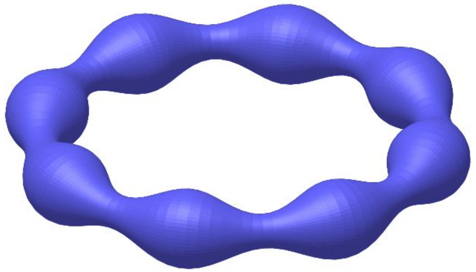
Is Flux Tube a Bosonic String? What Does Lattice Say?



Xmas Theoretical Physics Workshop @ Athens 2019

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Marie Skłodowska-Curie Individual Fellow
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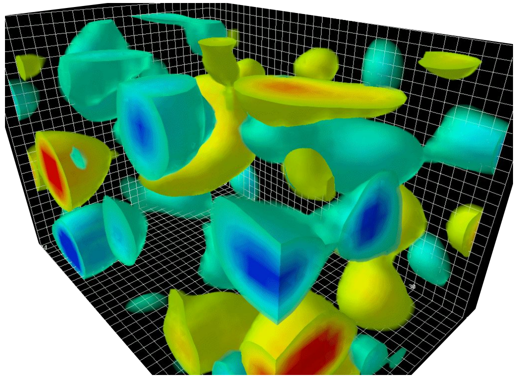
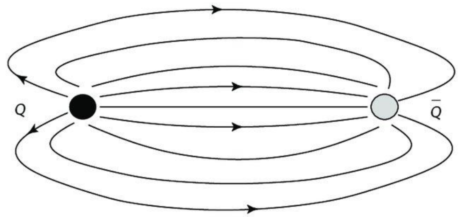


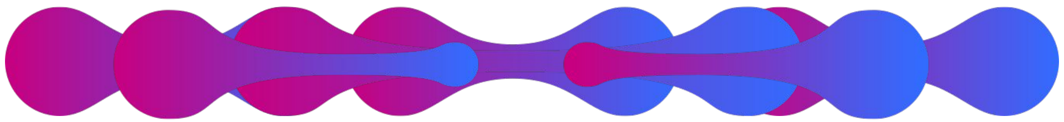
Courtesy to D. Leinweber

Work in collaboration with Mike Teper,
Based on arXiv:1103.5854, 1007.4720, 1702.03717, 1602.07634 and new upcoming results…

Question: Is Flux Tube a bosonic string?



In QCD quarks are confined in bound states by forming flux-tubes of chromo-magnetic and chromo-electric flux



$E \approx \sigma r$ Long flux tubes behave pretty much like strings

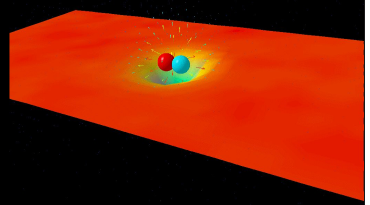
At some point they break (string breaking)

Let us think… there are $D-2$ obvious massless modes

Goldstone modes arising from the broken translation invariance in the $D-2$ directions transverse to the flux tube

There should be a Low Energy Effective String Theory model describing the energy spectrum of the flux tube

- In addition to string modes are there other kind of (massive) excitations?



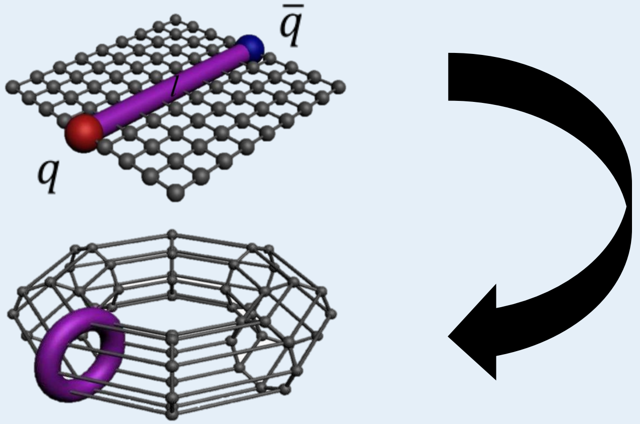
Question: Is Flux Tube a bosonic string?

- **Is there a theoretical description for the confining flux-tube in D=3+1 & D=2+1 ?**

- **Confining flux-tube:**

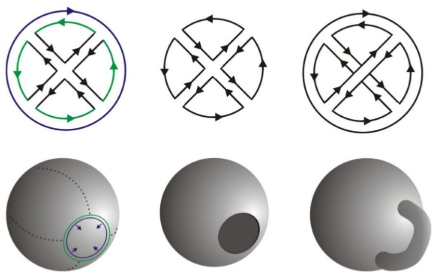
Open Flux-Tube

Closed Flux-Tube (Torelon)



Compactify

- **Pure gauge phenomena are also present...**
  - **Glueball – Flux-Tube mixing**
  - **Flux-tube – anti-Flux-Tube mixing**
- **Possible low-energy effective string theoretical description?**
  - **Cannot capture pure gauge phenomena!**
  - **Might be possible in the Large-*N* limit!**
- **Investigate Closed Flux tubes in the Large-*N* limit**



t' Hooft Coupling
Is kept fixed

$$\lambda = g^2 N$$

Have a look at S. Coleman's (Aspects of Symmetry)

Effective String Theory for (long) strings

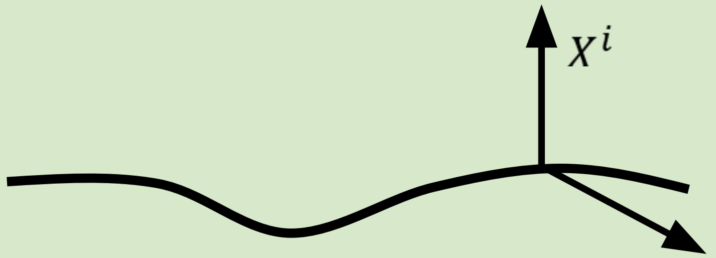
- Contributions by

M. Lüscher ’81, J. Polchinski & A. Strominger ‘90, M. Lüscher & P. Weisz ’04, O. Aharony et al ’07 – 11, S. Dubovski et al ’12 – 19

- Quantize the Bosonic String (Nambu-Goto String)

$$S_{\rm NG} = -\ell_s^{-2} \int d^2\sigma \sqrt{-\det \gamma} \qquad \gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta}\eta_{\mu\nu}$$

$\ell_s^{-1} = \sqrt{\sigma}$ (string tension)



The spectrum of a closed bosonic string compactified around a torus is:

$$E^2_{N_L,N_R,q,w} = (\sigma l w)^2 + 8\pi\sigma\left(\frac{N_L + N_R}{2} - \frac{D-2}{24}\right) + \left(\frac{2\pi q}{l}\right)^2$$

The spectrum is described by:

1. The winding number w (w=1, 2, . . .),
2. The winding momentum $p_{||} = 2\pi q/l$ with $q = 0, \pm 1, \pm 2, \ldots$.
3. The transverse momentum $p_\perp$ ($p_\perp = 0$),
4. $N_L = \sum_{k>0} n_L(k)k$ and $N_R = \sum_{k'>0} n_R(k')k'$
5. Level-matching constrain: $N_L - N_R = qw$.

$$(\alpha^{i_1}_{-k_1})^{n_L(k_1)} \ldots (\alpha^{i_{m_L}}_{-k_{m_L}})^{n_R(k_{m_L})} (\bar{\alpha}^{i'_1}_{-k'_1})^{n_R(k'_1)} \ldots (\bar{\alpha}^{i'_{m_R}}_{-k'_{m_R}})^{n_R(k'_{m_R})} |0\rangle$$

- Example: $\alpha_{-2}\alpha_{-1}\bar{\alpha}_{-1}|0\rangle$
  - ⋆ $N_L = 3$
  - ⋆ $N_R = 1$
  - ⋆ $q = 2$

Effective String Theory for (long) strings

- Nambu-Goto string is Lorentz invariance only in $D = 26$
- Gauge-Theory flux tubes must be described by a worldsheet theory that respects Lorentz in $D = 4$
- We can perturbatively calculate the spectrum of the flux-tube with Lorentz Invariance

$$S_{eff} = \sigma r \tau + \int_0^\tau dt \int_0^r dx \frac{1}{2}\partial h \partial h + \sum_{n=2} c_n \int_0^\tau dt \int_0^r dx (\partial h)^{2n} + \ldots$$

$E_n \overset{l\to\infty}{=}$	σl	linear confinement
	$+\ \frac{4\pi}{l}\left(n - \frac{D-2}{24}\right)$	Lüscher 1980, Polchinski&Strominger 1991
	$-\ \frac{8\pi^2}{\sigma l^3}\left(n - \frac{D-2}{24}\right)^2$	Lüscher&Weisz 2004, Drummond 2004
	$+\ \frac{32\pi^3}{\sigma^2 l^5}\left(n - \frac{D-2}{24}\right)^3$	Aharony&Karzbrun 2009

The Nambu-Goto energy for $w = 1$ and $q = 0$, in dimensionless units is written as:

$$\frac{E_n(l)}{\sqrt{\sigma}} = l\sqrt{\sigma}\left(1 + \frac{8\pi}{\sigma l^2}\left(n - \frac{1}{12}\right)\right)^{\frac{1}{2}} \quad \text{with} \quad n = \frac{N_L + N_R}{2}$$

becomes tachyonic for $\sigma l^2 < 2\pi/3 \equiv (1.45)^2$.

The above expression can be expanded in $1/l\sqrt{\sigma}$ for:

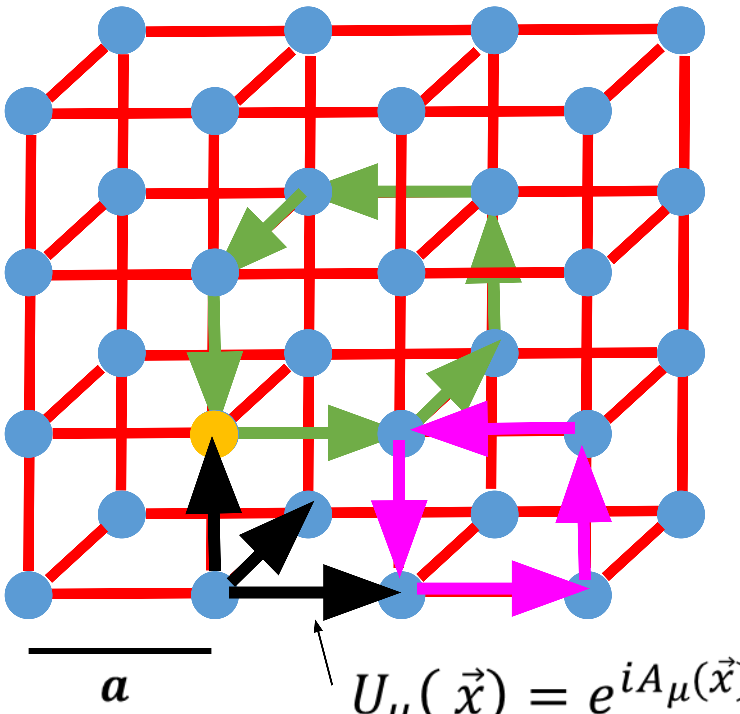
$$l\sqrt{\sigma} > l_c^{N.G}\sqrt{\sigma} = \left\{8\pi\left(n - \frac{1}{12}\right)\right\}^{\frac{1}{2}}$$

Relation to Nambu-Goto:

$$\frac{E_n(l)}{\sqrt{\sigma}} \overset{l\to\infty}{=} l\sqrt{\sigma} + \frac{c_1^{N.G}}{l\sqrt{\sigma}} + \frac{c_2^{N.G}}{(l\sqrt{\sigma})^3} + \frac{c_3^{N.G}}{(l\sqrt{\sigma})^5} + \mathcal{O}\left(\frac{1}{(l\sqrt{\sigma})^7}\right)$$

The Lattice

- Lattice is a mathematical trick
  - We discretised the Space time to a Hyper Cubic Lattice
  - We obtain physics at lattice spacing $a \to 0$
  - a is the minimum length (cutoff)
- It defines Quantum Field Theory as a limiting process
- Allows numerical calculations:
  - In hadron scales $g^2/4\pi \sim O(10)$ while $e^2/4\pi \sim 1/137$
  - There is no small parameter for expansion



$$U_\mu(\vec{x}) = e^{iA_\mu(\vec{x})} \in SU(N)$$

- Expectation value:

Gauge Invariant

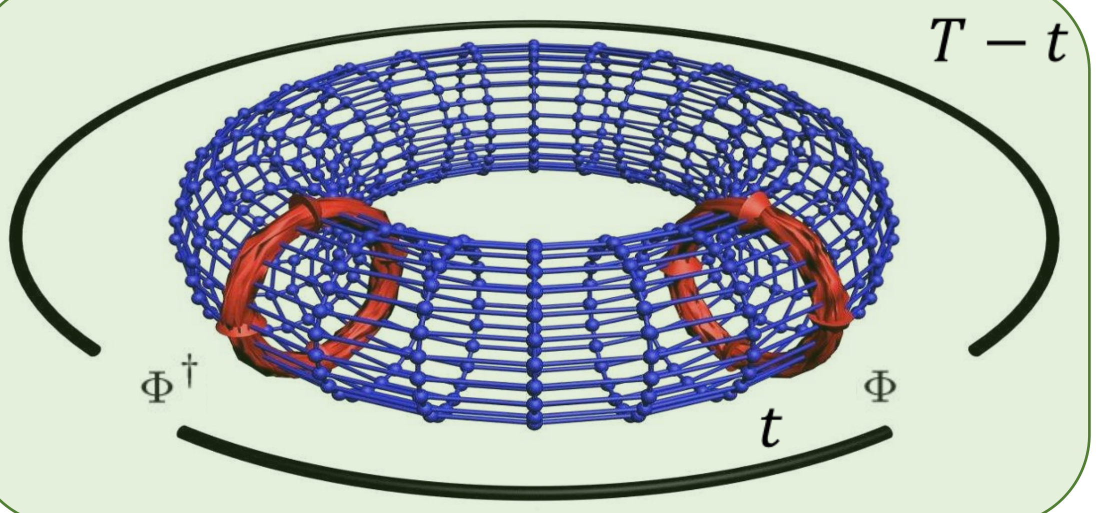
$$\langle O(A,\bar{q},q)\rangle = \frac{1}{Z}\int \prod_x dA(x)d\bar{q}(x)dq(x)O(A,q,\bar{q})e^{-S_{QCD}(A,\psi,\bar{\psi})}]$$

