	887	1009	627	888	1003	632	893	997	633	904	999	620	894	992	637	908	993	622	1227
453	635	882	1006	636	896	991	629	886	1008	631	897	995	625	902	996	619	903	1001	621	890	1012	614	907	1002	615	892	1016	1229
451	828	655	1040	842	640	1041	832	657	1034	843	644	1036	848	645	1030	849	650	1024	836	661	1026	853	651	1019	838	665	1020	1231
449	1033	851	639	1018	852	653	1035	845	643	1022	847	654	1023	841	659	1028	835	660	1039	837	647	1029	830	664	1043	831	649	1233
1237	662	1017	844	663	1031	829	656	1021	846	658	1032	833	652	1037	834	646	1038	839	648	1025	850	641	1042	840	642	1027	854	445
1239	666	790	1067	680	775	1068	670	792	1061	681	779	1063	686	780	1057	687	785	1051	674	796	1053	691	786	1046	676	800	1047	443
1241	1060	689	774	1045	690	788	1062	683	778	1049	685	789	1050	679	794	1055	673	795	1066	675	782	1056	668	799	1070	669	784	441
1243	797	1044	682	798	1058	667	791	1048	684	793	1059	671	787	1064	672	781	1065	677	783	1052	688	776	1069	678	777	1054	692	439
1245	720	1087	716	734	1072	717	724	1089	710	735	1076	712	740	1077	706	741	1082	700	728	1093	702	745	1083	695	730	1097	696	437
1247	709	743	1071	694	744	1085	711	737	1075	698	739	1086	699	733	1091	704	727	1092	715	729	1079	705	722	1096	719	723	1081	435
1249	1094	693	736	1095	707	721	1088	697	738	1090	708	725	1084	713	726	1078	714	731	1080	701	742	1073	718	732	1074	703	746	433
1251	531	817	1175	545	802	1176	535	819	1169	546	806	1171	551	807	1165	552	812	1159	539	823	1161	556	813	1154	541	827	1155	431
1253	1168	554	801	1153	555	815	1170	548	805	1157	550	816	1158	544	821	1163	538	822	1174	540	809	1164	533	826	1178	534	811	429
1255	824	1152	547	825	1166	532	818	1156	549	820	1167	536	814	1172	537	808	1173	542	810	1160	553	803	1177	543	804	1162	557	427
1257	558	1195	770	572	1180	771	562	1197	764	573	1184	766	578	1185	760	579	1190	754	566	1201	756	583	1191	749	568	1205	750	425
1259	763	581	1179	748	582	1193	765	575	1183	752	577	1194	753	571	1199	758	565	1200	769	567	1187	759	560	1204	773	561	1189	423
1261	1202	747	574	1203	761	559	1196	751	576	1198	762	563	1192	767	564	1186	768	569	1188	755	580	1181	772	570	1182	757	584	421
450	476	474	472	470	468	466	464	462	460	458	456	454	452	1235	1236	1238	1240	1242	1244	1246	1248	1250	1252	1254	1256	1258	1260	448

The **block-bordered** magic squares of order 29 given in Examples 29.1 and 29.2 are of equal magic sums, i.e., $S_{29\times 29} := 24389$.

29.3 Minimum Perfect Square Entries Sum

The examples given in 29.1 and 29.2 satisfy the uniformity property (11), i.e., $\left\langle 29, 29^2, 29^3, 29^4 \right\rangle$. However, the sum of the magic square entries of 29.2 is not a **minimum perfect square entries sum**. Choosing $m = 21, 22, \ldots, 29$ and $p = 29$ in equation (13), we get

$$1.\quad L\left(21^2+\frac{29^2-1}{2},29^2\right)\to(29,21,861,12789,370881)$$

$$2.\quad L\left(22^2+\frac{29^2-1}{2},29^2\right)\to(29,64,904,14036,407044)$$

$$\dots \quad \dots \qquad\qquad \dots \quad \dots$$

$$9.\quad L\left(29^2+\frac{29^2-1}{2},29^2\right)\to(29,421,1261,24389,707281)\Rightarrow\left\langle 29,29^2,29^3,29^4\right\rangle$$

The values written above are for $m=21,22,\dots,29$. The 9^{th} value for $m=29$ satisfy the uniformity property (11). This case is already studied in Example 29.1. The first value for $m=21$ give **minimum perfect square** magic square. In this case, we have magic square of order 29 with sum of all entries $370881=609^2$ which is a perfect square, but it does not satisfy the uniformity property (11). The magic square given below has a **minimum perfect square entries sum**.

Example 29.3. *For the* ***consecutive natural number*** *entries* $\{21,22,23,\dots,860,861\}$ *a* ***block-bordered*** *magic square of order 29 is given by*

