A strengthened form of the strong Goldbach conjecture

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Abstract. Based on a strengthened form of the strong Goldbach conjecture, this paper constitutes an antinomy within ZFC.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Theorem. $\mathbb{N}_4 \neq \mathbb{N}_4$.

Proof. We define the set S_g := { (pk, mk, qk) | k, m $\in \mathbb{N}$; p, q $\in \mathbb{P}_3$, p < q; m = (p + q) / 2 }.

SSGB is equivalent to saying that every integer x ≥ 4 is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers x ≥ 4 appear as m in a middle component mk of S_g.

There are two possibilities for S_g, exactly one of which must occur: Either there is an n $\in \mathbb{N}_4$ in addition to all the numbers m defined in S_g or there is not. The latter corresponds to SSGB and the former corresponds to the negation ¬SSGB.

The set S_g has the following property: The whole range of $\mathbb{N}_3$ can be expressed by the triple components of S_g, since every integer x ≥ 3 can be written as some pk with k = 1 when x is prime, as some pk with k ≠ 1 when x is composite and not a power of 2, or as (3 + 5)k / 2 when x is a power of 2; p $\in \mathbb{P}_3$, k $\in \mathbb{N}$.

We can split S_g into two complementary subsets: For any y $\in \mathbb{N}_3$, $S_g = S_g+(y) \cup S_g-(y)$, with

$S_g+(y)$:= { (pk', mk', qk') $\in S_g$ | $\exists$ k $\in \mathbb{N}$ pk' = yk $\vee$ mk' = yk $\vee$ qk' = yk } and

$S_g-(y)$:= { (pk', mk', qk') $\in S_g$ | $\forall$ k $\in \mathbb{N}$ pk' ≠ yk $\wedge$ mk' ≠ yk $\wedge$ qk' ≠ yk }.

In the case of ¬SSGB, there is at least one n $\in \mathbb{N}_4$ different from all the numbers m that are defined in S_g. In the case of SSGB, there is no such n. The following steps work regardless of the choice of n if there is more than one n.

According to the above three types of expression by S_g triple components, for n we have

(C) $\forall$ k $\in \mathbb{N}$ $\exists$ (pk', mk', qk') $\in S_g$ nk = pk' $\vee$ nk = mk' = 4k'.

Moreover, due to the definition of S_g, we have

(M) $\nexists$ p, q $\in \mathbb{P}_3$, p < q   n = (p + q) / 2.

Because the properties (C) and (M) hold for any n given by ¬SSGB, under the assumption ¬SSGB the set S_g can be written as the union of the following triples, which would otherwise be impossible.

(i) S_g triples of the form (pk' = nk, mk', qk') with k' = k in case n is prime, due to (C)

(ii) S_g triples of the form (pk' = nk, mk', qk') with k' ≠ k in case n is composite and not a power of 2, due to (C)

(iii) S_g triples of the form (3k', 4k' = nk, 5k') in case n is a power of 2, due to (C)

(iv) all remaining S_g triples of the form (pk' = nk, mk', qk'), (pk', mk' = nk, qk') or (pk', mk', qk' = nk)

and

(v) S_g triples of the form (pk' ≠ nk, mk' ≠ nk, qk' ≠ nk), i.e. those S_g triples where none of the nk's equals a component.

Let S_g+ be shorthand for S_g+(n) and let S_g- be shorthand for S_g-(n). Then, as S_g+ denotes the union of the triples of types (i) to (iv) and S_g- denotes the union of the triples of type (v), we can state

¬SSGB => ((S_g = S_g+ ∪ S_g-) or ¬(C) or ¬(M)).

Since (C) and (M) are true, we get

¬SSGB => S_g = S_g+ ∪ S_g-.

S_g+ ∪ S_g- is independent of n, since for every n it equals S_g. So, we can write

(1) $\forall$ y $\in \mathbb{N}_3$ ¬SSGB => S_g = S_g+(y) ∪ S_g-(y).

Under the assumption SSGB there is no n, which only means that the numbers m defined in S_g take all integer values x ≥ 4. So, in addition to (1), here we also have

(2) $\forall$ y $\in \mathbb{N}_3$ SSGB => S_g = S_g+(y) ∪ S_g-(y).

We note that the statements (1) and (2) were each derived from an assumption. That is, they were not stated trivially as

P => S_g = S_g+(y) ∪ S_g-(y) and ¬P => S_g = S_g+(y) ∪ S_g-(y), which, because of $S_g = S_g+(y) \cup S_g-(y)$, holds for every proposition P. In other words, both (1) and (2) are true non-vacuously. (See also the general principle in the final Note.)

Because of the rule "*($\forall$ x P(x)) and ($\forall$ x Q(x)) <=> $\forall$ x (P(x) and Q(x))*", by (1) and (2) for each set $S_g+(y) \cup S_g-(y)$, $y \in \mathbb{N}_3$, we have

SSGB => $S_g = S_g+(y) \cup S_g-(y)$

and

$\neg$SSGB => $S_g = S_g+(y) \cup S_g-(y)$.

For each k ≥ 1, we define M(k) := { mk | (pk, mk, qk) $\in S_g$ }. Then, for some set M,

(1') SSGB => M(1) = M

and

(2') $\neg$SSGB => M(1) = M.

On the other hand, under the assumption SSGB the numbers m defined in S_g take all integer values x ≥ 4 whereas under $\neg$SSGB they don't. By this, we get

(3) SSGB => M(1) = $\mathbb{N}_4$

and

(4) $\neg$SSGB => M(1) ≠ $\mathbb{N}_4$.

Since the statements (1'), (2'), (3), (4) are all true non-vacuously, by transitivity we obtain the contradiction $\mathbb{N}_4 \neq \mathbb{N}_4$.

□

Note. The proof is based on the following general principle.

Let P be a proposition and let A, B, C, D be sets with C ≠ D. If we can establish the following statements non-vacuously, i.e. the premises are all true,

(1) P => A = B

(2) $\neg$P => A = B

(3) P => A = C

(4) $\neg P \Rightarrow A = D$,

then, by transitivity, we have a contradiction.