Yang-Mills theory on the Lattice

$$S_L = \beta \sum_p \{1 - \frac{1}{N}\mathrm{ReTr}U_p\}$$

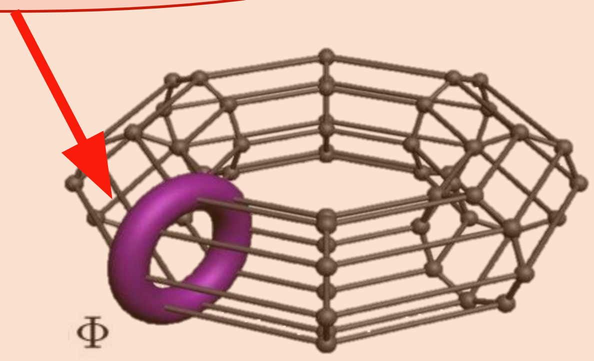
$$\beta = \frac{2N}{g^2}(D = 3+1)$$

Lattice Calculation: Correlation Function



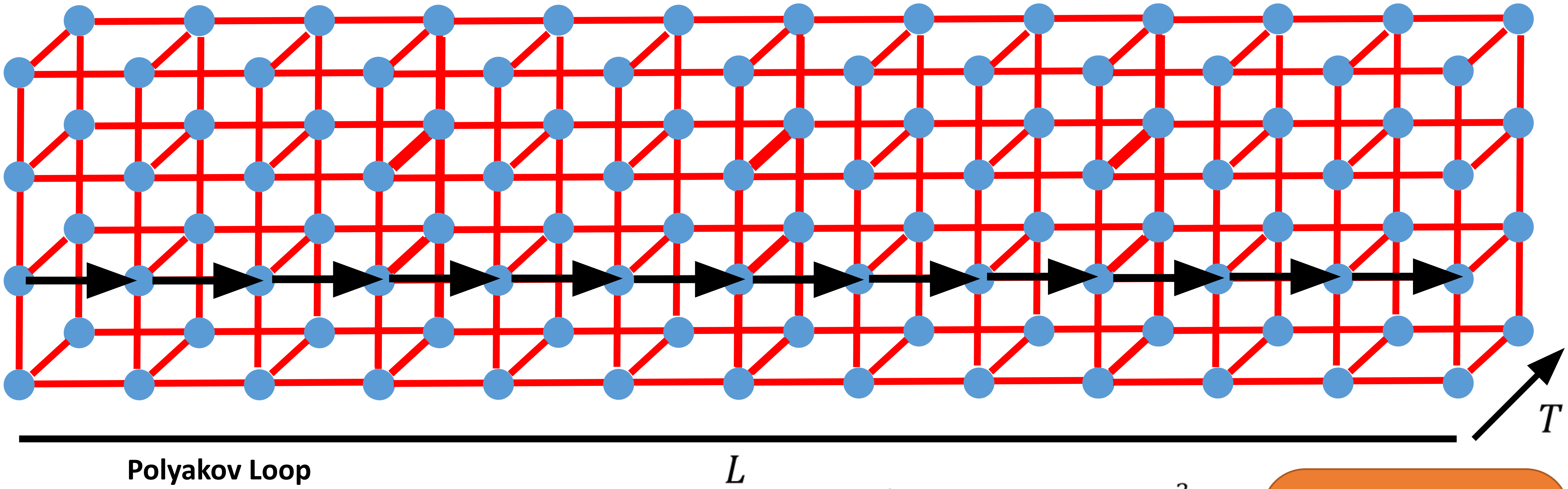
$$
\begin{aligned}
C(t) &= \langle \Phi^\dagger(t)\Phi(0)\rangle \\
&= \langle \Phi^\dagger(0)e^{-Ht}\Phi(0)\rangle \\
&= |\langle 0|\Phi(0)|vac\rangle|^2 e^{-E_0 t} \\
&\quad + \sum_{n=1} |\langle n|\Phi(0)|vac\rangle|^2 e^{-E_n t} \\
&\xrightarrow{t\to\infty} |\langle 0|\Phi(0)|vac\rangle|^2 e^{-E_0 t}
\end{aligned}
$$

- Construct a large basis of Operators $\Phi_i : i = 1, 2, \ldots$ RIGHT QUANTUM NUMBERS
- Calculate the correlation function (Matrix) $C_{ij}(t) = \langle \Phi_i^\dagger(t)\Phi_j(0)\rangle$
- Diagonalize the matrix $C^{-1}(t=0)C(t=a)$
- Extract the eigenvectors
- Extract the correlator for each state $(\sim \mathrm{e}^{-E_n t})$
- By fitting the results, we extract the mass (energy) for each state

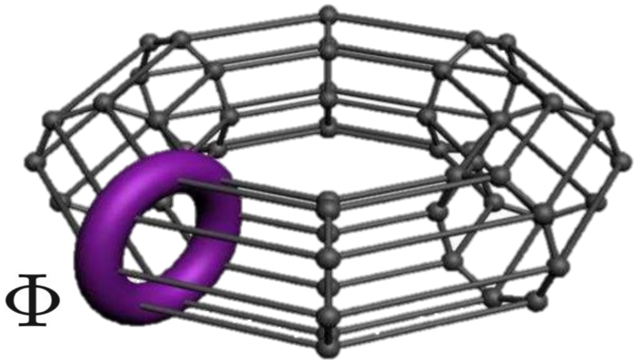


Operators Building

We build the path order product of links along the spatial direction → Polyakov Loop



Polyakov Loop



As N increases computational time increases as N^3

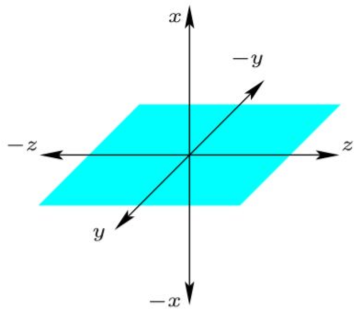
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \quad \text{For } N = 3, \textit{ we have } N^3 \textit{ multiplications}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}$$

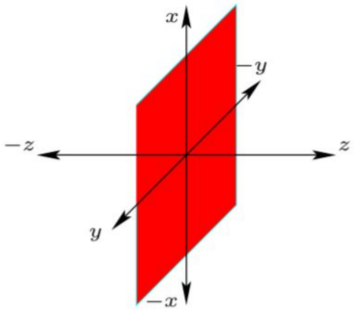
What
Quantum Numbers
Can we give to this
Object???

Parity

$\mathcal{P}$-Parity reflection plane



$\mathcal{R}$-Parity reflection plane



In D=2+1

In D=3+1

Translation to Strings (phonons) in D=2+1

$P_{\mathcal{P}}$ Parity:

- Under $P_{\mathcal{P}}$ parity $(x_{||}, x_{\perp}) \rightarrow (x_{||}, -x_{\perp})$ and, therefore,

$\alpha_{-k} \longleftrightarrow -\alpha_{-k}$ and $\bar{\alpha}_{-k} \longleftrightarrow -\bar{\alpha}_{-k}$.

- The parity of a state is given:

$$P_{\mathcal{P}} = (-1)^{number\ of\ phonons}$$

- For instance:
  - Even number of phonons, for example $\alpha_{-2}\bar{\alpha}_{-1}|0\rangle$, transforms as $P_{\mathcal{P}} = +$.
  - Odd number of phonons, for example $\alpha_{-1}|0\rangle$, transforms as $P_{\mathcal{P}} = -$.

$P_{\mathcal{R}}$ Parity:

- Under $P_{\mathcal{R}}$ Parity: $\alpha_{-k} \longleftrightarrow \bar{\alpha}_{-k}$
- Only useful in the $q = 0$ sector
- The only non-null pair of states with $P_{\mathcal{R}} = \pm$ is for $P_{\mathcal{P}} = -$:

$$\{\alpha_{-2}\bar{\alpha}_{-1}\bar{\alpha}_{-1} \pm \alpha_{-1}\alpha_{-1}\bar{\alpha}_{-2}\}|0\rangle$$

- This is quite heavy!
- In practice this Quantum Number is of minor utility.

Spin – Intrinsic Angular Momentum

On the Lattice things are a bit different….

→ Lattice Symmetry of Rotations about the string axis.

→ $C_{4\nu} \bigotimes Z(\mathcal{R})$ for zero longitudinal momentum.

- Rotations of $\pi/2 \rightarrow$ angular momentum J
- Reflections in orthogonal plane ($\mathcal{P}$-Parity)
- Reflections about the mid-point on the principal axis ($\mathcal{R}$-Parity)

→ 10 irreducible representations ≡ 10 correlation matrices

→ $C_{4\nu}$ for non-zero longitudinal momentum.

- Rotations of $\pi/2 \rightarrow$ angular momentum J
- Reflections in orthogonal plane ($\mathcal{P}$-Parity)

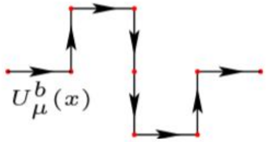
→ 5 irreducible representations ≡ 5 correlation matrices

→ $A_1 \equiv (J = 0, 4, \ldots, 4N,\ \ P_{\mathcal{P}} = +)$, $A_2 \equiv (J = 0, 4, \ldots, 4N,\ \ P_{\mathcal{P}} = -)$

→ $E \equiv (J = 1, 3, \ldots, 2N + 1)$

→ $B_1 \equiv (J = 2, 6, \ldots, 4N + 2,\ \ P_{\mathcal{P}} = +)$, $B_2 \equiv (J = 2, 6, \ldots, 4N + 2,\ \ P_{\mathcal{P}} = -)$

Imposing Spin in an operator i.e



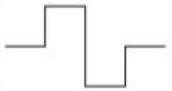
→ We can then form an operator of spin J:

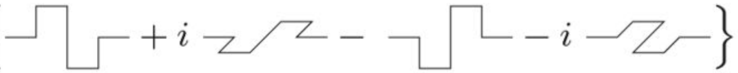
$$\text{Continuum}: \phi(J) = \int d\theta e^{iJ\theta} \phi_\theta$$

$$\text{Lattice}: \phi_L(J) = \sum_n e^{iJn\frac{\pi}{2}} \phi_{n\frac{\pi}{2}}$$

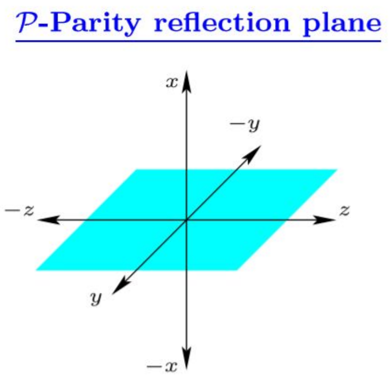
→ Example $J = 1$:

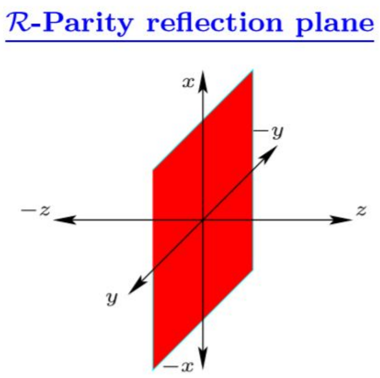
$$\phi_L(J = 1) = i\phi_{\frac{\pi}{2}} - \phi_\pi - i\phi_{\frac{3\pi}{2}} + \phi_{2\pi}$$

→ If $\phi_{\theta=0} \equiv \mathrm{Tr}\{$  $\}$

$\phi_L(J = 1) = \mathrm{Tr}\{$  $\}$

Parity & Spin





In D=2+1

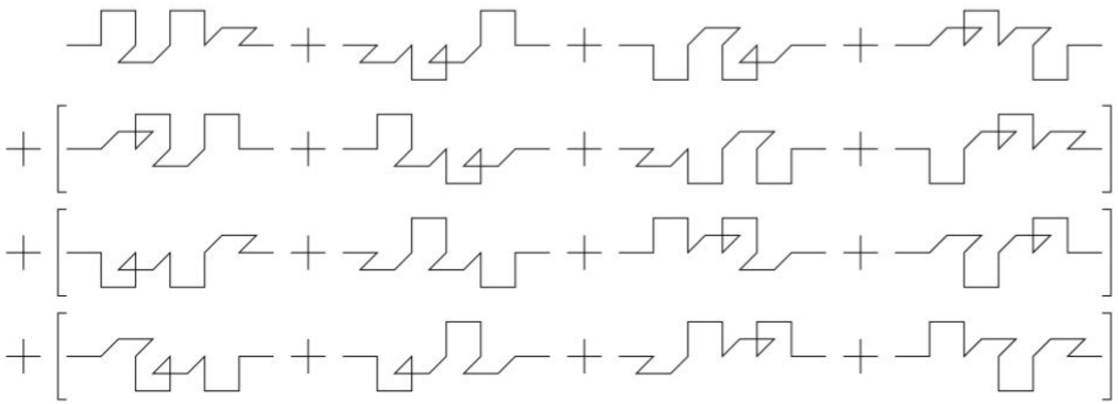
In D=3+1

Translation to Strings (phonons) in D=3+1
We have two transverse directions

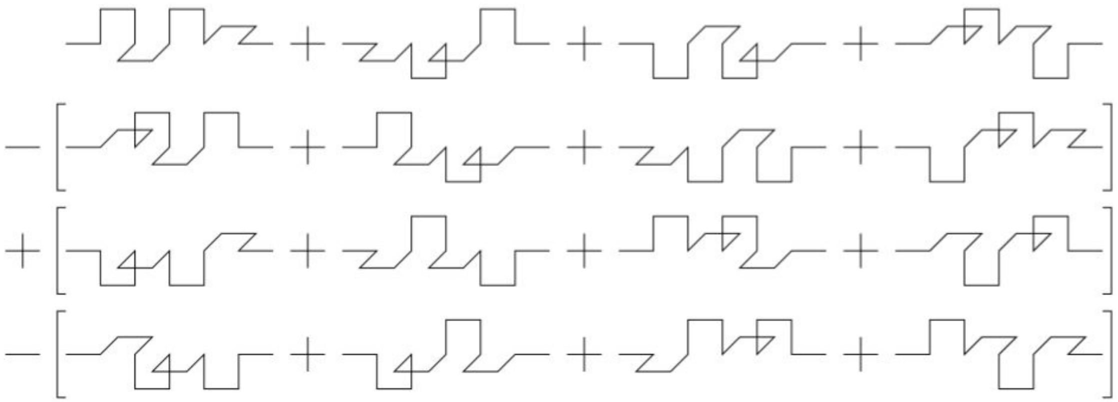
- $\rightarrow$ Define α^+_{-k} and α^-_{-k} as (x, y are the transverse directions):
  - $\alpha^+_{-k} = \alpha^x_{-k} + i\alpha^y_{-k}$
  - $\alpha^-_{-k} = \alpha^x_{-k} - i\alpha^y_{-k}$
- $\rightarrow$ Spin J.
  - $J = \mid \#(+) - \#(-) \mid$
- $\rightarrow$ $\mathcal{P}$-Parity
  - Under $\mathcal{P}$-Parity: $\alpha^+_{-k} \overset{P_{\mathcal{P}}}{\longleftrightarrow} \alpha^-_{-k}$ & $\bar{\alpha}^+_{-k} \overset{P_{\mathcal{P}}}{\longleftrightarrow} \bar{\alpha}^-_{-k}$
- $\rightarrow$ $\mathcal{R}$-Parity
  - Under $\mathcal{R}$-Parity: $\alpha^\pm_{-k} \overset{P_{\mathcal{R}}}{\longleftrightarrow} \bar{\alpha}^\pm_{-k}$
- Example: $(\alpha^+_{-1}\bar{\alpha}^+_{-1} \pm \alpha^-_{-1}\bar{\alpha}^-_{-1}) \mid 0\rangle$
  - $J = 2$
  - $P_{\mathcal{P}} = \pm$
  - $P_{\mathcal{R}} = +$

Parity & Spin Example of Operators

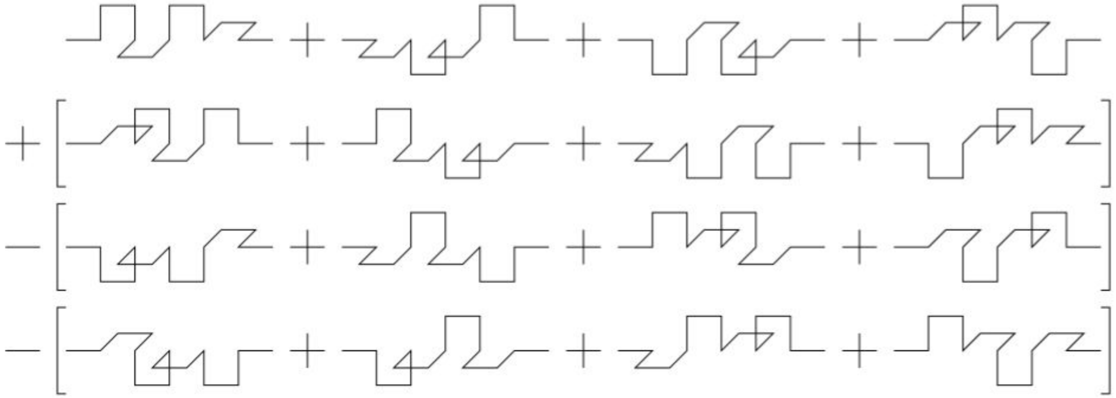
Example $J = 0,\ P_{\mathcal{P}} = +,\ P_{\mathcal{R}} = +$