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834	806	808	810	812	814	816	818	820	822	824	826	828	830	47	46	44	42	40	38	36	34	32	30	28	26	24	22	832
75	698	525	100	712	510	101	702	527	94	713	514	96	718	515	90	719	520	84	706	531	86	723	521	79	708	535	80	807
73	93	721	509	78	722	523	95	715	513	82	717	524	83	711	529	88	705	530	99	707	517	89	700	534	103	701	519	809
71	532	77	714	533	91	699	526	81	716	528	92	703	522	97	704	516	98	709	518	85	720	511	102	710	512	87	724	811
69	725	120	478	739	105	479	729	122	472	740	109	474	745	110	468	746	115	462	733	126	464	750	116	457	735	130	458	813
67	471	748	104	456	749	118	473	742	108	460	744	119	461	738	124	466	732	125	477	734	112	467	727	129	481	728	114	815
65	127	455	741	128	469	726	121	459	743	123	470	730	117	475	731	111	476	736	113	463	747	106	480	737	107	465	751	817
63	536	579	208	550	564	209	540	581	202	551	568	204	556	569	198	557	574	192	544	585	194	561	575	187	546	589	188	819
61	201	559	563	186	560	577	203	553	567	190	555	578	191	549	583	196	543	584	207	545	571	197	538	588	211	539	573	821
59	586	185	552	587	199	537	580	189	554	582	200	541	576	205	542	570	206	547	572	193	558	565	210	548	566	195	562	823
57	590	228	505	604	213	506	594	230	499	605	217	501	610	218	495	611	223	489	598	234	491	615	224	484	600	238	485	825
55	498	613	212	483	614	226	500	607	216	487	609	227	488	603	232	493	597	233	504	599	220	494	592	237	508	593	222	827
53	235	482	606	236	496	591	229	486	608	231	497	595	225	502	596	219	503	601	221	490	612	214	507	602	215	492	616	829
51	428	255	640	442	240	641	432	257	634	443	244	636	448	245	630	449	250	624	436	261	626	453	251	619	438	265	620	831
49	633	451	239	618	452	253	635	445	243	622	447	254	623	441	259	628	435	260	639	437	247	629	430	264	643	431	249	833
837	262	617	444	263	631	429	256	621	446	258	63	433	252	637	434	246	638	439	248	625	450	241	642	440	242	627	454	45
839	266	390	667	280	375	668	270	392	661	281	379	663	286	380	657	287	385	651	274	396	653	291	386	646	276	400	647	43
841	660	289	374	645	290	388	662	283	378	649	285	389	650	279	394	655	273	395	666	275	382	656	268	399	670	269	384	41
843	397	644	282	398	658	267	391	648	284	393	659	271	387	664	272	381	665	277	383	652	288	376	669	278	377	654	292	39
845	320	687	316	334	672	317	324	689	310	335	676	312	340	677	306	341	682	300	328	693	302	345	683	295	330	697	296	37
847	309	343	671	294	344	685	311	337	675	298	339	686	299	333	691	304	327	692	315	329	679	305	322	696	319	323	681	35
849	694	293	336	695	307	321	688	297	338	690	308	325	684	313	326	678	314	331	680	301	342	673	318	332	674	303	346	33
851	131	417	775	145	402	776	135	419	769	146	406	771	151	407	765	152	412	759	139	423	761	156	413	754	141	427	755	31
853	768	154	401	753	155	415	770	148	405	757	150	416	758	144	421	763	138	422	774	140	409	764	133	426	778	134	411	29
855	424	752	147	425	766	132	418	756	149	420	767	136	414	772	137	408	773	142	410	760	153	403	777	143	404	762	157	27
857	158	795	370	172	780	371	162	797	364	173	784	366	178	785	360	179	790	354	166	801	356	183	791	349	168	805	350	25
859	363	181	779	348	182	793	365	175	783	352	177	794	353	171	799	358	165	800	369	167	787	359	160	804	373	161	789	23
861	802	347	174	803	361	159	796	351	176	798	362	163	792	367	164	786	368	169	788	355	180	781	372	170	782	357	184	21
50	76	74	72	70	68	66	64	62	60	58	56	54	52	835	836	838	840	842	844	846	848	850	852	854	856	858	860	48

The above Example 29.3 is with magic sum $S_{29\times 29} = 12789$, and the sum of all entries is $T_{841} := 29 \times 12789 = 370881 = 609^2$. This magic square of order 29 is a **minimum perfect square entries sum**. The blocks of order 3 are **semi-magic** squares of order 3 with equal **semi-magic** sums, $Sm_{3\times 3} = 1323$. It is also **perfect square entries sum**, i.e. $T_9 := 3 \times 1323 = 3969 = 63^2$.

29.4 Pythagorean Triples

According to Examples 29.1, 29.2 and 29.3, we have two perfect square entries sums, i.e., $T_{841} := 24389 \times 29 = 707281 = 841^2$ and $T_{841} := 29 \times 12789 = 370881 = 609^2$. Below are **Pythagorean triples** for the numbers 841 and 609:

$$
\begin{array}{lll}
\mathbf{841^2} + 12180^2 := 12209^2 & \mathbf{609^2} + 1188^2 := 1335^2 & \mathbf{609^2} + 8820^2 := 8841^2 \\
\mathbf{841^2} + 353640^2 := 353641^2 & \mathbf{609^2} + 2088^2 := 2175^2 & \mathbf{609^2} + 20600^2 := 20609^2 \\
\mathbf{609^2} + 200^2 := 641^2 & \mathbf{609^2} + 2912^2 := 2975^2 & \mathbf{609^2} + 26488^2 := 26495^2 \\
\mathbf{609^2} + 580^2 := 841^2 & \mathbf{609^2} + 3760^2 := 3809^2 & \mathbf{609^2} + 61812^2 := 61815^2 \\
\mathbf{609^2} + 812^2 := 1015^2 & \mathbf{609^2} + 6380^2 := 6409^2 & \mathbf{609^2} + 185440^2 := 185441^2
\end{array}
$$

Remark 29.1. *The above 15* ***Pythagorean triples*** *are with numbers 841 and 609. These numbers are due to Examples 29.1 and 29.3. According to Result 2, there are only 3 triples generating magic square with* ***perfect square entries sum****:*

$$
\begin{aligned}
(\mathbf{609, 200, 641}) &\Rightarrow 641 - 200 &&= 441 = 21^2 \\
(\mathbf{609, 3760, 3809}) &\Rightarrow 3809 - 3760 &&= 49 = 7^2 \\
(\mathbf{609, 20600, 20609}) &\Rightarrow 20609 - 20600 &&= 9 = 3^2.
\end{aligned}
$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(609, 200, 641)$ *generating a* ***perfect square entries sum*** *magic square of order 21 with the* ***odd number entries****,* $\{401, 403, \ldots, 1279, 1281\}$ *with magic sum,* $S_{21\times21} := 17661$*. The entries total sum is* $T_{441} := 16 \times 17661 = 370881 = 609^2$.

2. *The triple* $(609, 3760, 3809)$ *generating a* ***perfect square entries sum*** *magic square of order 7 with the* ***odd number entries****,* $\{7521, 7523, \ldots, 7615, 7617\}$ *with magic sum,* $S_{7\times7} := 52983$*. The entries total sum is* $T_{49} := 40 \times 52983 = 370881 = 609^2$.

3. *The triple* $(609, 20600, 20609)$ *generating a* ***perfect square entries sum*** *magic square of order 3 with the* ***odd number entries****,* $\{41201, 41203, \ldots, 41215, 41217\}$ *with magic sum,* $S_{3\times 3} := 123627$*. The entries total sum is* $T_9 := 3 \times 123627 = 370881 = 609^2$.

