A strengthened form of the strong Goldbach conjecture

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Abstract. Based on a strengthened form of the strong Goldbach conjecture, this paper constitutes an antinomy within ZFC.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Theorem. $\mathbb{N}_4 \neq \mathbb{N}_4$.

Proof. We define the set $S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \}$.

SSGB is equivalent to saying that every integer x ≥ 4 is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers x ≥ 4 appear as m in a middle component mk of S_g. So, by the definitions we have

SSGB <=> $\forall x \in \mathbb{N}_4 \quad \exists (pk, mk, qk) \in S_g \quad x = m$.

$\neg$SSGB <=> $\exists x \in \mathbb{N}_4 \quad \forall (pk, mk, qk) \in S_g \quad x \neq m$.

The set S_g has the following property: The whole range of $\mathbb{N}_3$ can be expressed by the triple components of S_g, since every integer x ≥ 3 can be written as some pk with k = 1 when x is prime, as some pk with k ≠ 1 when x is composite and not a power of 2, or as (3 + 5)k / 2 when x is a power of 2; $p \in \mathbb{P}_3$, $k \in \mathbb{N}$.

In the case of $\neg$SSGB, there is at least one $n \in \mathbb{N}_4$ different from all the numbers m that are defined in S_g. In the case of SSGB, there is no such n. The following steps work regardless of the choice of n if there is more than one n.

According to the above three types of expression by S_g triple components, for n we have

(C) $\forall k \in \mathbb{N} \quad \exists (pk', mk', qk') \in S_g \quad nk = pk' \lor nk = mk' = 4k'$.

Moreover, due to the definition of S_g, we have

(M) $\nexists\, p, q \in \mathbb{P}_3, p < q \quad n = (p + q) / 2$.

The properties (C) and (M) hold for any n given by ¬SSGB. We will show that under the assumption ¬SSGB the set S_g can be written as the union of the following triples, which would be impossible without having (C) and (M).

(i) S_g triples of the form (pk' = nk, mk', qk') with k' = k in case n is prime, due to (C)

(ii) S_g triples of the form (pk' = nk, mk', qk') with k' ≠ k in case n is composite and not a power of 2, due to (C)

(iii) S_g triples of the form (3k', 4k' = nk, 5k') in case n is a power of 2, due to (C)

(iv) all remaining S_g triples of the form (pk' = nk, mk', qk'), (pk', mk' = nk, qk') or (pk', mk', qk' = nk)

and

(v) S_g triples of the form (pk' ≠ nk, mk' ≠ nk, qk' ≠ nk), i.e. those S_g triples where none of the nk's equals a component.

We can formalize this as follows.

We split S_g into two complementary subsets: For any $y \in \mathbb{N}_3$, $S_g = S_g+(y) \cup S_g-(y)$, where

$S_g+(y) := \{ (pk', mk', qk') \in S_g \mid \exists\, k \in \mathbb{N}\quad pk' = yk \;\vee\; mk' = yk \;\vee\; qk' = yk \}$ and

$S_g-(y) := \{ (pk', mk', qk') \in S_g \mid \forall\, k \in \mathbb{N}\quad pk' \neq yk \;\wedge\; mk' \neq yk \;\wedge\; qk' \neq yk \}$.

Let S_g+ be shorthand for $S_g+(n)$ and let S_g- be shorthand for $S_g-(n)$. Then, as S_g+ denotes the union of the triples of the above types (i) to (iv) and S_g- denotes the union of the triples of type (v), we can state

¬SSGB => (($S_g = S_g+ \cup S_g-$) or ¬(C) or ¬(M)).

Since (C) and (M) are true, we get

¬SSGB => $S_g = S_g+ \cup S_g-$.

$S_g+ \cup S_g-$ is independent of n, since for every n it equals S_g. So, we can write

(1) $\forall\, y \in \mathbb{N}_3$ ¬SSGB => $S_g = S_g+(y) \cup S_g-(y)$.

Under the assumption SSGB there is no n, which only means that the numbers m defined in S_g take all integer values $x \geq 4$. So, in addition to (1), here we also have

(2) $\forall\, y \in \mathbb{N}_3$ SSGB => $S_g = S_g+(y) \cup S_g-(y)$.

Because of the rule "*($\forall x$ $P(x)$) and ($\forall x$ $Q(x)$) <=> $\forall x$ ($P(x)$ and $Q(x)$)*", by (1) and (2) for each set $S_g+(y) \cup S_g-(y)$, $y \in \mathbb{N}_3$, we have

$\neg$SSGB => $S_g = S_g+(y) \cup S_g-(y)$

and

SSGB => $S_g = S_g+(y) \cup S_g-(y)$.

For each $k \geq 1$, we define $M(k) := \{ mk \mid (pk, mk, qk) \in S_g \}$. Then, for some set M,

(1') $\neg$SSGB => M(1) = M

and

(2') SSGB => M(1) = M.

On the other hand, under the assumption SSGB the numbers m defined in S_g take all integer values $x \geq 4$ whereas under $\neg$SSGB they don't. By this, we get

(3) $\neg$SSGB => $M(1) \neq \mathbb{N}_4$

and

(4) SSGB => $M(1) = \mathbb{N}_4$.

The statements (1'), (2'), (3), (4) were each derived from an assumption. That is, they were derived without using the tautology "False => Q" or the tautology "Q => True", i.e. none of the four proofs uses that SSGB or $\neg$SSGB is false, and the proofs for (1) and (2) don't use $S_g = S_g+(y) \cup S_g-(y)$.

This means that in each individual proof of (1') to (4) the premise is assumed to be true. Therefore, from (1') and (3) we obtain $M \neq \mathbb{N}_4$ and from (2') and (4) we obtain $M = \mathbb{N}_4$, which results in the contradiction $\mathbb{N}_4 \neq \mathbb{N}_4$.

□

Note. The proof is based on the following general principle. Let's suppose:

There exists a proposition P and there exist sets A, B, C, D with C ≠ D such that

(1) P => A = B

(2) $\neg$P => A = B

(3) P => A = C

(4) $\neg$P => A = D.

Each of the four statements has a proof that does not rely on the tautology "False => Q" or the tautology "Q => A = A". Furthermore, the definition of the sets B, C and D is not related to the proposition P.

Then, (1) to (4) lead to the conclusion B = C and B = D, and therefore to a contradiction.