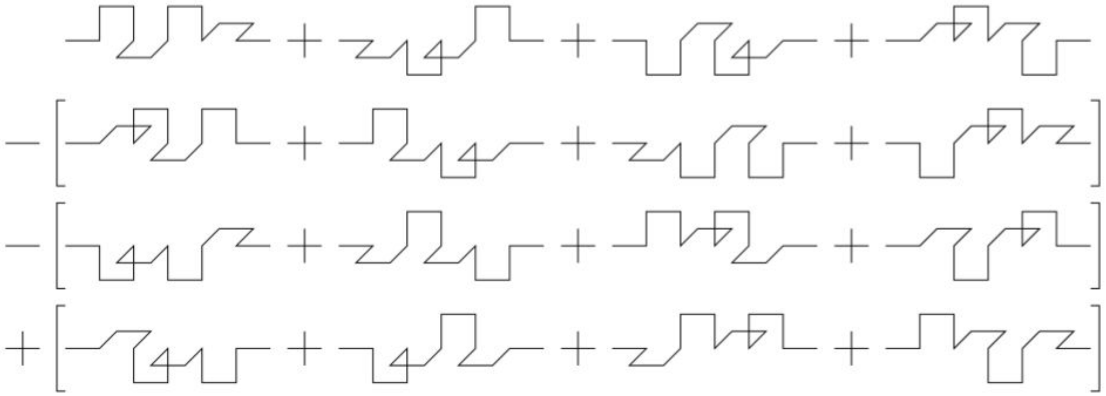
Example $J = 0,\ P_{\mathcal{P}} = +,\ P_{\mathcal{R}} = -$



Example $J = 0,\ P_{\mathcal{P}} = -,\ P_{\mathcal{R}} = +$



Example $J = 0,\ P_{\mathcal{P}} = -,\ P_{\mathcal{R}} = -$

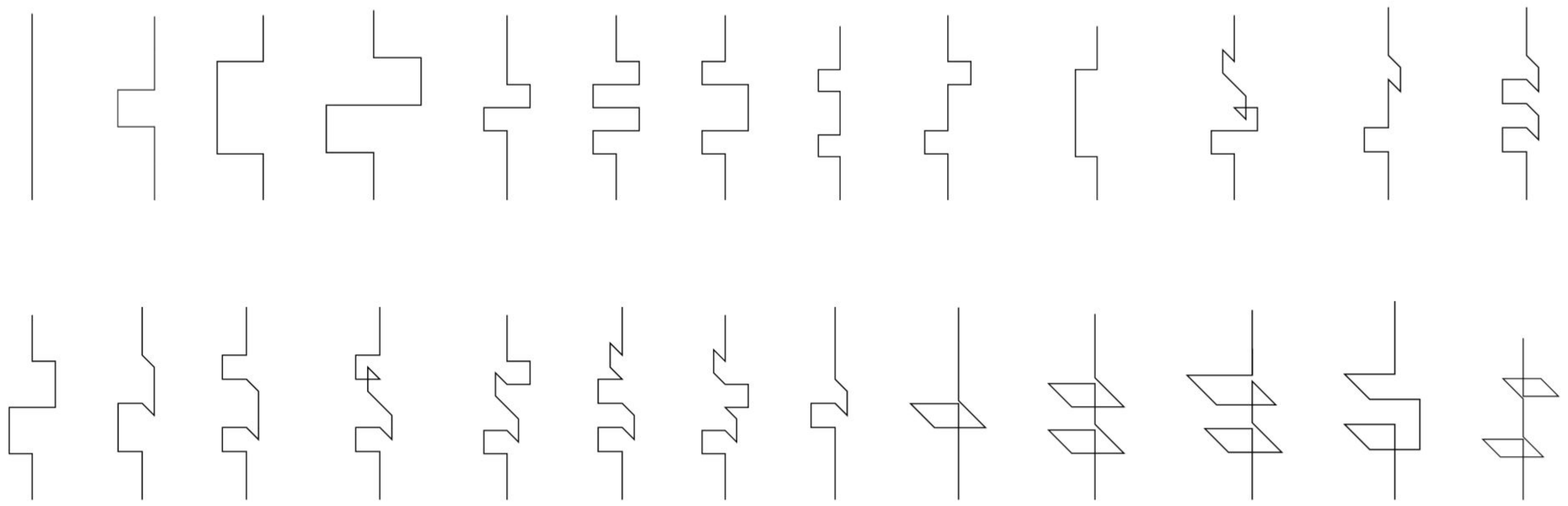


Parity & Spin Example of Operators

~ 1000 different operators with transverse deformations

$$C_{ij}(t) = \langle \Phi_i^\dagger(t)\Phi_j(0)\rangle = \begin{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} & \cdot & \cdot & \cdot \\ \cdot & \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} & \cdot & \cdot \\ \cdot & \cdot & \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} & \cdot \\ \cdot & \cdot & \cdot & \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \end{bmatrix}$$

Examples of transverse deformations in 2 & 3 spatial dimensions:



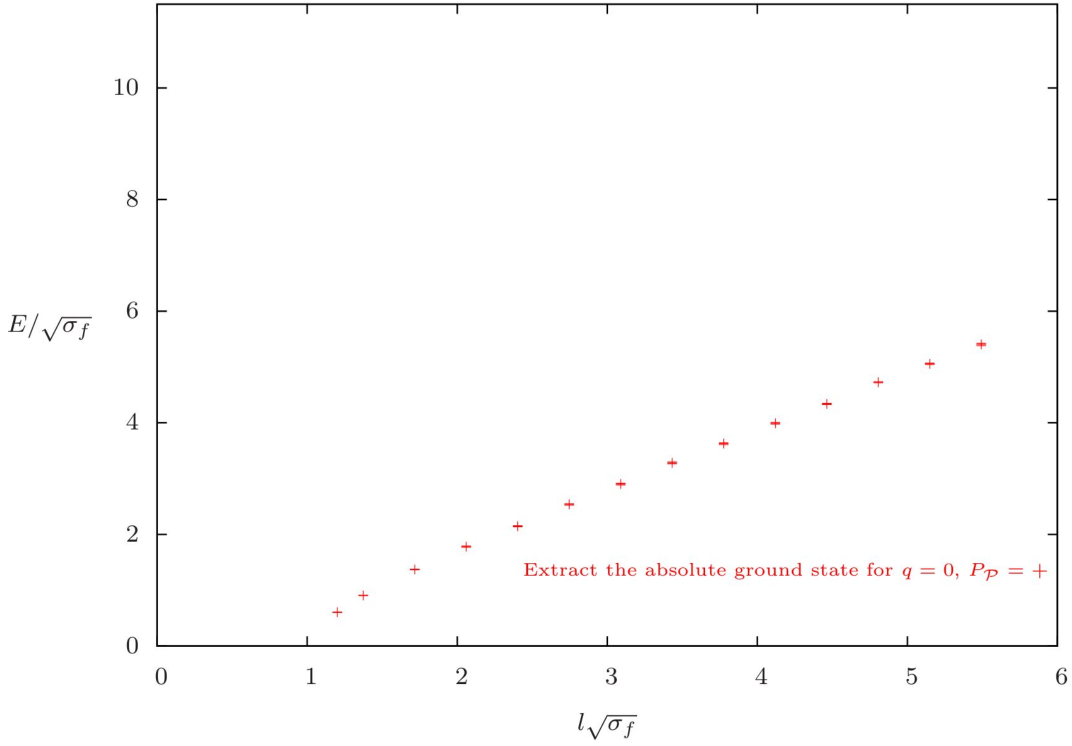
Results: the D=2+1 case

String States & Quantum Numbers

N_L, N_R	q	P	String State
$N_L = N_R = 0$	0	+	$\|0\rangle$
$N_L = 1, N_R = 0$	1	−	$a_1\|0\rangle$
$N_L = N_R = 1$	0	+	$a_1 a_{-1}\|0\rangle$
$N_L = 2, N_R = 0$	2	+ −	$a_1 a_1\|0\rangle$ $a_2\|0\rangle$
$N_L = 2, N_R = 1$	1	+ −	$a_2 a_{-1}\|0\rangle$ $a_1 a_1 a_{-1}\|0\rangle$
$N_L = 3, N_R = 0$	3	+ − −	$a_2 a_1\|0\rangle$ $a_3\|0\rangle$ $a_1 a_1 a_1\|0\rangle$
$N_L = N_R = 2$	0	+ + − −	$a_2 a_{-2}\|0\rangle$ $a_1 a_1 a_{-1} a_{-1}\|0\rangle$ $a_2 a_{-1} a_{-1}\|0\rangle$ $a_1 a_1 a_{-2}\|0\rangle$

Results for SU(6)

Positive Parity, q=0



Lattice Channels:

Quantum numbers	Ground state	1st excited state	2nd excited state	3rd excited state	4th excited state
	●				

Results: the D=2+1 case

String States & Quantum Numbers

N_L, N_R	q	P	String State
$N_L = N_R = 0$	0	$+$	$\vert 0\rangle$
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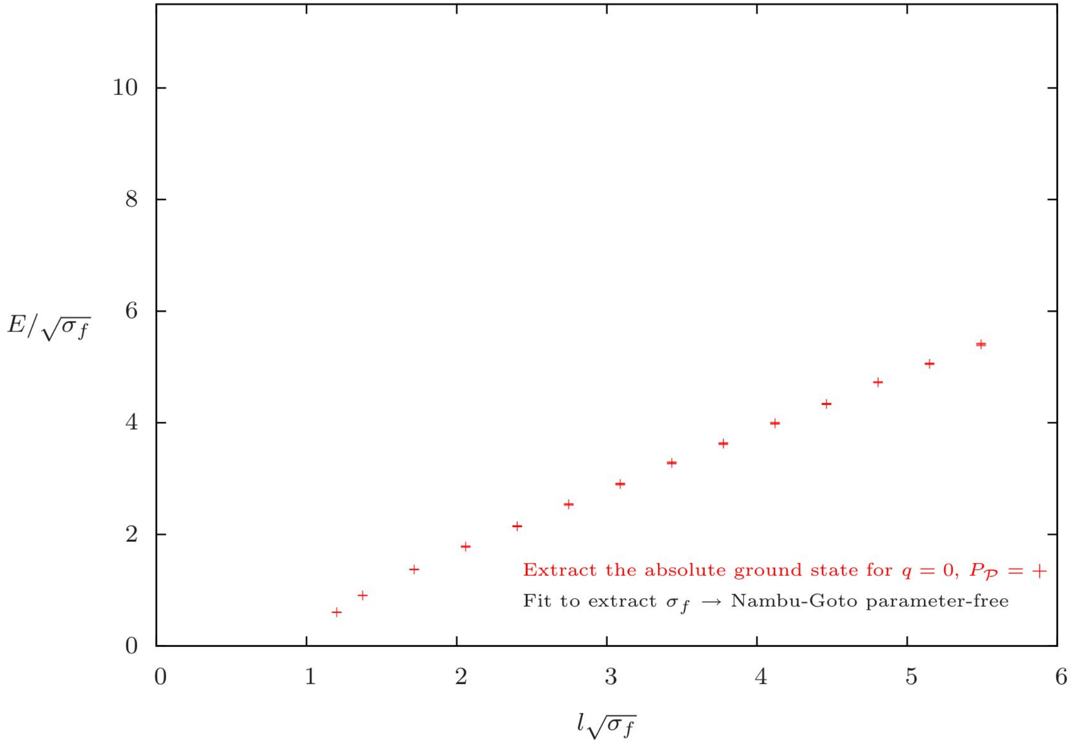
$N_L = N_R = 1$	0	$+$	$a_1 a_{-1} \vert 0\rangle$
$N_L = 2, N_R = 0$	2	$+$ $-$	$a_1 a_1 \vert 0\rangle$ $a_2 \vert 0\rangle$
$N_L = 2, N_R = 1$	1	$+$ $-$	$a_2 a_{-1} \vert 0\rangle$ $a_1 a_1 a_{-1} \vert 0\rangle$
$N_L = 3, N_R = 0$	3	$+$ $-$ $-$	$a_2 a_1 \vert 0\rangle$ $a_3 \vert 0\rangle$ $a_1 a_1 a_1 \vert 0\rangle$
$N_L = N_R = 2$	0	$+$ $+$ $-$ $-$	$a_2 a_{-2} \vert 0\rangle$ $a_1 a_1 a_{-1} a_{-1} \vert 0\rangle$ $a_2 a_{-1} a_{-1} \vert 0\rangle$ $a_1 a_1 a_{-2} \vert 0\rangle$

Results for SU(6)

Positive Parity, q=0



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Results: the D=2+1 case

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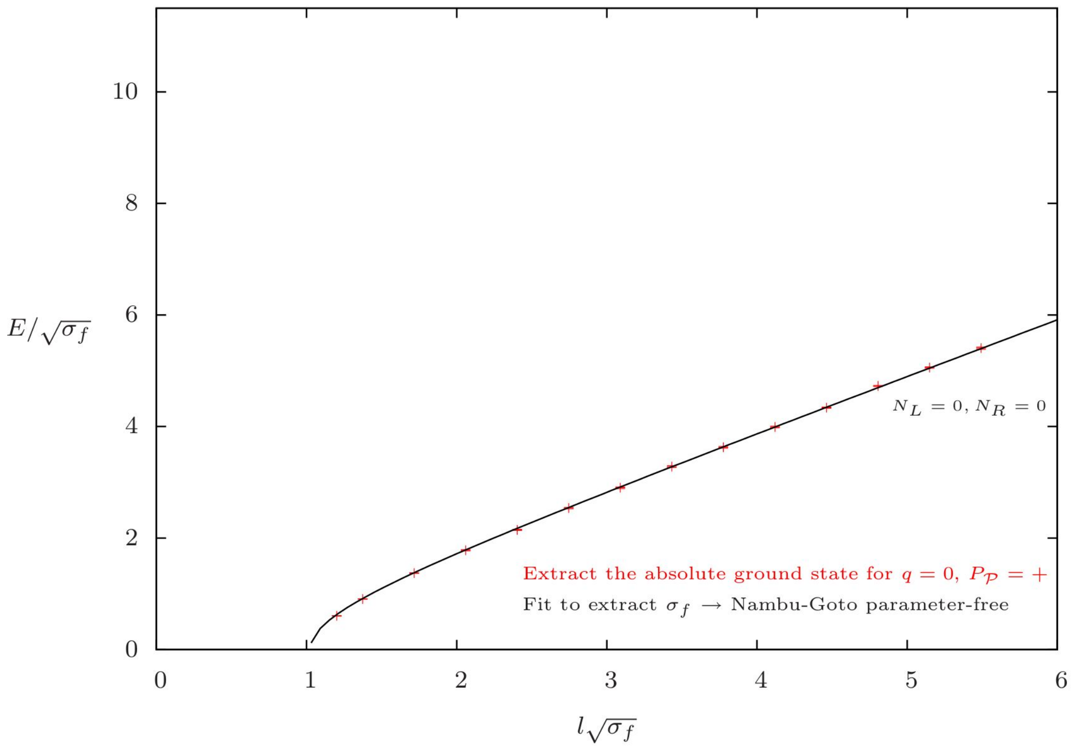
$N_L = 1, N_R = 0$	1	−	$a_1\|0\rangle$
$N_L = N_R = 1$	0	+	$a_1 a_{-1}\|0\rangle$
$N_L = 2, N_R = 0$	2	+ −	$a_1 a_1\|0\rangle$ $a_2\|0\rangle$
$N_L = 2, N_R = 1$	1	+ −	$a_2 a_{-1}\|0\rangle$ $a_1 a_1 a_{-1}\|0\rangle$
$N_L = 3, N_R = 0$	3	+ − −	$a_2 a_1\|0\rangle$ $a_3\|0\rangle$ $a_1 a_1 a_1\|0\rangle$
$N_L = N_R = 2$	0	+ + − −	$a_2 a_{-2}\|0\rangle$ $a_1 a_1 a_{-1} a_{-1}\|0\rangle$ $a_2 a_{-1} a_{-1}\|0\rangle$ $a_1 a_1 a_{-2}\|0\rangle$

Results for SU(6)

Positive Parity, q=0



Lattice Channels:

Quantum numbers	Ground state	1st excited state	2nd excited state	3rd excited state	4th excited state
	●				