30 Magic Square of Order 30

According to equation (7), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (3), we can only obtain a magic square of order 30 with this property if we use odd number entries. Taking $k = 30$ in equation (3), we get

$$\begin{aligned} & 1+3+5+\ldots+(2\times 30^2-1) = 30^4 \\ \Rightarrow & 1+3+5+\ldots+1797+1799 \quad = 160000 = 400^2 = 20^4. \end{aligned}$$

Example 30.1. *A* ***block-wise*** *magic square of order 30 for the* ***consecutive odd numbers*** *entries* $\{1, 3, 5, ..., 1797, 1799\}$ *is given by*

																														27000
1	1701	1699	1601	99	299	3	1703	1697	1603	97	297	5	1705	1695	1605	95	295	7	1707	1693	1607	93	293	9	1709	1691	1609	91	291	27000
1499	399	1399	401	501	1201	1497	397	1397	403	503	1203	1495	395	1395	405	505	1205	1493	393	1393	407	507	1207	1491	391	1391	409	509	1209	27000
1199	1101	701	799	999	601	1197	1103	703	797	997	603	1195	1105	705	795	995	605	1193	1107	707	793	993	607	1191	1109	709	791	991	609	27000
899	699	1001	1099	801	901	897	697	1003	1097	803	903	895	695	1005	1095	805	905	893	693	1007	1093	807	907	891	691	1009	1091	809	909	27000
301	1299	499	1301	1401	599	303	1297	497	1303	1403	597	305	1295	495	1305	1405	595	307	1293	493	1307	1407	593	309	1291	491	1309	1409	591	27000
1501	201	101	199	1599	1799	1503	203	103	197	1597	1797	1505	205	105	195	1595	1795	1507	207	107	193	1593	1793	1509	209	109	191	1591	1791	27000
11	1711	1689	1611	89	289	13	1713	1687	1613	87	287	15	1715	1685	1615	85	285	17	1717	1683	1617	83	283	19	1719	1681	1619	81	281	27000
1489	389	1389	411	511	1211	1487	387	1387	413	513	1213	1485	385	1385	415	515	1215	1483	383	1383	417	517	1217	1481	381	1381	419	519	1219	27000
1189	1111	711	789	989	611	1187	1113	713	787	987	613	1185	1115	715	785	985	615	1183	1117	717	783	983	617	1181	1119	719	781	981	619	27000
889	689	1011	1089	811	911	887	687	1013	1087	813	913	885	685	1015	1085	815	915	883	683	1017	1083	817	917	881	681	1019	1081	819	919	27000
311	1289	489	1311	1411	589	313	1287	487	1313	1413	587	315	1285	485	1315	1415	585	317	1283	483	1317	1417	583	319	1281	481	1319	1419	581	27000
1511	211	111	189	1589	1789	1513	213	113	187	1587	1787	1515	215	115	185	1585	1785	1517	217	117	183	1583	1783	1519	219	119	181	1581	1781	27000
21	1721	1679	1621	79	279	23	1723	1677	1623	77	277	25	1725	1675	1625	75	275	27	1727	1673	1627	73	273	29	1729	1671	1629	71	271	27000
1479	379	1379	421	521	1221	1477	377	1377	423	523	1223	1475	375	1375	425	525	1225	1473	373	1373	427	527	1227	1471	371	1371	429	529	1229	27000
1179	1121	721	779	979	621	1177	1123	723	777	977	623	1175	1125	725	775	975	625	1173	1127	727	773	973	627	1171	1129	729	771	971	629	27000
879	679	1021	1079	821	921	877	677	1023	1077	823	923	875	675	1025	1075	825	925	873	673	1027	1073	827	927	871	671	1029	1071	829	929	27000
321	1279	479	1321	1421	579	323	1277	477	1323	1423	577	325	1275	475	1325	1425	575	327	1273	473	1327	1427	573	329	1271	471	1329	1429	571	27000
1521	221	121	179	1579	1779	1523	223	123	177	1577	1777	1525	225	125	175	1575	1775	1527	227	127	173	1573	1773	1529	229	129	171	1571	1771	27000
31	1731	1669	1631	69	269	33	1733	1667	1633	67	267	35	1735	1665	1635	65	265	37	1737	1663	1637	63	263	39	1739	1661	1639	61	261	27000
1469	369	1369	431	531	1231	1467	367	1367	433	533	1233	1465	365	1365	435	535	1235	1463	363	1363	437	537	1237	1461	361	1361	439	539	1239	27000
1169	1131	731	769	969	631	1167	1133	733	767	967	633	1165	1135	735	765	965	635	1163	1137	737	763	963	637	1161	1139	739	761	961	639	27000
869	669	1031	1069	831	931	867	667	1033	1067	833	933	865	665	1035	1065	835	935	863	663	1037	1063	837	937	861	661	1039	1061	839	939	27000
331	1269	469	1331	1431	569	333	1267	467	1333	1433	567	335	1265	465	1335	1435	565	337	1263	463	1337	1437	563	339	1261	461	1339	1439	561	27000
1531	231	131	169	1569	1769	1533	233	133	167	1567	1767	1535	235	135	165	1565	1765	1537	237	137	163	1563	1763	1539	239	139	161	1561	1761	27000
41	1741	1659	1641	59	259	43	1743	1657	1643	57	257	45	1745	1655	1645	55	255	47	1747	1653	1647	53	253	49	1749	1651	1649	51	251	27000
1459	359	1359	441	541	1241	1457	357	1357	443	543	1243	1455	355	1355	445	545	1245	1453	353	1353	447	547	1247	1451	351	1351	449	549	1249	27000
1159	1141	741	759	959	641	1157	1143	743	757	957	643	1155	1145	745	755	955	645	1153	1147	747	753	953	647	1151	1149	749	751	951	649	27000
859	659	1041	1059	841	941	857	657	1043	1057	843	943	855	655	1045	1055	845	945	853	653	1047	1053	847	947	851	651	1049	1051	849	949	27000
341	1259	459	1341	1441	559	343	1257	457	1343	1443	557	345	1255	455	1345	1445	555	347	1253	453	1347	1447	553	349	1251	451	1349	1449	551	27000
1541	241	141	159	1559	1759	1543	243	143	157	1557	1757	1545	245	145	155	1555	1755	1547	247	147	153	1553	1753	1549	249	149	151	1551	1751	27000
27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000	27000

In this case the magic sum is $S_{30\times 30} = 27000 = 30^3$, and the sum of all entries is $T_{900} := 30 \times 27000 = 810000 = 900^2 = 30^4$. Block of order 6 magic squares equal magic sum $S_{6\times 6} = 5400$. It satisfy the uniformity property (11), i.e., $\left\langle 30, 30^2, 30^3, 30^4 \right\rangle$.

30.1 Pythagorean Triples

According to Example 30.1, we have perfect square entries sum, i.e., $T_{900} := 30 \times 27000 = 810000 = 900^2$. Below are **Pythagorean triples** for the number 900:

$$\begin{array}{lll}
900^2 + 95^2 := 905^2 & 900^2 + 1767^2 := 1983^2 & 900^2 + 8075^2 := 8125^2 \\
900^2 + 165^2 := 915^2 & 900^2 + 1925^2 := 2125^2 & 900^2 + 10105^2 := 10145^2 \\
900^2 + 301^2 := 949^2 & 900^2 + 2160^2 := 2340^2 & 900^2 + 11232^2 := 11268^2 \\
900^2 + 375^2 := 975^2 & 900^2 + 2419^2 := 2581^2 & 900^2 + 13485^2 := 13515^2 \\
900^2 + 480^2 := 1020^2 & 900^2 + 2625^2 := 2775^2 & 900^2 + 16863^2 := 16887^2 \\
900^2 + 560^2 := 1060^2 & 900^2 + 3315^2 := 3435^2 & 900^2 + 20240^2 := 20260^2 \\
900^2 + 675^2 := 1125^2 & 900^2 + 3696^2 := 3804^2 & 900^2 + 22491^2 := 22509^2 \\
900^2 + 945^2 := 1305^2 & 900^2 + 4000^2 := 4100^2 & 900^2 + 33744^2 := 33756^2 \\
900^2 + 1088^2 := 1412^2 & 900^2 + 4455^2 := 4545^2 & 900^2 + 40495^2 := 40505^2 \\
900^2 + 1200^2 := 1500^2 & 900^2 + 5589^2 := 5661^2 & 900^2 + 50621^2 := 50629^2 \\
900^2 + 1365^2 := 1635^2 & 900^2 + 6720^2 := 6780^2 & 900^2 + 67497^2 := 67503^2 \\
900^2 + 1495^2 := 1745^2 & 900^2 + 7473^2 := 7527^2 & 900^2 + 101248^2 := 101252^2 \\
 & & 900^2 + 202499^2 := 202501^2
\end{array}$$

Remark 30.1. *The above 37 **Pythagorean triples** are with number 900. This number is due to Examples 30.1. According to Result 2, there are only 12 triples generating magic square with **perfect square entries sum**:*

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$$
\begin{aligned}
(\mathbf{900, 301, 949}) &\Rightarrow 949 - 900 &&= 49 = 7^2 \\
(\mathbf{900, 675, 1125}) &\Rightarrow 1125 - 900 &&= 225 = 15^2 \\
(\mathbf{900, 1088, 1412}) &\Rightarrow 1412 - 1088 &&= 324 = 18^2 \\
(\mathbf{900, 1925, 2125}) &\Rightarrow 2125 - 900 &&= 1225 = 35^2 \\
(\mathbf{900, 2419, 2581}) &\Rightarrow 2581 - 900 &&= 1681 = 41^2 \\
(\mathbf{900, 4000, 4100}) &\Rightarrow 4100 - 4000 &&= 100 = 10^2 \\
(\mathbf{900, 5589, 5661}) &\Rightarrow 5661 - 900 &&= 4761 = 69^2 \\
(\mathbf{900, 8075, 8125}) &\Rightarrow 8125 - 900 &&= 7225 = 85^2 \\
(\mathbf{900, 11232, 11268}) &\Rightarrow 11268 - 11232 &&= 36 = 6^2 \\
(\mathbf{900, 22491, 22509}) &\Rightarrow 22509 - 900 &&= 21609 = 147^2 \\
(\mathbf{900, 50621, 50629}) &\Rightarrow 50629 - 900 &&= 49729 = 223^2 \\
(\mathbf{900, 202499, 202501}) &\Rightarrow 202501 - 900 &&= 201601 = 449^2.
\end{aligned}
$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(900, 301, 949)$ *generating a* ***perfect square entries sum*** *magic square of order 7 with the* ***odd number entries***, $\{1801, 1803, \ldots, 1895, 1897\}$ *with magic sum,* $\boldsymbol{S}_{7\times 7} := 12943$. *The entries total sum is* $\boldsymbol{T}_{49} := 7 \times 12943 = 90601 = 301^2$.

2. *The triple* $(900, 675, 1125)$ *generating a* ***perfect square entries sum*** *magic square of order 15 with the* ***odd number entries***, $\{1801, 1803, \ldots, 2247, 2249\}$ *with magic sum,* $\boldsymbol{S}_{15\times 15} := 30375$. *The entries total sum is* $\boldsymbol{T}_{225} := 15 \times 30375 = 455625 = 675^2$.

3. *The triple* $(900, 1088, 1412)$ *generating a* ***perfect square entries sum*** *magic square of order 18 with the* ***odd number entries***, $\{2177, 2179, \ldots, 2821, 2823\}$ *with magic sum,* $\boldsymbol{S}_{18\times 18} := 45000$. *The entries total sum is* $\boldsymbol{T}_{324} := 18 \times 45000 = 810000 = 900^2$.

4. *The triple* $(900, 1925, 2125)$ *generating a* ***perfect square entries sum*** *magic square of order 35 with the* ***odd number entries***, $\{1801, 1803, \ldots, 4247, 4249\}$ *with magic sum,* $\boldsymbol{S}_{35\times 35} := 105875$. *The entries total sum is*

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$T_{1225} := 35 \times 105875 = 3705625 = 1925^2$.

5. *The triple* $(900, 2419, 2581)$ *generating a* ***perfect square entries sum*** *magic square of order 41 with the* ***odd number entries***, $\{1801, 1803, \ldots, 5159, 5161\}$ *with magic sum,* $S_{41\times 41} := 142721$. *The entries total sum is* $T_{1681} := 41 \times 142721 = 5851561 = 2419^2$.

6. *The triple* $(900, 4000, 4100)$ *generating a* ***perfect square entries sum*** *magic square of order 10 with the* ***odd number entries***, $\{8001, 8003, \ldots, 8197, 8199\}$ *with magic sum,* $S_{10\times 10} := 81000$. *The entries total sum is* $T_{100} := 10 \times 81000 = 810000 = 900^2$.

7. *The triple* $(900, 5589, 5661)$ *generating a* ***perfect square entries sum*** *magic square of order 69 with the* ***odd number entries***, $\{1801, 1803, \ldots, 11319, 11321\}$ *with magic sum,* $S_{69\times 69} := 452709$. *The entries total sum is* $T_{4761} := 69 \times 452709 = 31236921 = 5589^2$.