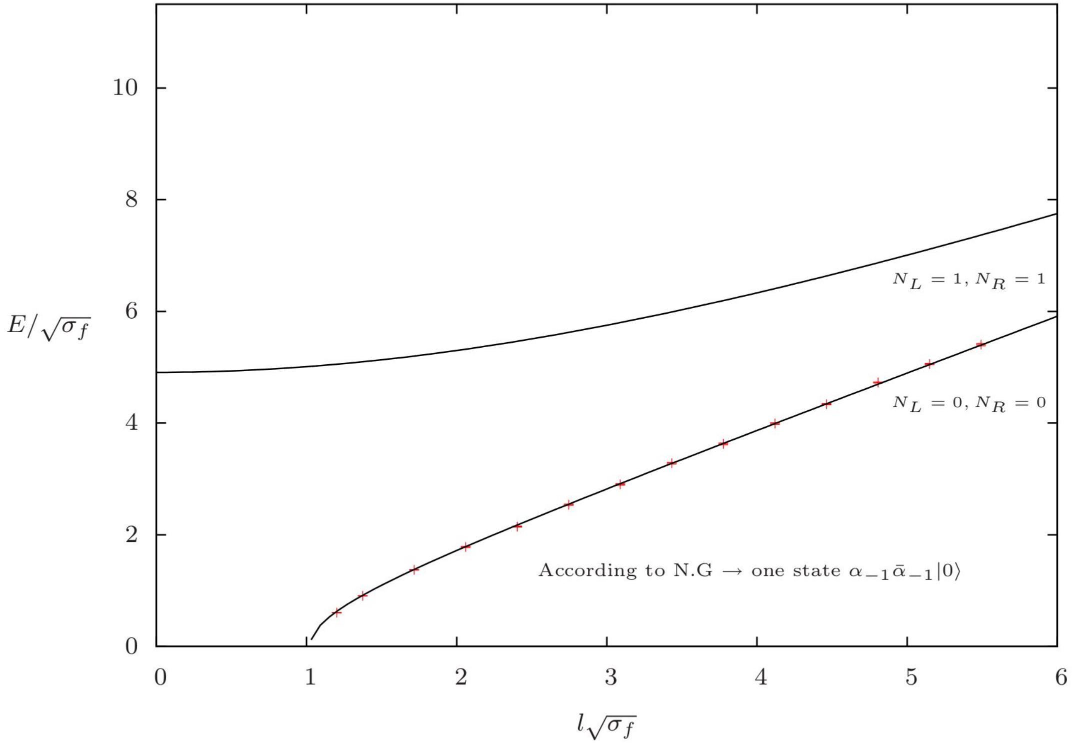
Results: the D=2+1 case

String States & Quantum Numbers

N_L, N_R	q	P	String State
$N_L = N_R = 0$	0	+	$\|0\rangle$
$N_L = 1, N_R = 0$	1	−	$a_1\|0\rangle$
$N_L = N_R = 1$	0	+	$a_1 a_{-1}\|0\rangle$
$N_L = 2, N_R = 0$	2	+ −	$a_1 a_1\|0\rangle$ $a_2\|0\rangle$
$N_L = 2, N_R = 1$	1	+ −	$a_2 a_{-1}\|0\rangle$ $a_1 a_1 a_{-1}\|0\rangle$
$N_L = 3, N_R = 0$	3	+ − −	$a_2 a_1\|0\rangle$ $a_3\|0\rangle$ $a_1 a_1 a_1\|0\rangle$
$N_L = N_R = 2$	0	+ + − −	$a_2 a_{-2}\|0\rangle$ $a_1 a_1 a_{-1} a_{-1}\|0\rangle$ $a_2 a_{-1} a_{-1}\|0\rangle$ $a_1 a_1 a_{-2}\|0\rangle$

Results for SU(6)

Positive Parity, q=0



Lattice Channels:

Quantum numbers	Ground state	1st excited state	2nd excited state	3rd excited state	4th excited state
	●				

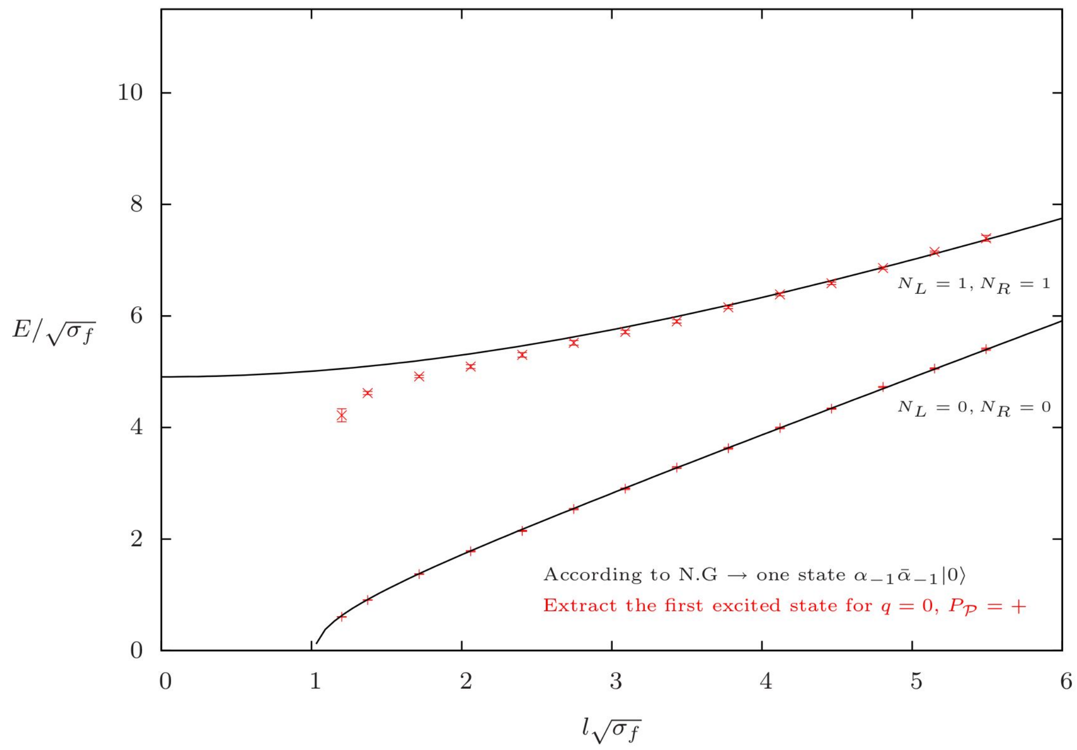
Results: the D=2+1 case

String States & Quantum Numbers

N_L, N_R	q	P	String State
$N_L = N_R = 0$	0	$+$	$\|0\rangle$
$N_L = 1, N_R = 0$	1	$-$	$a_1\|0\rangle$
$N_L = N_R = 1$	0	$+$	$a_1 a_{-1}\|0\rangle$
$N_L = 2, N_R = 0$	2	$+$ $-$	$a_1 a_1\|0\rangle$ $a_2\|0\rangle$
$N_L = 2, N_R = 1$	1	$+$ $-$	$a_2 a_{-1}\|0\rangle$ $a_1 a_1 a_{-1}\|0\rangle$
$N_L = 3, N_R = 0$	3	$+$ $-$ $-$	$a_2 a_1\|0\rangle$ $a_3\|0\rangle$ $a_1 a_1 a_1\|0\rangle$
$N_L = N_R = 2$	0	$+$ $+$ $-$ $-$	$a_2 a_{-2}\|0\rangle$ $a_1 a_1 a_{-1} a_{-1}\|0\rangle$ $a_2 a_{-1} a_{-1}\|0\rangle$ $a_1 a_1 a_{-2}\|0\rangle$

Results for SU(6)

Positive Parity, q=0



Lattice Channels:

Quantum numbers	Ground state	1st excited state	2nd excited state	3rd excited state	4th excited state
	●	●			

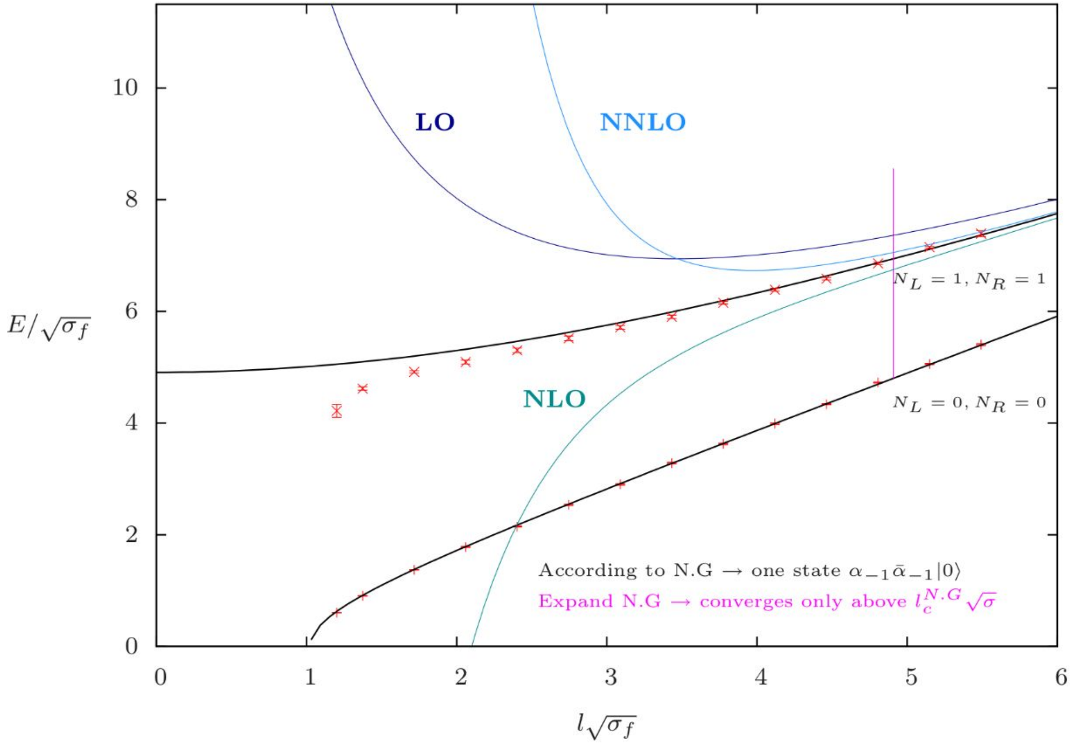
Results: the D=2+1 case

String States & Quantum Numbers

N_L, N_R	q	P	String State
$N_L = N_R = 0$	0	+	$\|0\rangle$
$N_L = 1, N_R = 0$	1	−	$a_1\|0\rangle$
$N_L = N_R = 1$	0	+	$a_1 a_{-1}\|0\rangle$
$N_L = 2, N_R = 0$	2	+ −	$a_1 a_1\|0\rangle$ $a_2\|0\rangle$
$N_L = 2, N_R = 1$	1	+ −	$a_2 a_{-1}\|0\rangle$ $a_1 a_1 a_{-1}\|0\rangle$
$N_L = 3, N_R = 0$	3	+ − −	$a_2 a_1\|0\rangle$ $a_3\|0\rangle$ $a_1 a_1 a_1\|0\rangle$
$N_L = N_R = 2$	0	+ + − −	$a_2 a_{-2}\|0\rangle$ $a_1 a_1 a_{-1} a_{-1}\|0\rangle$ $a_2 a_{-1} a_{-1}\|0\rangle$ $a_1 a_1 a_{-2}\|0\rangle$

Results for SU(6)

Positive Parity, *q*=0



Lattice Channels:

Quantum numbers	Ground state	1^{st} excited state	2^{nd} excited state	3^{rd} excited state	4^{th} excited state
	●	●			

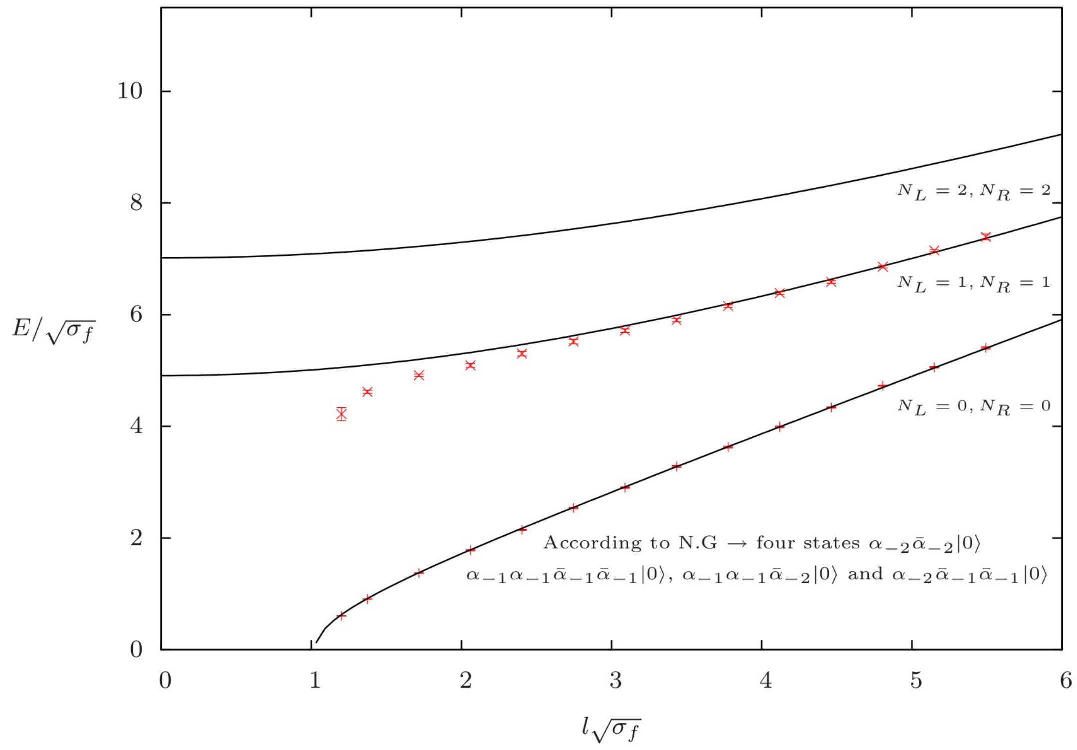
Results: the D=2+1 case

String States & Quantum Numbers

N_L, N_R	q	P	String State
$N_L = N_R = 0$	0	$+$	$\|0\rangle$
$N_L = 1, N_R = 0$	1	$-$	$a_1\|0\rangle$
$N_L = N_R = 1$	0	$+$	$a_1 a_{-1}\|0\rangle$
$N_L = 2, N_R = 0$	2	$+$ $-$	$a_1 a_1\|0\rangle$ $a_2\|0\rangle$
$N_L = 2, N_R = 1$	1	$+$ $-$	$a_2 a_{-1}\|0\rangle$ $a_1 a_1 a_{-1}\|0\rangle$
$N_L = 3, N_R = 0$	3	$+$ $-$ $-$	$a_2 a_1\|0\rangle$ $a_3\|0\rangle$ $a_1 a_1 a_1\|0\rangle$
$N_L = N_R = 2$	0	$+$ $+$ $-$ $-$	$a_2 a_{-2}\|0\rangle$ $a_1 a_1 a_{-1} a_{-1}\|0\rangle$ $a_2 a_{-1} a_{-1}\|0\rangle$ $a_1 a_1 a_{-2}\|0\rangle$

Results for SU(6)

Positive Parity, q=0



Lattice Channels:

Quantum numbers	Ground state	1st excited state	2nd excited state	3rd excited state	4th excited state
	●	●			