8. *The triple* $(900, 8075, 8125)$ *generating a* ***perfect square entries sum*** *magic square of order 85 with the* ***odd number entries***, $\{1801, 1803, \ldots, 16247, 16249\}$ *with magic sum,* $S_{85\times 85} := 767125$. *The entries total sum is* $T_{7225} := 85 \times 767125 = 65205625 = 8075^2$.

9. *The triple* $(900, 11232, 11268)$ *generating a* ***perfect square entries sum*** *magic square of order 6 with the* ***odd number entries***, $\{22465, 22467, \ldots, 22533, 22535\}$ *with magic sum,* $S_{6\times 6} := 135000$. *The entries total sum is* $T_{36} := 6 \times 135000 = 810000 = 900^2$.

10. *The triple* $(900, 22491, 22509)$ *generating a* ***perfect square entries sum*** *magic square of order 147 with the* ***odd number entries***, $\{1801, 1803, \ldots, 45015, 45017\}$ *with magic sum,* $S_{147\times 147} := 3441123$. *The entries total sum is* $T_{21609} := 147 \times 3441123 = 505845081 = 22491^2$.

11. *The triple* $(900, 50621, 50629)$ *generating a* ***perfect square entries sum*** *magic square of order 223 with the* ***odd number entries***, $\{1801, 1803, \ldots, 101255, 101257\}$ *with magic sum,* $S_{223\times 223} := 11490967$. *The entries total sum is* $T_{49729} := 223 \times 11490967 = 2562485641 = 50621^2$.

12. *The triple* $(900, 202499, 202501)$ *generating a* ***perfect square entries sum*** *magic square of order 449 with the* ***odd number entries***, $\{1801, 1803, \ldots, 404999, 405001\}$ *with magic sum,* $S_{449\times 449} := 91327049$. *The*

entries total sum is $\boldsymbol{T}_{201601} := 449 \times 91327049 = 41005845001 = 202499^2$.

31 Magic Square of Order 31

According to equations (3) and (8), one can obtain **perfect square entries sum magic squares** of order 31 in two ways: first using **consecutive odd numbers**, and the second using **consecutive natural numbers**. For the **consecutive natural numbers** there are again two ways: the first one is of **uniformity**, and the second is of **minimum perfect square entries sum**.

31.1 Consecutive Odd Numbers

Taking $k = 31$ in equation (3), we get

$$\begin{aligned} & 1+3+5+\ldots+(2\times 31^2-1) = 31^4 \\ \Rightarrow & 1+3+5+\ldots+1919+1921 \quad = 923521 = 961^2 = 31^4 \end{aligned}$$

Example 31.1. *For the **consecutive odd number** entries* $\{1, 3, 5, \ldots, 1919, 1921\}$ *a **block-bordered** magic square of order 31 is given by*

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63	89	81	65	1885	1893	61	1877	93	117	1913	1909	69	1917	113	1869	105	77	101	109	85	1905	1901	1881	1873	1921	97	1897	1889	73	1863
1871	1743	1723	159	1711	1707	139	1731	1719	1699	143	131	1735	163	1727	155	147	173	171	1739	1715	127	1691	151	167	1695	135	1703	123	1747	51
1875	165	1475	1129	279	1503	1099	281	1483	1133	267	1505	1107	271	1515	1109	259	1517	1119	247	1491	1141	251	1525	1121	237	1495	1149	239	1757	47
1899	137	265	1521	1097	235	1523	1125	269	1509	1105	243	1513	1127	245	1501	1137	255	1489	1139	277	1493	1113	257	1479	1147	285	1481	1117	1785	23
103	1745	1143	233	1507	1145	261	1477	1131	241	1511	1135	263	1485	1123	273	1487	1111	275	1497	1115	249	1519	1101	283	1499	1103	253	1527	177	1819
1903	1741	1529	319	1035	1557	289	1037	1537	323	1023	1559	297	1027	1569	299	1015	1571	309	1003	1545	331	1007	1579	311	993	1549	339	995	181	19
71	125	1021	1575	287	991	1577	315	1025	1563	295	999	1567	317	1001	1555	327	1011	1543	329	1033	1547	303	1013	1533	337	1041	1535	307	1797	1851
111	1733	333	989	1561	335	1017	1531	321	997	1565	325	1019	1539	313	1029	1541	301	1031	1551	305	1005	1573	291	1039	1553	293	1009	1581	189	1811
1879	1693	1151	1237	495	1179	1207	497	1159	1241	483	1181	1215	487	1191	1217	475	1193	1227	463	1167	1249	467	1201	1229	453	1171	1257	455	229	43
67	129	481	1197	1205	451	1199	1233	485	1185	1213	459	1189	1235	461	1177	1245	471	1165	1247	493	1169	1221	473	1155	1255	501	1157	1225	1793	1855
1883	1697	1251	449	1183	1253	477	1153	1239	457	1187	1243	479	1161	1231	489	1163	1219	491	1173	1223	465	1195	1209	499	1175	1211	469	1203	225	39
115	133	1259	535	1089	1287	505	1091	1267	539	1077	1289	513	1081	1299	515	1069	1301	525	1057	1275	547	1061	1309	527	1047	1279	555	1049	1789	1807
119	161	1075	1305	503	1045	1307	531	1079	1293	511	1053	1297	533	1055	1285	543	1065	1273	545	1087	1277	519	1067	1263	553	1095	1265	523	1761	1803
1895	1729	549	1043	1291	551	1071	1261	537	1051	1295	541	1073	1269	529	1083	1271	517	1085	1281	521	1059	1303	507	1093	1283	509	1063	1311	193	27
79	145	935	589	1359	963	559	1361	943	593	1347	965	567	1351	975	569	1339	977	579	1327	951	601	1331	985	581	1317	955	609	1319	1777	1843
95	1737	1345	981	557	1315	983	585	1349	969	565	1323	973	587	1325	961	597	1335	949	599	1357	953	573	1337	939	607	1365	941	577	185	1827
91	1725	603	1313	967	605	1341	937	591	1321	971	595	1343	945	583	1353	947	571	1355	957	575	1329	979	561	1363	959	563	1333	987	197	1831
87	149	611	859	1413	639	829	1415	619	863	1401	641	837	1405	651	839	1393	653	849	1381	627	871	1385	661	851	1371	631	879	1373	1773	1835
1911	169	1399	657	827	1369	659	855	1403	645	835	1377	649	857	1379	637	867	1389	625	869	1411	629	843	1391	615	877	1419	617	847	1753	11
99	153	873	1367	643	875	1395	613	861	1375	647	865	1397	621	853	1407	623	841	1409	633	845	1383	655	831	1417	635	833	1387	663	1769	1823
1865	157	719	1453	711	747	1423	713	727	1457	699	749	1431	703	759	1433	691	761	1443	679	735	1465	683	769	1445	669	739	1473	671	1765	57
1891	1701	697	765	1421	667	767	1449	701	753	1429	675	757	1451	677	745	1461	687	733	1463	709	737	1437	689	723	1471	717	725	1441	221	31
83	1709	1467	665	751	1469	693	721	1455	673	755	1459	695	729	1447	705	731	1435	707	741	1439	681	763	1425	715	743	1427	685	771	213	1839
1915	1721	341	913	1629	369	883	1631	349	917	1617	371	891	1621	381	893	1609	383	903	1597	357	925	1601	391	905	1587	361	933	1589	201	7
1867	1717	1615	387	881	1585	389	909	1619	375	889	1593	379	911	1595	367	921	1605	355	923	1627	359	897	1607	345	931	1635	347	901	205	55
1887	1713	927	1583	373	929	1611	343	915	1591	377	919	1613	351	907	1623	353	895	1625	363	899	1599	385	885	1633	365	887	1603	393	209	35
107	1705	395	1669	819	423	1639	821	403	1673	807	425	1647	811	435	1649	799	437	1659	787	411	1681	791	445	1661	777	415	1689	779	217	1815
75	141	805	441	1637	775	443	1665	809	429	1645	783	433	1667	785	421	1677	795	409	1679	817	413	1653	797	399	1687	825	401	1657	1781	1847
1907	121	1683	773	427	1685	801	397	1671	781	431	1675	803	405	1663	813	407	1651	815	417	1655	789	439	1641	823	419	1643	793	447	1801	15
1919	175	199	1763	211	215	1783	191	203	223	1779	1791	187	1759	195	1767	1775	1749	1751	183	207	1795	231	1771	1755	227	1787	219	1799	179	3
59	1833	1841	1857	37	29	1861	45	1829	1805	9	13	1853	5	1809	53	1817	1845	1821	1813	1837	17	21	41	49	1	1825	25	33	1849	1859