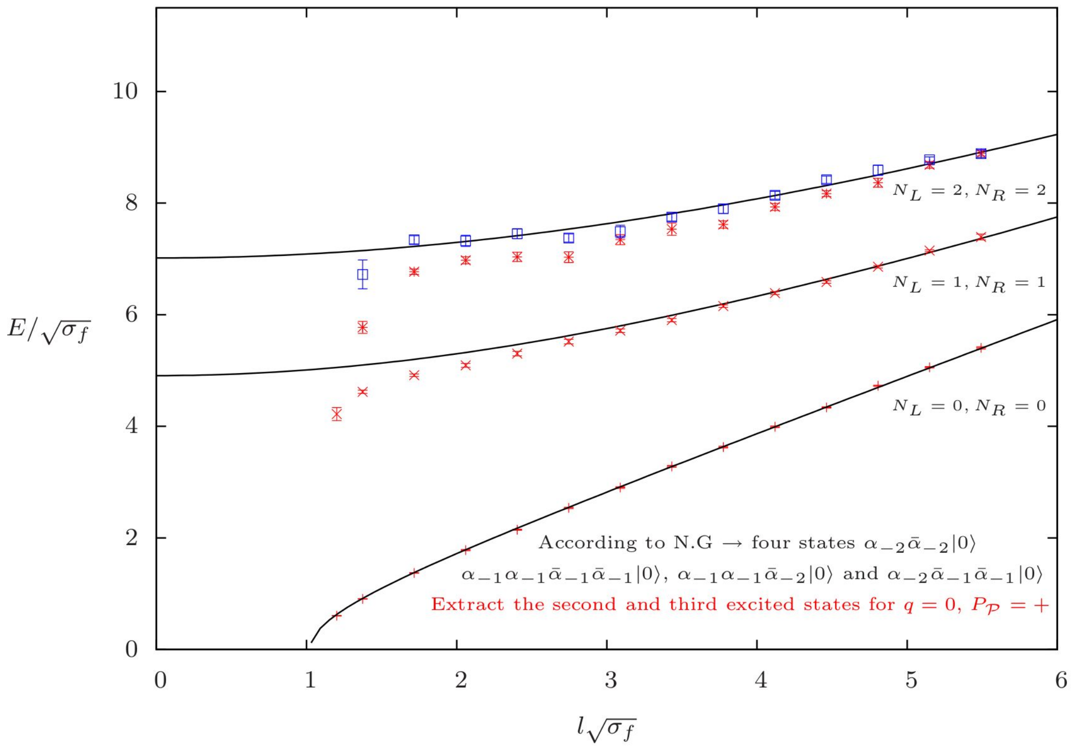
Results: the D=2+1 case

String States & Quantum Numbers

N_L, N_R	q	P	String State
$N_L = N_R = 0$	0	+	$\|0\rangle$
$N_L = 1, N_R = 0$	1	−	$a_1\|0\rangle$
$N_L = N_R = 1$	0	+	$a_1 a_{-1}\|0\rangle$
$N_L = 2, N_R = 0$	2	+ −	$a_1 a_1\|0\rangle$ $a_2\|0\rangle$
$N_L = 2, N_R = 1$	1	+ −	$a_2 a_{-1}\|0\rangle$ $a_1 a_1 a_{-1}\|0\rangle$
$N_L = 3, N_R = 0$	3	+ − −	$a_2 a_1\|0\rangle$ $a_3\|0\rangle$ $a_1 a_1 a_1\|0\rangle$
$N_L = N_R = 2$	0	+ + − −	$a_2 a_{-2}\|0\rangle$ $a_1 a_1 a_{-1} a_{-1}\|0\rangle$ $a_2 a_{-1} a_{-1}\|0\rangle$ $a_1 a_1 a_{-2}\|0\rangle$

Results for SU(6)

Positive Parity, *q*=0



Lattice Channels:

Quantum numbers	Ground state	1st excited state	2nd excited state	3rd excited state	4th excited state
	●	●	●	●	

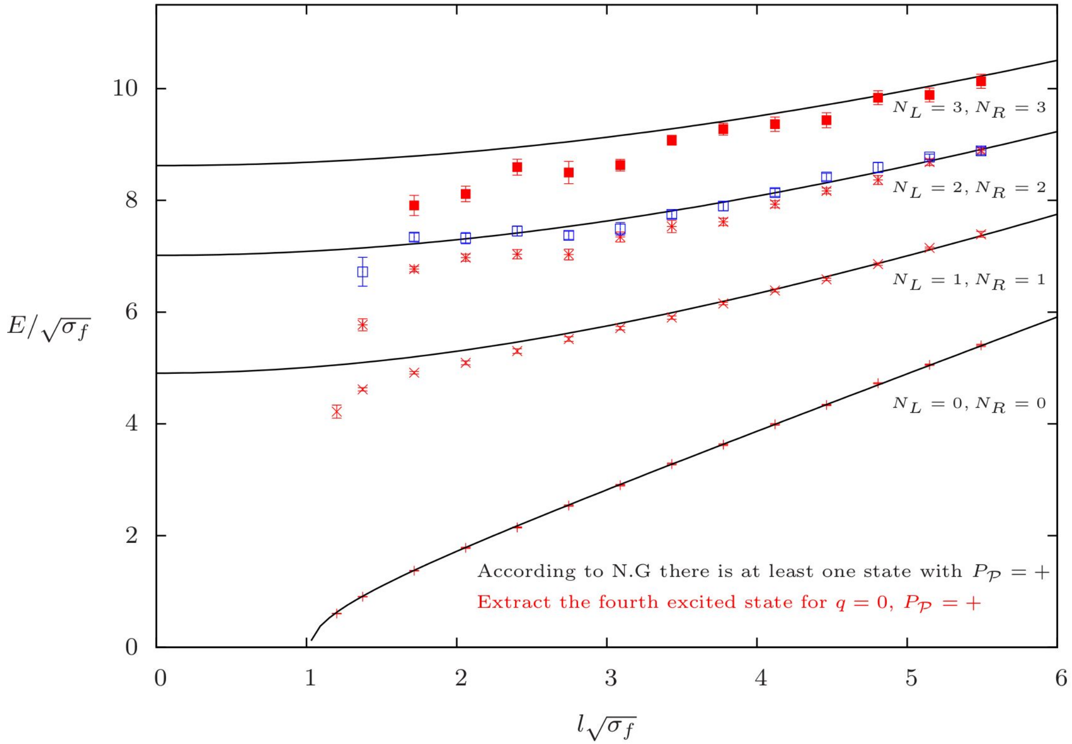
Results: the D=2+1 case

String States & Quantum Numbers

N_L, N_R	q	P	String State
$N_L = N_R = 0$	0	+	$\|0\rangle$
$N_L = 1, N_R = 0$	1	−	$a_1\|0\rangle$
$N_L = N_R = 1$	0	+	$a_1 a_{-1}\|0\rangle$
$N_L = 2, N_R = 0$	2	+ −	$a_1 a_1\|0\rangle$ $a_2\|0\rangle$
$N_L = 2, N_R = 1$	1	+ −	$a_2 a_{-1}\|0\rangle$ $a_1 a_1 a_{-1}\|0\rangle$
$N_L = 3, N_R = 0$	3	+ − −	$a_2 a_1\|0\rangle$ $a_3\|0\rangle$ $a_1 a_1 a_1\|0\rangle$
$N_L = N_R = 2$	0	+ + − −	$a_2 a_{-2}\|0\rangle$ $a_1 a_1 a_{-1} a_{-1}\|0\rangle$ $a_2 a_{-1} a_{-1}\|0\rangle$ $a_1 a_1 a_{-2}\|0\rangle$

Results for SU(6)

Positive Parity, *q*=0



Lattice Channels:

Quantum numbers	Ground state	1st excited state	2nd excited state	3rd excited state	4th excited state
	●	●	●	●	●

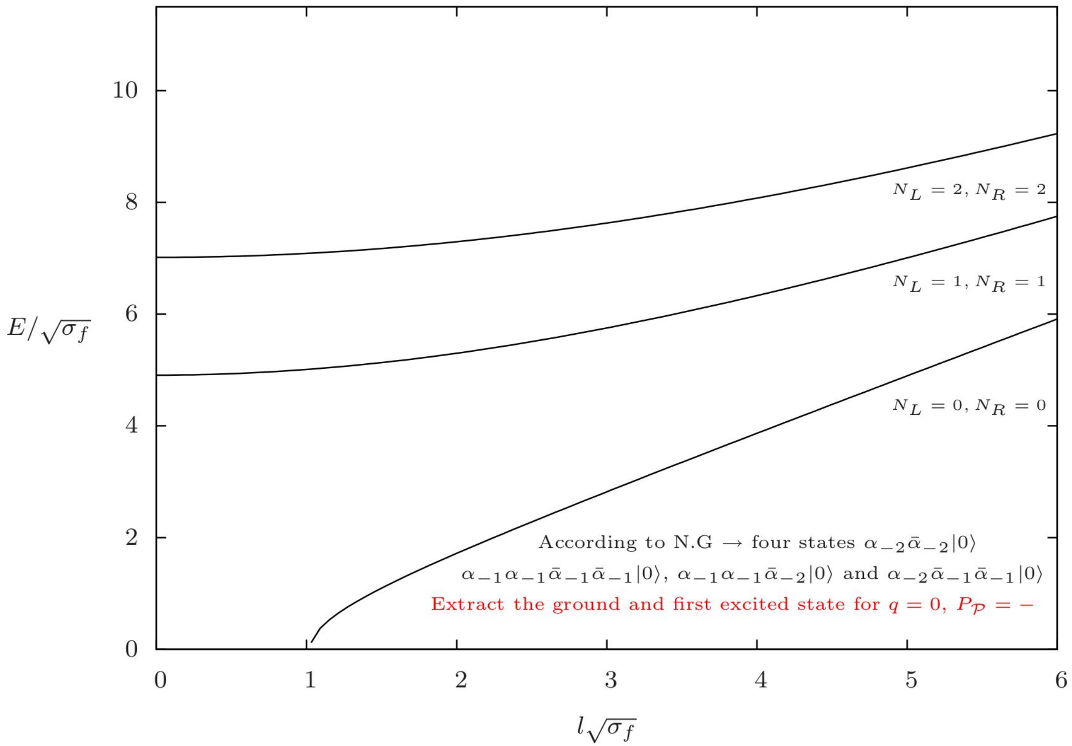
Results: the D=2+1 case

String States & Quantum Numbers

N_L, N_R	q	P	String State
$N_L = N_R = 0$	0	+	$\|0\rangle$
$N_L = 1, N_R = 0$	1	−	$a_1\|0\rangle$
$N_L = N_R = 1$	0	+	$a_1 a_{-1}\|0\rangle$
$N_L = 2, N_R = 0$	2	+ −	$a_1 a_1\|0\rangle$ $a_2\|0\rangle$
$N_L = 2, N_R = 1$	1	+ −	$a_2 a_{-1}\|0\rangle$ $a_1 a_1 a_{-1}\|0\rangle$
$N_L = 3, N_R = 0$	3	+ − −	$a_2 a_1\|0\rangle$ $a_3\|0\rangle$ $a_1 a_1 a_1\|0\rangle$
$N_L = N_R = 2$	0	+ + − −	$a_2 a_{-2}\|0\rangle$ $a_1 a_1 a_{-1} a_{-1}\|0\rangle$ $a_2 a_{-1} a_{-1}\|0\rangle$ $a_1 a_1 a_{-2}\|0\rangle$

Results for SU(6)

Negative Parity, *q*=0



Lattice Channels:

Quantum numbers	Ground state	1st excited state	2nd excited state	3rd excited state	4th excited state
	●	●	●	●	●

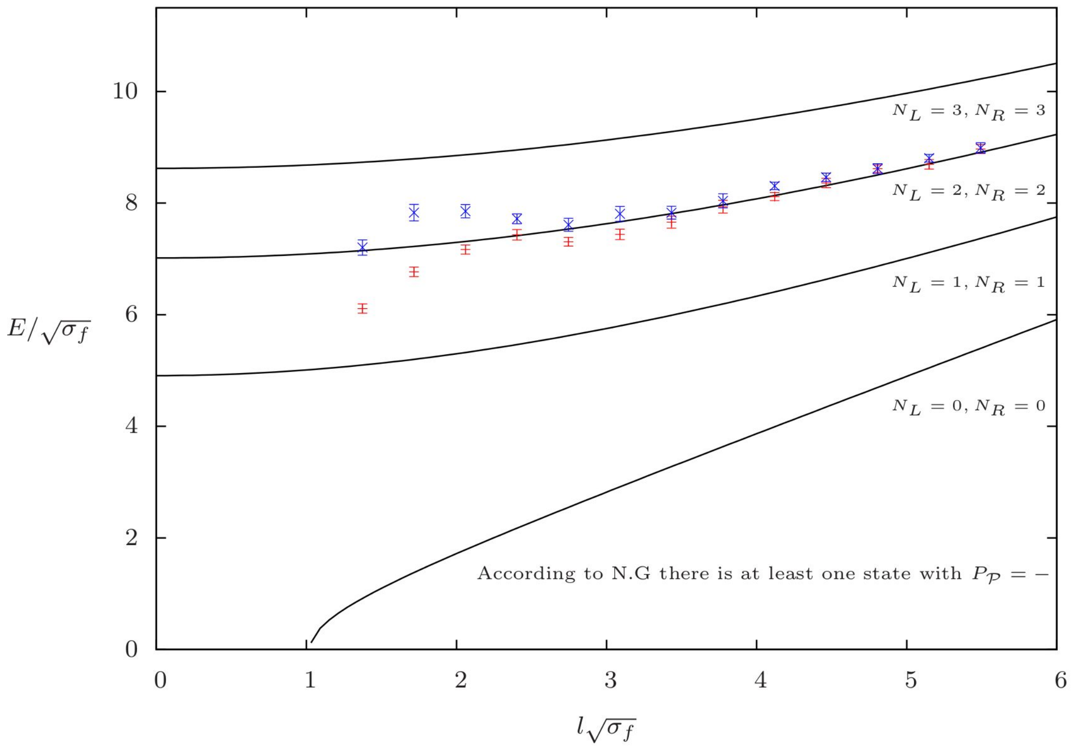
Results: the D=2+1 case

String States & Quantum Numbers

N_L, N_R	q	P	String State
$N_L = N_R = 0$	0	+	$\|0\rangle$
$N_L = 1, N_R = 0$	1	−	$a_1\|0\rangle$
$N_L = N_R = 1$	0	+	$a_1 a_{-1}\|0\rangle$
$N_L = 2, N_R = 0$	2	+ −	$a_1 a_1\|0\rangle$ $a_2\|0\rangle$
$N_L = 2, N_R = 1$	1	+ −	$a_2 a_{-1}\|0\rangle$ $a_1 a_1 a_{-1}\|0\rangle$
$N_L = 3, N_R = 0$	3	+ − −	$a_2 a_1\|0\rangle$ $a_3\|0\rangle$ $a_1 a_1 a_1\|0\rangle$
$N_L = N_R = 2$	0	+ + − −	$a_2 a_{-2}\|0\rangle$ $a_1 a_1 a_{-1} a_{-1}\|0\rangle$ $a_2 a_{-1} a_{-1}\|0\rangle$ $a_1 a_1 a_{-2}\|0\rangle$

Results for SU(6)

Negative Parity, *q*=0



Lattice Channels:

Quantum numbers	Ground state	1^{st} excited state	2^{nd} excited state	3^{rd} excited state	4^{th} excited state
	●	●	●	●	●
	●	●			