The above **block-bordered** magic square of order 31 is with inner part as **block-wise** magic square of order 27 with blocks of orders 3 as **semi-magic** squares with equal **semi-magic** sums. Below are

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respective magic sums:

$$S_{31\times31} = 29791$$
$$S_{29\times29} = 27869$$
$$S_{27\times27} = 25947$$
$$Sm_{3\times3} = 2883$$

The total entries sum is $\boldsymbol{T}_{961} := 29791 = 923521 = 961^2$. It satisfy the uniformity property (11), i.e., $\left\langle 31, 31^2, 31^3, 31^4 \right\rangle$

31.2 Consecutive Natural Numbers

According to equation (8),

$$G := \frac{n(n+1)}{2} - \frac{(n-961)(n-960)}{2} = 961(n-480)$$

Taking $n = 1441$, we get a perfect square, i.e.,

$$G := T(1441) - T(480) = \frac{1441 \times 1442}{2} - \frac{480 \times 481}{2} = 1441 \times 721 - 240 \times 481$$
$$= 923521 = 961^2 = 31^4.$$

Simplifying, we get

$$481 + 482 + 483 + \ldots + 1440 + 1441 = 923521 = 961^2 = 31^4.$$

This gives a perfect square entries sum for the 961 **consecutive natural numbers** from to 481 to 1441.

Example 31.2. *For the* ***consecutive natural number*** *entries* $\{481, 482, 483, \ldots, 1440, 1441\}$ *a* ***block-bordered*** *magic square of order 31 is given by*

512	525	521	513	1423	1427	511	1419	527	539	1437	1435	515	1439	537	1415	533	519	531	535	523	1433	1431	1421	1417	1441	529	1429	1425	517	1412
1416	1352	1342	560	1336	1334	550	1346	1340	1330	552	546	1348	562	1344	558	554	567	566	1350	1338	544	1326	556	564	1328	548	1332	542	1354	506
1418	563	1218	1045	620	1232	1030	621	1222	1047	614	1233	1034	616	1238	1035	610	1239	1040	604	1226	1051	606	1243	1041	599	1228	1055	600	1359	504
1430	549	613	1241	1029	598	1242	1043	615	1235	1033	602	1237	1044	603	1231	1049	608	1225	1050	619	1227	1037	609	1220	1054	623	1221	1039	1373	492
532	1353	1052	597	1234	1053	611	1219	1046	601	1236	1048	612	1223	1042	617	1224	1036	618	1229	1038	605	1240	1031	622	1230	1032	607	1244	569	1390
1432	1351	1245	640	998	1259	625	999	1249	642	992	1260	629	994	1265	630	988	1266	635	982	1253	646	984	1270	636	977	1255	650	978	571	490
516	543	991	1268	624	976	1269	638	993	1262	628	980	1264	639	981	1258	644	986	1252	645	997	1254	632	987	1247	649	1001	1248	634	1379	1406
536	1347	647	975	1261	648	989	1246	641	979	1263	643	990	1250	637	995	1251	631	996	1256	633	983	1267	626	1000	1257	627	985	1271	575	1386
1420	1327	1056	1099	728	1070	1084	729	1060	1101	722	1071	1088	724	1076	1089	718	1077	1094	712	1064	1105	714	1081	1095	707	1066	1109	708	595	502
514	545	721	1079	1083	706	1080	1097	723	1073	1087	710	1075	1098	711	1069	1103	716	1063	1104	727	1065	1091	717	1058	1108	731	1059	1093	1377	1408
1422	1329	1106	705	1072	1107	719	1057	1100	709	1074	1102	720	1061	1096	725	1062	1090	726	1067	1092	713	1078	1085	730	1068	1086	715	1082	593	500
538	547	1110	748	1025	1124	733	1026	1114	750	1019	1125	737	1021	1130	738	1015	1131	743	1009	1118	754	1011	1135	744	1004	1120	758	1005	1375	1384
540	561	1018	1133	732	1003	1134	746	1020	1127	736	1007	1129	747	1008	1123	752	1013	1117	753	1024	1119	740	1014	1112	757	1028	1113	742	1361	1382
1428	1345	755	1002	1126	756	1016	1111	749	1006	1128	751	1017	1115	745	1022	1116	739	1023	1121	741	1010	1132	734	1027	1122	735	1012	1136	577	494
520	553	948	775	1160	962	760	1161	952	777	1154	963	764	1156	968	765	1150	969	770	1144	956	781	1146	973	771	1139	958	785	1140	1369	1402
528	1349	1153	971	759	1138	972	773	1155	965	763	1142	967	774	1143	961	779	1148	955	780	1159	957	767	1149	950	784	1163	951	769	573	1394
526	1343	782	1137	964	783	1151	949	776	1141	966	778	1152	953	772	1157	954	766	1158	959	768	1145	970	761	1162	960	762	1147	974	579	1396
524	555	786	910	1187	800	895	1188	790	912	1181	801	899	1183	806	900	1177	807	905	1171	794	916	1173	811	906	1166	796	920	1167	1367	1398
1436	565	1180	809	894	1165	810	908	1182	803	898	1169	805	909	1170	799	914	1175	793	915	1186	795	902	1176	788	919	1190	789	904	1357	486
530	557	917	1164	802	918	1178	787	911	1168	804	913	1179	791	907	1184	792	901	1185	797	903	1172	808	896	1189	798	897	1174	812	1365	1392
1413	559	840	1207	836	854	1192	837	844	1209	830	855	1196	832	860	1197	826	861	1202	820	848	1213	822	865	1203	815	850	1217	816	1363	509
1426	1331	829	863	1191	814	864	1205	831	857	1195	818	859	1206	819	853	1211	824	847	1212	835	849	1199	825	842	1216	839	843	1201	591	496
522	1335	1214	813	856	1215	827	841	1208	817	858	1210	828	845	1204	833	846	1198	834	851	1200	821	862	1193	838	852	1194	823	866	587	1400
1438	1341	651	937	1295	665	922	1296	655	939	1289	666	926	1291	671	927	1285	672	932	1279	659	943	1281	676	933	1274	661	947	1275	581	484
1414	1339	1288	674	921	1273	675	935	1290	668	925	1277	670	936	1278	664	941	1283	658	942	1294	660	929	1284	653	946	1298	654	931	583	508
1424	1337	944	1272	667	945	1286	652	938	1276	669	940	1287	656	934	1292	657	928	1293	662	930	1280	673	923	1297	663	924	1282	677	585	498
534	1333	678	1315	890	692	1300	891	682	1317	884	693	1304	886	698	1305	880	699	1310	874	686	1321	876	703	1311	869	688	1325	870	589	1388
518	551	883	701	1299	868	702	1313	885	695	1303	872	697	1314	873	691	1319	878	685	1320	889	687	1307	879	680	1324	893	681	1309	1371	1404
1434	541	1322	867	694	1323	881	679	1316	871	696	1318	882	683	1312	887	684	1306	888	689	1308	875	700	1301	892	690	1302	877	704	1381	488
1440	568	580	1362	586	588	1372	576	582	592	1370	1376	574	1360	578	1364	1368	1355	1356	572	584	1378	596	1366	1358	594	1374	590	1380	570	482
510	1397	1401	1409	499	495	1411	503	1395	1383	485	487	1407	483	1385	507	1389	1403	1391	1387	1399	489	491	501	505	481	1393	493	497	1405	1410

The above **block-bordered** magic square of order 31 is with inner part as **block-wise** magic square of order 27 with blocks of orders 3 as **semi-magic** squares with equal **semi-magic** sums. Below are respective magic sums:

$$S_{31\times 31} = 29791$$
$$S_{29\times 29} = 27869$$
$$S_{27\times 27} = 25947$$
$$Sm_{3\times 3} = 2883$$

Both the Examples 31.1 and 31.2, the magic sum is same. The total entries sum is $\boldsymbol{T}_{961} := 29791 = 923521 = 961^2$. It satisfy the uniformity property (11), i.e., $\left\langle 31, 31^2, 31^3, 31^4 \right\rangle$

31.3 Minimum Perfect Square Entries Sum

The examples given in 31.1 and 31.2 satisfy the uniformity property (11), i.e., $\left\langle 31, 31^2, 31^3, 31^4 \right\rangle$. However, the sum of the magic square entries of 31.2 is not a **minimum perfect square entries sum**. Choosing $m = 22, 23, \ldots, 31$ and $p = 31$ in equation (13), we get

$$1. \quad L\left(22^2 + \frac{31^2-1}{2}, 31^2\right) \rightarrow (31, 4, 964, 15004, 465124)$$
$$2. \quad L\left(23^2 + \frac{31^2-1}{2}, 31^2\right) \rightarrow (31, 49, 1009, 16399, 508369)$$
$$\ldots \quad \ldots \quad\quad \ldots \quad \ldots$$
$$10. \quad L\left(31^2 + \frac{31^2-1}{2}, 31^2\right) \rightarrow (31, 481, 1441, 29791, 923521) \Rightarrow \left\langle 31, 31^2, 31^3, 31^4 \right\rangle$$

The values written above are for $m = 22, 23, \ldots, 31$. The 10^{th} value for $m = 31$ satisfy the uniformity property (11). This case is already studied in Example 31.1. The first value for $m = 22$ give **minimum perfect square** magic square. In this case, we have magic square of order 31 with sum of all entries $465124 = 682^2$ which is a perfect square, but it does not satisfy the uniformity property (11). The magic square given below has a **minimum perfect square entries sum**.

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Example 31.3. *For the* ***consecutive natural number*** *entries* $\{4,5,6,\ldots,963,964\}$*, a* ***block-bordered*** *magic square of order 19 is given by*