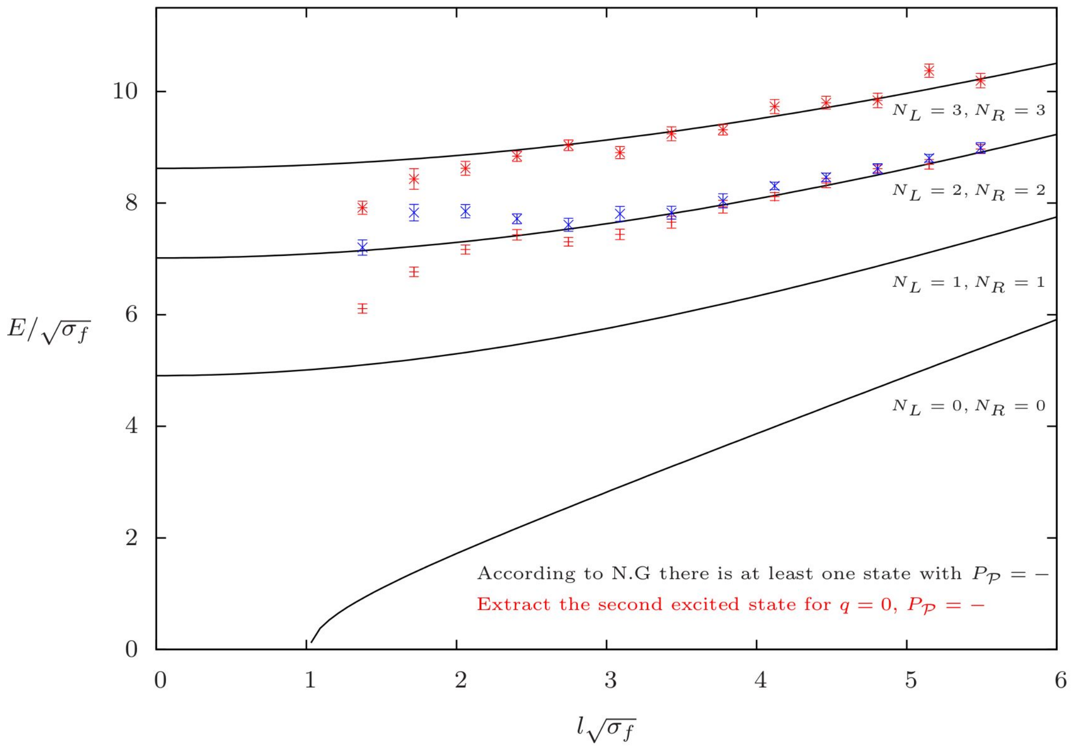
Results: the D=2+1 case

String States & Quantum Numbers

N_L, N_R	q	P	String State
$N_L = N_R = 0$	0	+	$\|0\rangle$
$N_L = 1, N_R = 0$	1	−	$a_1\|0\rangle$
$N_L = N_R = 1$	0	+	$a_1 a_{-1}\|0\rangle$
$N_L = 2, N_R = 0$	2	+ −	$a_1 a_1\|0\rangle$ $a_2\|0\rangle$
$N_L = 2, N_R = 1$	1	+ −	$a_2 a_{-1}\|0\rangle$ $a_1 a_1 a_{-1}\|0\rangle$
$N_L = 3, N_R = 0$	3	+ − −	$a_2 a_1\|0\rangle$ $a_3\|0\rangle$ $a_1 a_1 a_1\|0\rangle$
$N_L = N_R = 2$	0	+ + − −	$a_2 a_{-2}\|0\rangle$ $a_1 a_1 a_{-1} a_{-1}\|0\rangle$ $a_2 a_{-1} a_{-1}\|0\rangle$ $a_1 a_1 a_{-2}\|0\rangle$

Results for SU(6)

Negative Parity, *q*=0



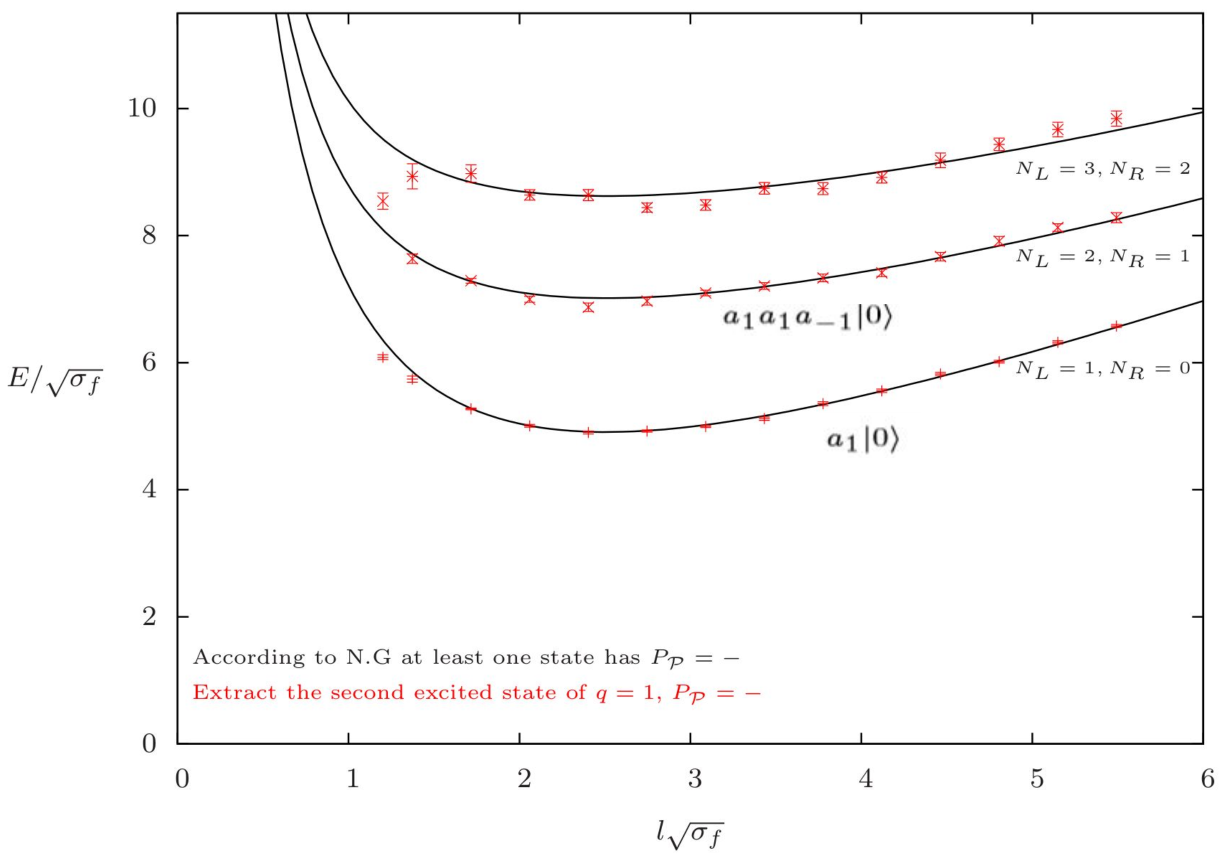
Lattice Channels:

Quantum numbers	Ground state	1st excited state	2nd excited state	3rd excited state	4th excited state
	● (red)	● (red)	● (red)	● (blue)	● (red)
	● (red)	● (blue)	● (red)		

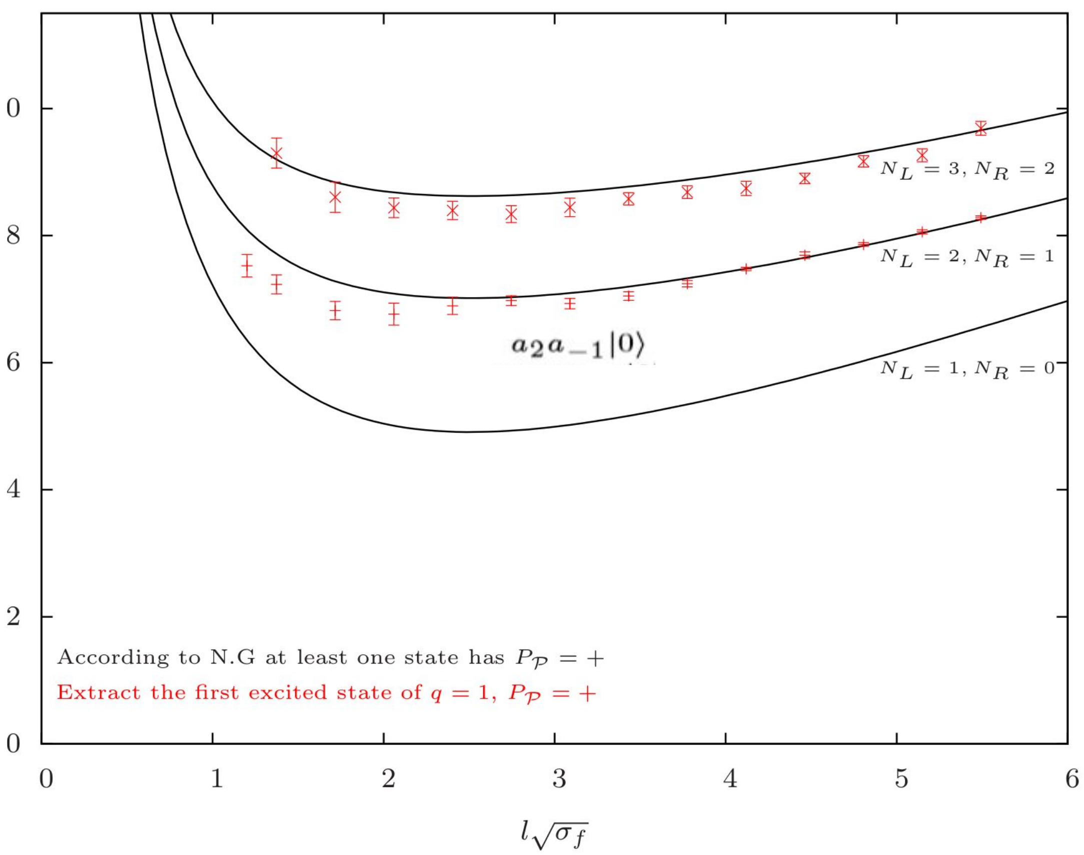
Results: the D=2+1 case

Results for SU(6)

Negative Parity, *q*=1



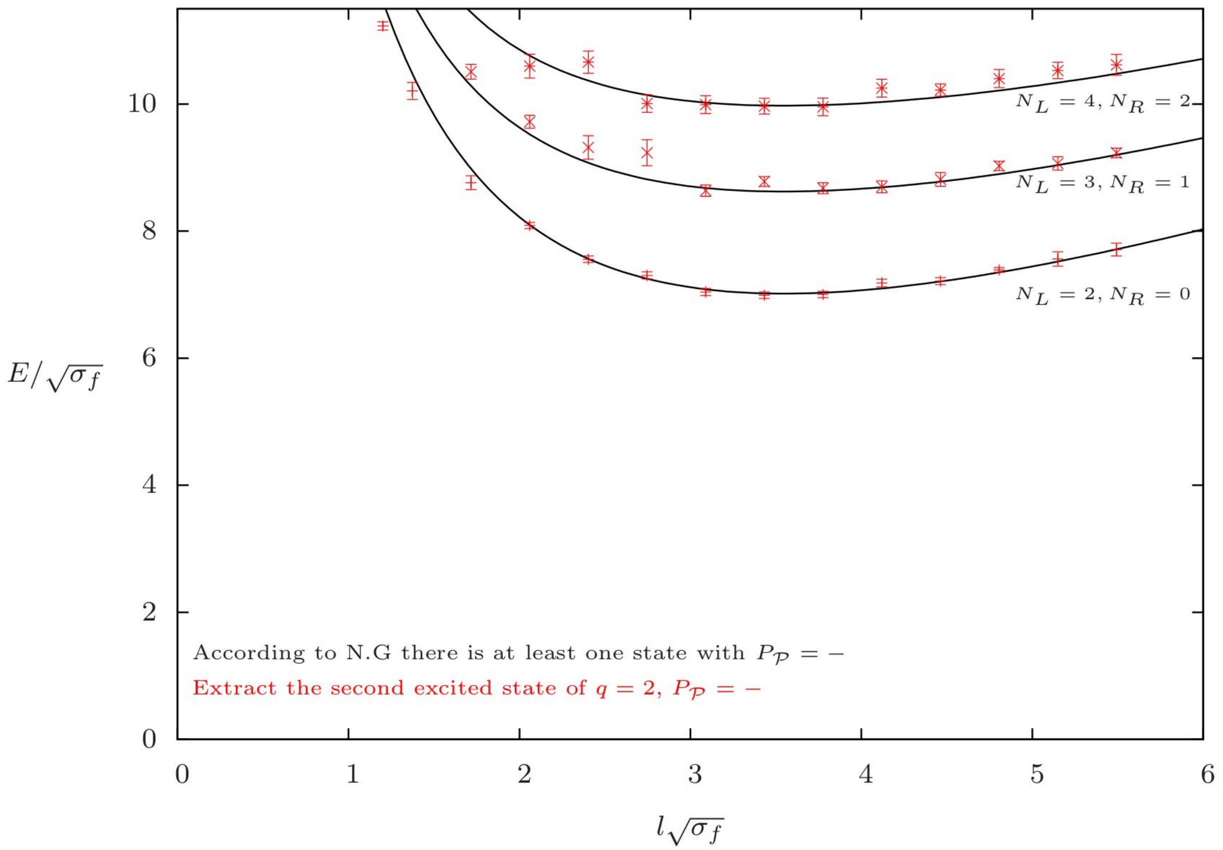
Positive Parity, *q*=1



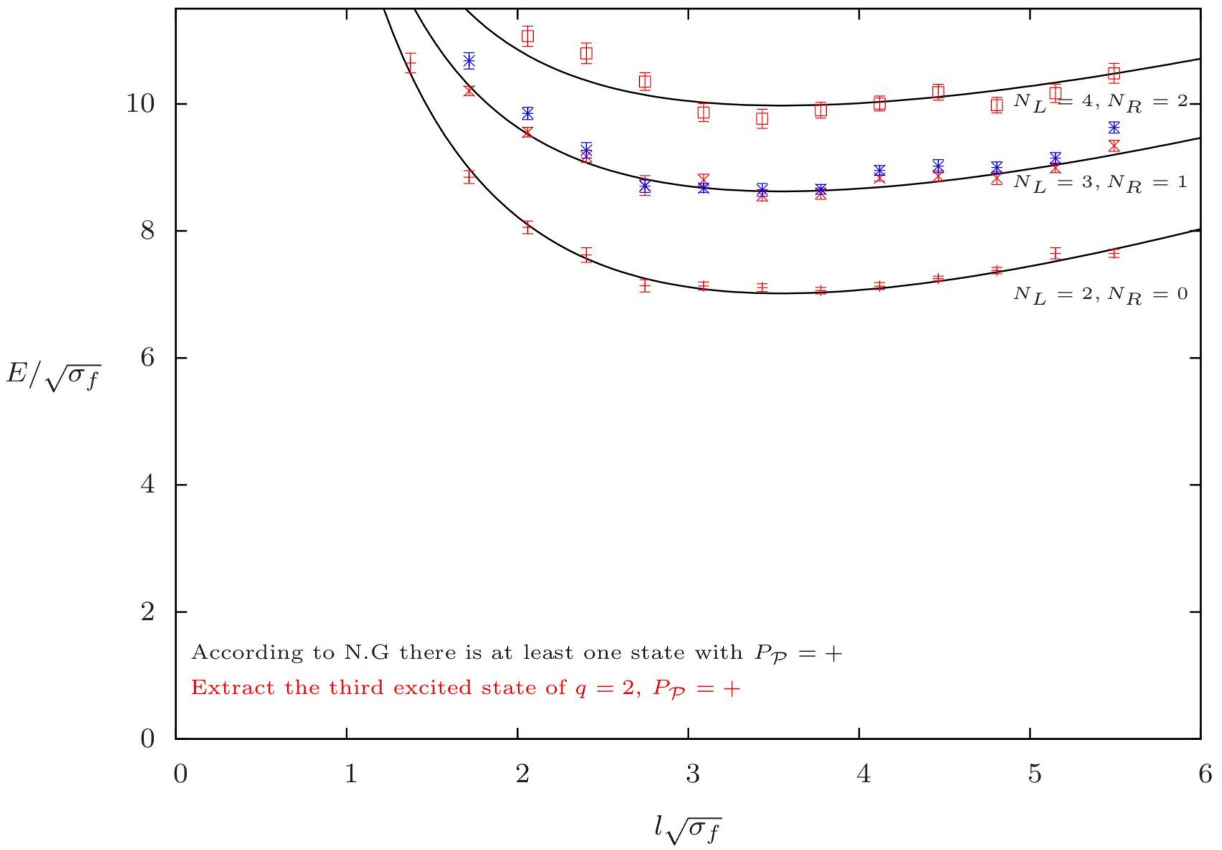
Results: the D=2+1 case

Results for SU(6)