35	48	44	36	946	950	34	942	50	62	960	958	38	962	60	938	56	42	54	58	46	956	954	944	940	964	52	952	948	40	935
939	875	865	83	859	857	73	869	863	853	75	69	871	85	867	81	77	90	89	873	861	67	849	79	87	851	71	855	65	877	29
941	86	741	568	143	755	553	144	745	570	137	756	557	139	761	558	133	762	563	127	749	574	129	766	564	122	751	578	123	882	27
953	72	136	764	552	121	765	566	138	758	556	125	760	567	126	754	572	131	748	573	142	750	560	132	743	577	146	744	562	896	15
55	876	575	120	757	576	134	742	569	124	759	571	135	746	565	140	747	559	141	752	561	128	763	554	145	753	555	130	767	92	913
955	874	768	163	521	782	148	522	772	165	515	783	152	517	788	153	511	789	158	505	776	169	507	793	159	500	778	173	501	94	13
39	66	514	791	147	499	792	161	516	785	151	503	787	162	504	781	167	509	775	168	520	777	155	510	770	172	524	771	157	902	929
59	870	170	498	784	171	512	769	164	502	786	166	513	773	160	518	774	154	519	779	156	506	790	149	523	780	150	508	794	98	909
943	850	579	622	251	593	607	252	583	624	245	594	611	247	599	612	241	600	617	235	587	628	237	604	618	230	589	632	231	118	25
37	68	244	602	606	229	603	620	246	596	610	233	598	621	234	592	626	239	586	627	250	588	614	240	581	631	254	582	616	900	931
945	852	629	228	595	630	242	580	623	232	597	625	243	584	619	248	585	613	249	590	615	236	601	608	253	591	609	238	605	116	23
61	70	633	271	548	647	256	549	637	273	542	648	260	544	653	261	538	654	266	532	641	277	534	658	267	527	643	281	528	898	907
63	84	541	656	255	526	657	269	543	650	259	530	652	270	531	646	275	536	640	276	547	642	263	537	635	280	551	636	265	884	905
951	868	278	525	649	279	539	634	272	529	651	274	540	638	268	545	639	262	546	644	264	533	655	257	550	645	258	535	659	100	17
43	76	471	298	683	485	283	684	475	300	677	486	287	679	491	288	673	492	293	667	479	304	669	496	294	662	481	308	663	892	925
51	872	676	494	282	661	495	296	678	488	286	665	490	297	666	484	302	671	478	303	682	480	290	672	473	307	686	474	292	96	917
49	866	305	660	487	306	674	472	299	664	489	301	675	476	295	680	477	289	681	482	291	668	493	284	685	483	285	670	497	102	919
47	78	309	433	710	323	418	711	313	435	704	324	422	706	329	423	700	330	428	694	317	439	696	334	429	689	319	443	690	890	921
959	88	703	332	417	688	333	431	705	326	421	692	328	432	693	322	437	698	316	438	709	318	425	699	311	442	713	312	427	880	9
53	80	440	687	325	441	701	310	434	691	327	436	702	314	430	707	315	424	708	320	426	695	331	419	712	321	420	697	335	888	915
936	82	363	730	359	377	715	360	367	732	353	378	719	355	383	720	349	384	725	343	371	736	345	388	726	338	373	740	339	886	32
949	854	352	386	714	337	387	728	354	380	718	341	382	729	342	376	734	347	370	735	358	372	722	348	365	739	362	366	724	114	19
45	858	737	336	379	738	350	364	731	340	381	733	351	368	727	356	369	721	357	374	723	344	385	716	361	375	717	346	389	110	923
961	864	174	460	818	188	445	819	178	462	812	189	449	814	194	450	808	195	455	802	182	466	804	199	456	797	184	470	798	104	7
937	862	811	197	444	796	198	458	813	191	448	800	193	459	801	187	464	806	181	465	817	183	452	807	176	469	821	177	454	106	31
947	860	467	795	190	468	809	175	461	799	192	463	810	179	457	815	180	451	816	185	453	803	196	446	820	186	447	805	200	108	21
57	856	201	838	413	215	823	414	205	840	407	216	827	409	221	828	403	222	833	397	209	844	399	226	834	392	211	848	393	112	911
41	74	406	224	822	391	225	836	408	218	826	395	220	837	396	214	842	401	208	843	412	210	830	402	203	847	416	204	832	894	927
957	64	845	390	217	846	404	202	839	394	219	841	405	206	835	410	207	829	411	212	831	398	223	824	415	213	825	400	227	904	11
963	91	103	885	109	111	895	99	105	115	893	899	97	883	101	887	891	878	879	95	107	901	119	889	881	117	897	113	903	93	5
33	920	924	932	22	18	934	26	918	906	8	10	930	6	908	30	912	926	914	910	922	12	14	24	28	4	916	16	20	928	933

The above example 31.3 has a magic sum $S_{31\times 31} = 15004$, and the sum of all entries is $T_{961} := 31 \times 15004 = 465124 = 682^2$. The Example 31.3 is **block-bordered** magic squares, where inner **block-wise** magic square of order 27 is **pandiagonal**. Moreover, blocks of order 3 are equal sum **semi-magic** squares, with **semi-magic** sums: $S_{3\times 3} = 1452$.

31.4 Pythagorean Triples

According to Examples 31.1, 31.2 and 31.3, we have two perfect square entries sums, i.e., $\boldsymbol{T}_{961} := 29791 = 923521 = 961^2$ and $\boldsymbol{T}_{961} := 31 \times 15004 = 465124 = 682^2$. Below are **Pythagorean triples** for the numbers 961 and 682:

$$\mathbf{961^2} + 14880^2 := 14911^2$$
$$\mathbf{961^2} + 461760^2 := 461761^2$$
$$\mathbf{682^2} + 840^2 := 1082^2$$
$$\mathbf{682^2} + 3720^2 := 3782^2$$
$$\mathbf{682^2} + 10560^2 := 10582^2$$
$$\mathbf{682^2} + 116280^2 := 116282^2$$

Remark 31.1. *The above 6* ***Pythagorean triples*** *are with numbers 625 and 450. These numbers are due to Examples 31.1 and 31.3. According to Result 2, there are only 2 triples generating magic square with* ***perfect square entries sum****:*

$$(\mathbf{682, 840, 1082}) \Rightarrow 1082 - 682 = 400 = 20^2$$
$$(\mathbf{682, 116280, 116282}) \Rightarrow 116282 - 682 = 115600 = 340^2.$$

Below are details of magic square calculated according expression (17) *given in Result 3:*

1. *The triple* $(682, 840, 1082)$ *generating a* ***perfect square entries sum*** *magic square of order 20 with the* ***odd number entries****,* $\{1365, 1367, \ldots, 2161, 2163\}$ *with magic sum,* $\boldsymbol{S}_{20\times 20} := 35280$*. The entries total sum is* $\boldsymbol{T}_{400} := 20 \times 35280 = 705600 = 840^2$.
2. *The triple* $(682, 116280, 116282)$ *generating a* ***perfect square entries sum*** *magic square of order 340 with the* ***odd number entries****,* $\{1365, 1367, \ldots, 232561, 232563\}$ *with magic sum,* $\boldsymbol{S}_{340\times 340} := 39767760$*. The entries total sum is* $\boldsymbol{T}_{115600} := 340 \times 39767760 = 13521038400 = 116280^2$.

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