Negative Parity, *q*=2



Positive Parity, *q*=2



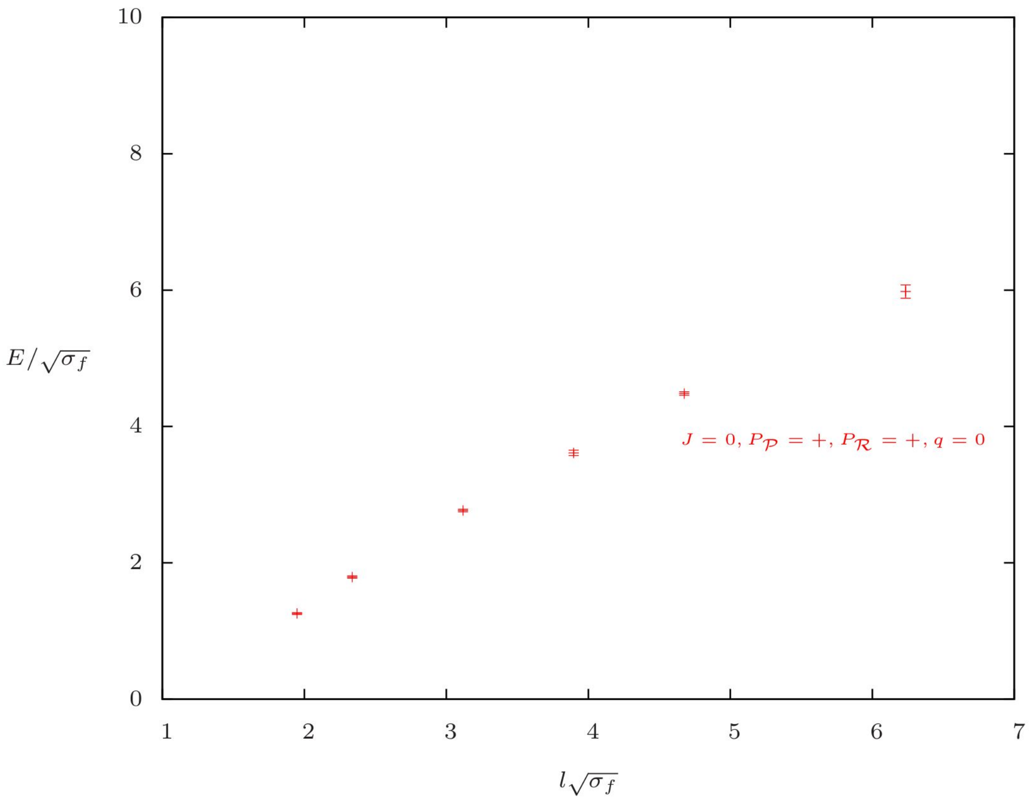
Results: the D=3+1 case

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	P_t	P_l	String State
$N_L = N_R = 0$	0	$+$	$+$	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$+$	$+$	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$+$	$+$	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$-$	$+$	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	0	$+$		$a_1^+ a_1^- \lvert 0\rangle$
$N_L = 2, N_R = 0$	1	$\pm$		$\left(a_2^+ \pm a_2^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$+$		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$-$		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$+$		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$-$		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$+$		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$-$		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	3	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

Results for SU(3) and ground states for several values of *q*

Absolute Ground State $J=0, P_{\mathcal{P}}=+, P_{\mathcal{R}}=+, q=0$.



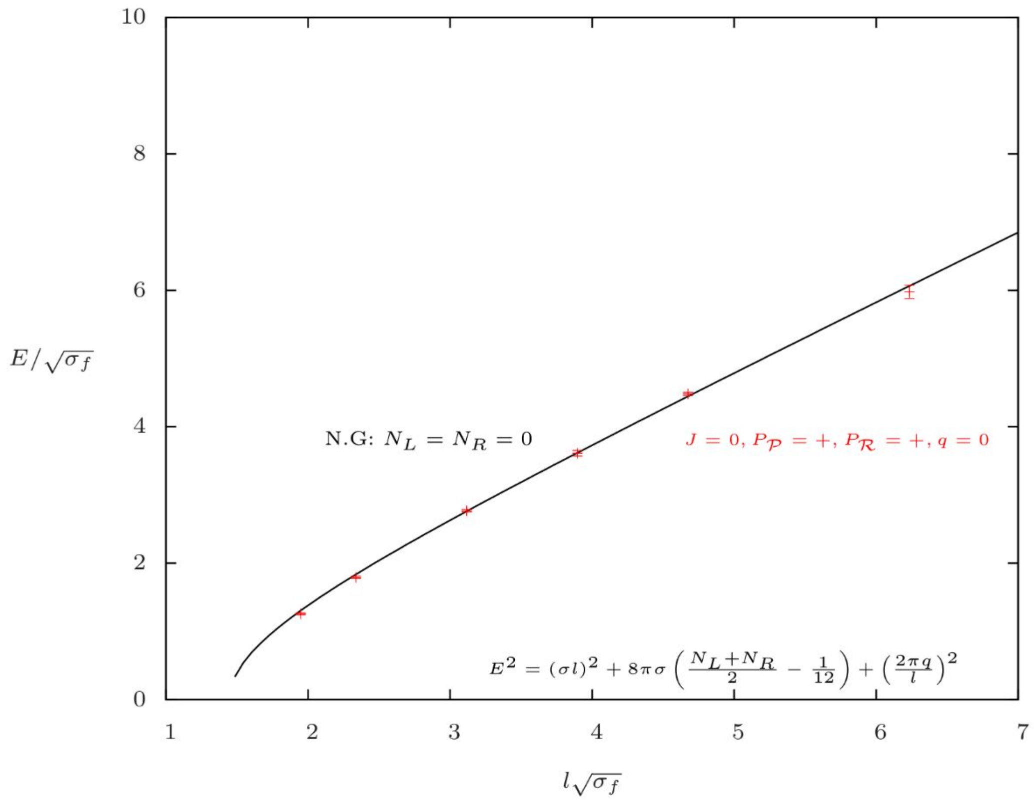
Results: the D=3+1 case

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	P_t	P_l	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	±		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	0	+		$a_1^+ a_1^- \lvert 0\rangle$
$N_L = 2, N_R = 0$	1	±		$\left(a_2^+ \pm a_2^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	+		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	−		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	+		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	−		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	±		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	±		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	+		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	−		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	3	±		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

Results for SU(3) and ground states for several values of *q*

Absolute Ground State $J = 0, P_{\mathcal{P}} = +, P_{\mathcal{R}} = +, q = 0.$



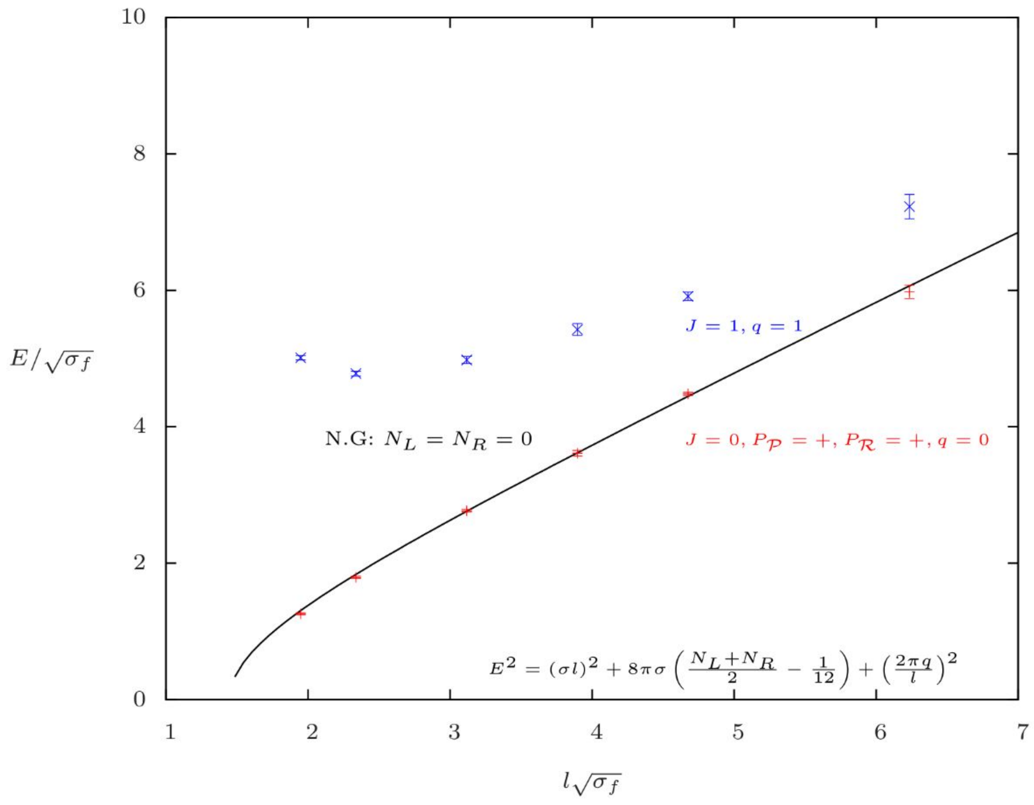
Results: the D=3+1 case

String States & Quantum Numbers

N_L, N_R	$\lvert J\rvert$	P_t	P_l	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$-$	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	0	+		$a_1^+ a_1^- \lvert 0\rangle$
$N_L = 2, N_R = 0$	1	$\pm$		$\left(a_2^+ \pm a_2^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	+		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$-$		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	+		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$-$		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	+		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$-$		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	3	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

Results for SU(3) and ground states for several values of q

Absolute Ground State $J = 1, q = 1$.



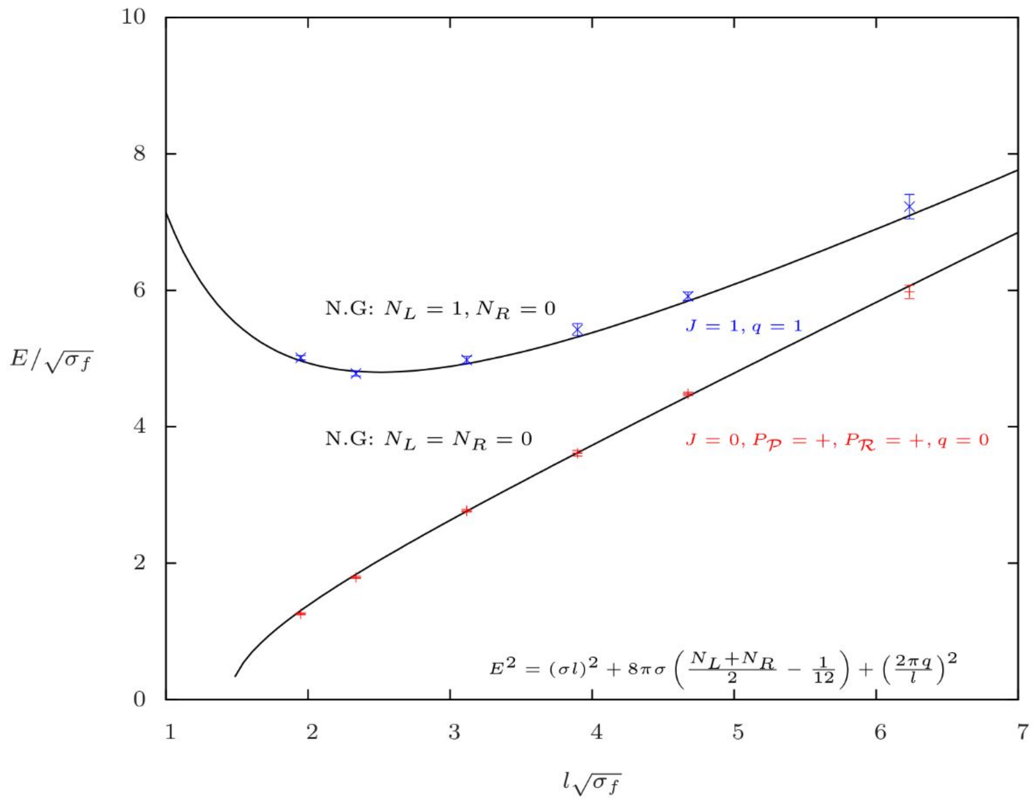
Results: the D=3+1 case

String States & Quantum Numbers

N_L, N_R	$\lvert J\rvert$	P_t	P_l	String State
$N_L = N_R = 0$	0	$+$	$+$	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$+$	$+$	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$+$	$+$	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$-$	$+$	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	0	$+$		$a_1^+ a_1^- \lvert 0\rangle$
$N_L = 2, N_R = 0$	1	$\pm$		$\left(a_2^+ \pm a_2^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$+$		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$-$		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$+$		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$-$		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$+$		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$-$		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	3	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

Results for SU(3) and ground states for several values of q

Absolute Ground State $J = 1, q = 1$.



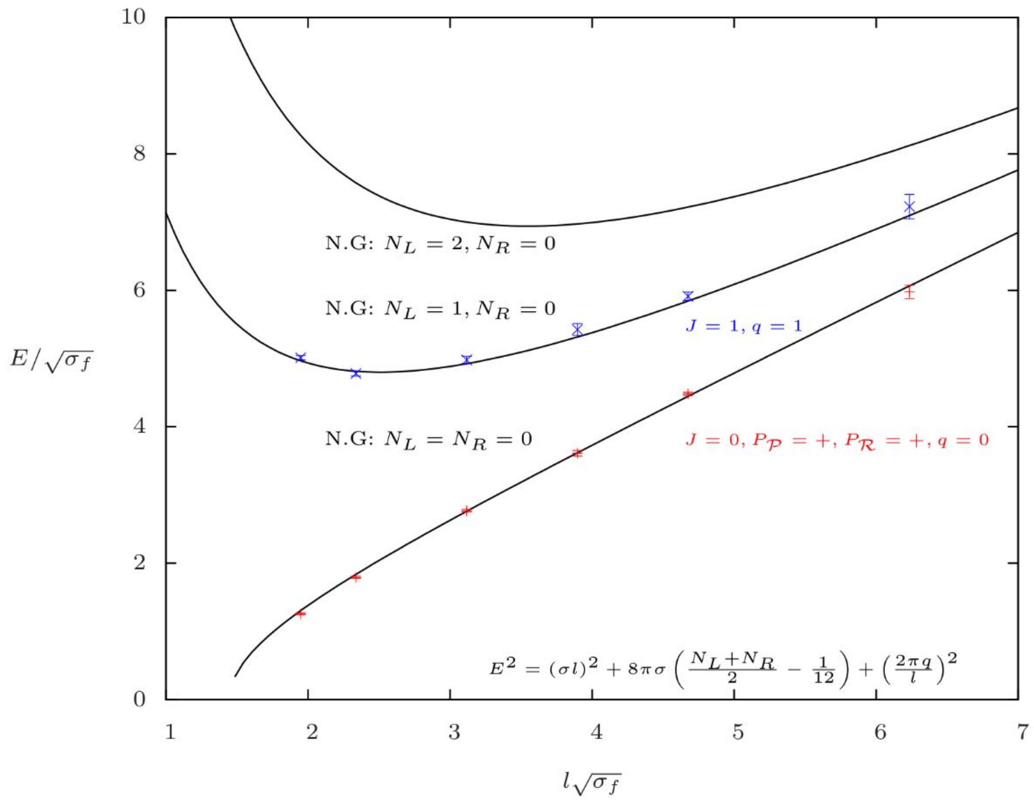
Results: the D=3+1 case

String States & Quantum Numbers

N_L, N_R	$\lvert J\rvert$	P_t	P_l	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	±		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	0	+		$a_1^+ a_1^- \lvert 0\rangle$
$N_L = 2, N_R = 0$	1	±		$\left(a_2^+ \pm a_2^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	+		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	−		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	+		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	−		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	±		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	±		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	+		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	−		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	3	±		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

Results for SU(3) and ground states for several values of q

Absolute Ground States for $N_L = 2, N_R = 0$.



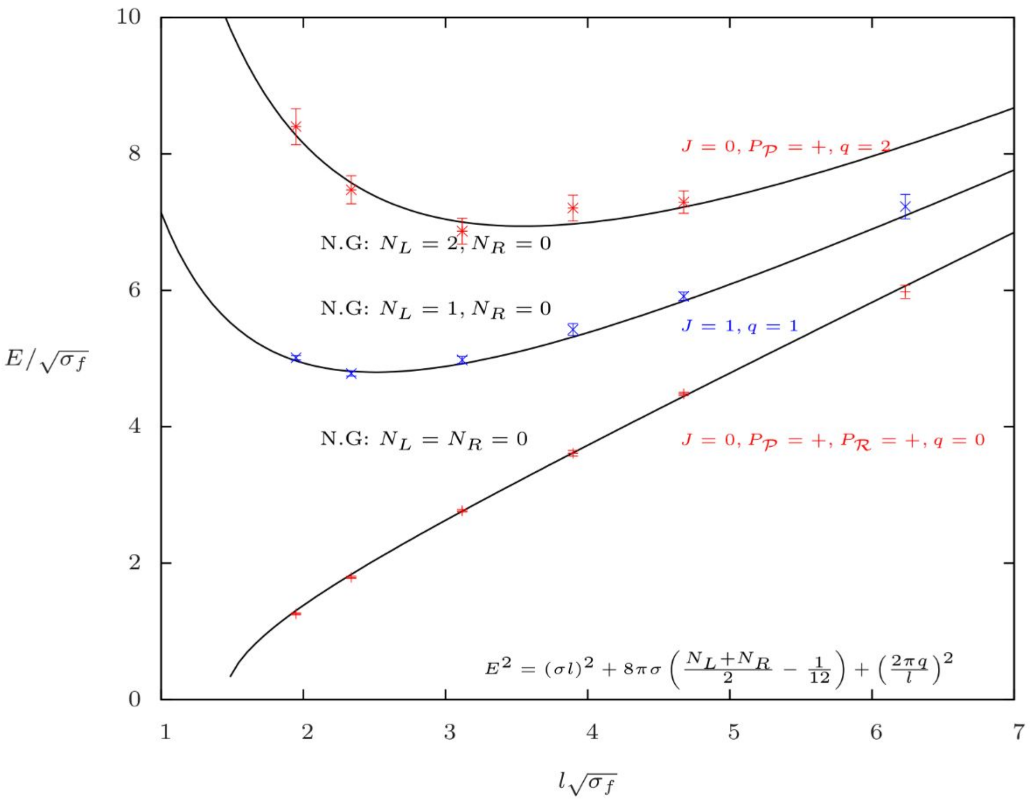
Results: the D=3+1 case

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	P_t	P_l	String State
$N_L = N_R = 0$	0	$+$	$+$	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$+$	$+$	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$+$	$+$	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$-$	$+$	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	0	$+$		$a_1^+ a_1^- \lvert 0\rangle$
$N_L = 2, N_R = 0$	1	$\pm$		$\left(a_2^+ \pm a_2^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$+$		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$-$		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$+$		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$-$		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$+$		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$-$		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	3	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

Results for SU(3) and ground states for several values of q

Absolute Ground States for $N_L = 2, N_R = 0$.



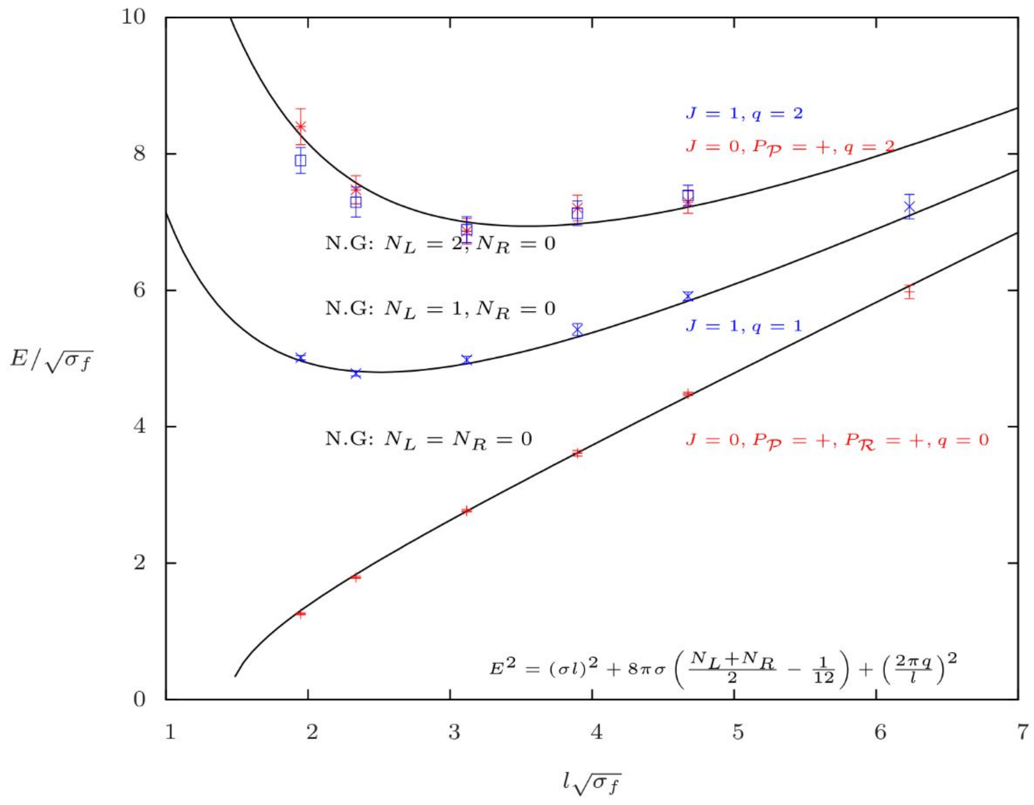
Results: the D=3+1 case

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	P_t	P_l	String State
$N_L = N_R = 0$	0	$+$	$+$	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$+$	$+$	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$+$	$+$	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$-$	$+$	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	0	$+$		$a_1^+ a_1^- \lvert 0\rangle$
$N_L = 2, N_R = 0$	1	$\pm$		$\left(a_2^+ \pm a_2^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$+$		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$-$		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$+$		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$-$		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$+$		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$-$		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	3	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

Results for SU(3) and ground states for several values of *q*

Absolute Ground States for $N_L = 2, N_R = 0$.



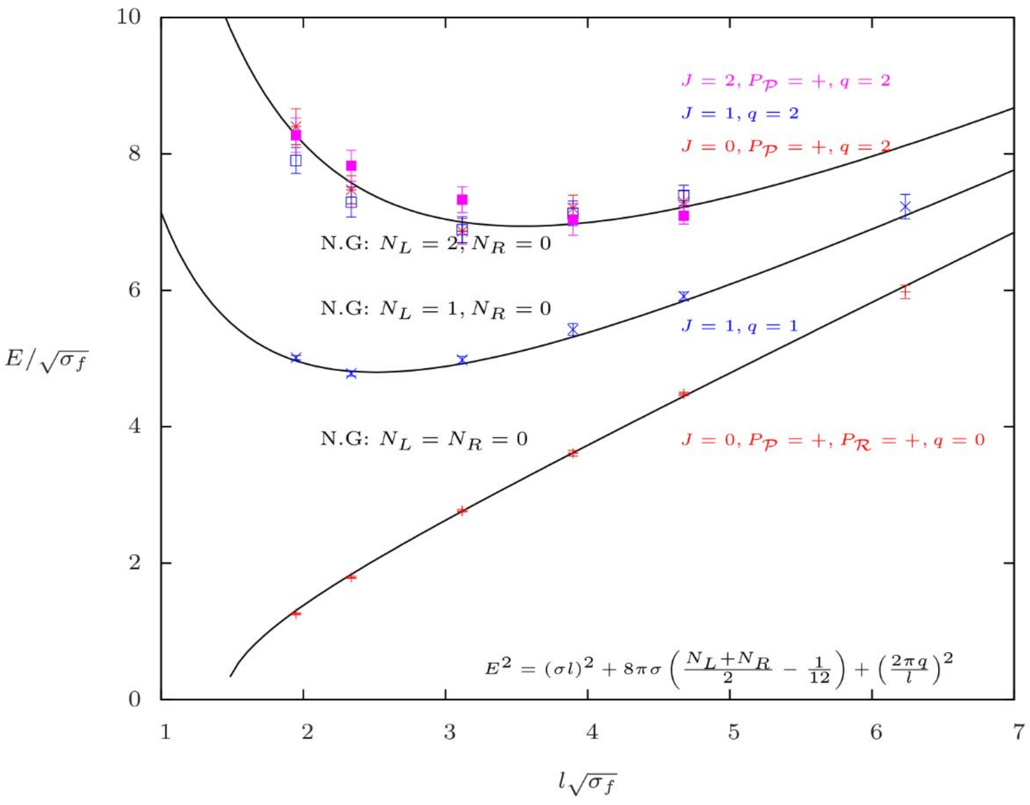
Results: the D=3+1 case

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	P_t	P_l	String State
$N_L = N_R = 0$	0	$+$	$+$	$\lvert 0 \rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right) \lvert 0 \rangle$
$N_L = N_R = 1$	0	$+$	$+$	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = N_R = 1$	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = N_R = 1$	2	$+$	$+$	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right) \lvert 0 \rangle$
$N_L = N_R = 1$	2	$-$	$+$	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 0$	0	$+$		$a_1^+ a_1^- \lvert 0 \rangle$
$N_L = 2, N_R = 0$	1	$\pm$		$\left(a_2^+ \pm a_2^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 0$	2	$+$		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 0$	2	$-$		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	0	$+$		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	0	$-$		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	2	$+$		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	2	$-$		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	3	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right) \lvert 0 \rangle$

Results for SU(3) and ground states for several values of q

Absolute Ground States for $N_L = 2, N_R = 0$.



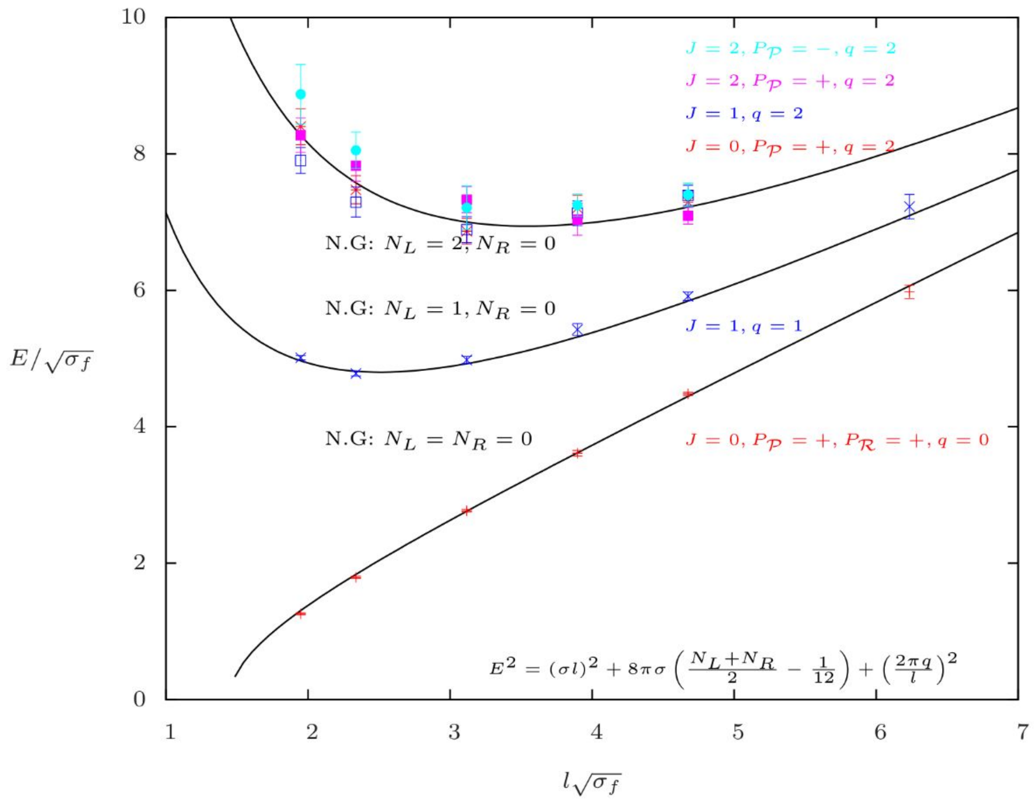
Results: the D=3+1 case

String States & Quantum Numbers

N_L, N_R	$\|J\|$	P_t	P_l	String State
$N_L = N_R = 0$	0	+	+	$\|0\rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right)\|0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\|0\rangle$
$N_L = N_R = 1$	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\|0\rangle$
$N_L = N_R = 1$	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\|0\rangle$
$N_L = N_R = 1$	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\|0\rangle$
$N_L = 2, N_R = 0$	0	+		$a_1^+ a_1^- \|0\rangle$
$N_L = 2, N_R = 0$	1	$\pm$		$\left(a_2^+ \pm a_2^-\right)\|0\rangle$
$N_L = 2, N_R = 0$	2	+		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\|0\rangle$
$N_L = 2, N_R = 0$	2	−		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\|0\rangle$
$N_L = 2, N_R = 1$	0	+		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\|0\rangle$
$N_L = 2, N_R = 1$	0	−		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\|0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\|0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\|0\rangle$
$N_L = 2, N_R = 1$	2	+		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\|0\rangle$
$N_L = 2, N_R = 1$	2	−		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\|0\rangle$
$N_L = 2, N_R = 1$	3	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\|0\rangle$

Results for SU(3) and ground states for several values of *q*

Absolute Ground States for $N_L = 2, N_R = 0$.



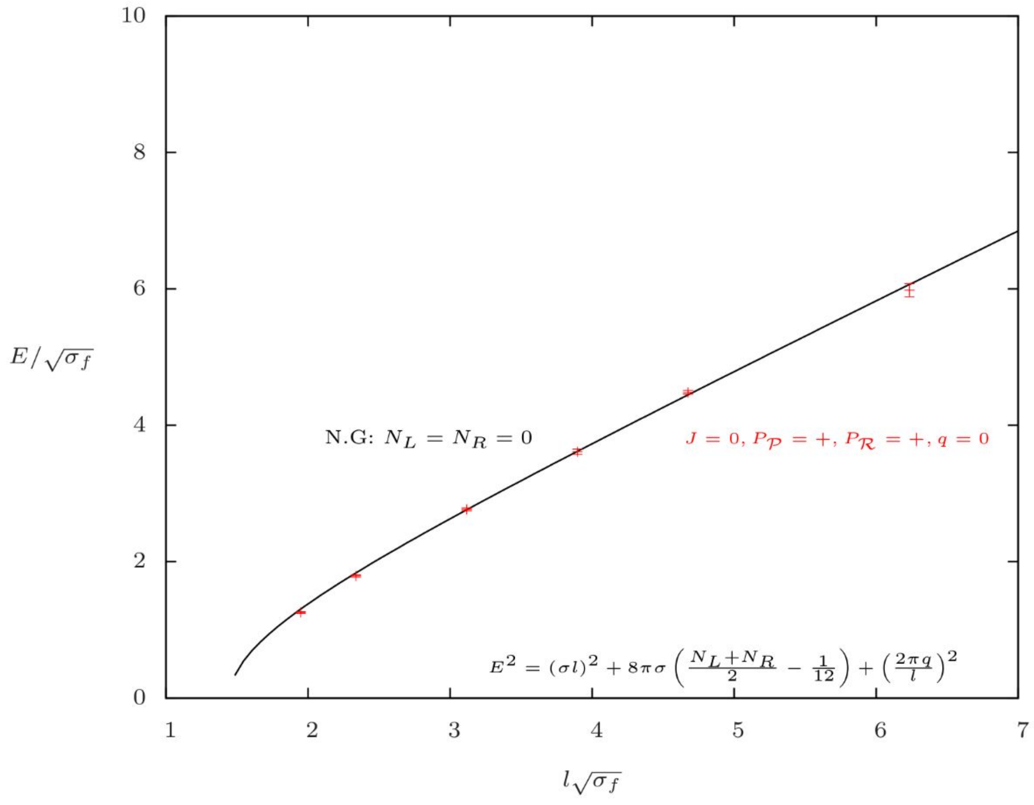
Results: the D=3+1 case

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	P_t	P_l	String State
$N_L = N_R = 0$	0	$+$	$+$	$\lvert 0 \rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right) \lvert 0 \rangle$
$N_L = N_R = 1$	0	$+$	$+$	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = N_R = 1$	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = N_R = 1$	2	$+$	$+$	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right) \lvert 0 \rangle$
$N_L = N_R = 1$	2	$-$	$+$	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 0$	0	$+$		$a_1^+ a_1^- \lvert 0 \rangle$
$N_L = 2, N_R = 0$	1	$\pm$		$\left(a_2^+ \pm a_2^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 0$	2	$+$		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 0$	2	$-$		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	0	$+$		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	0	$-$		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	2	$+$		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	2	$-$		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	3	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right) \lvert 0 \rangle$

Results for SU(3) and q=0

First Excited State for $N_L = N_R = 1$.



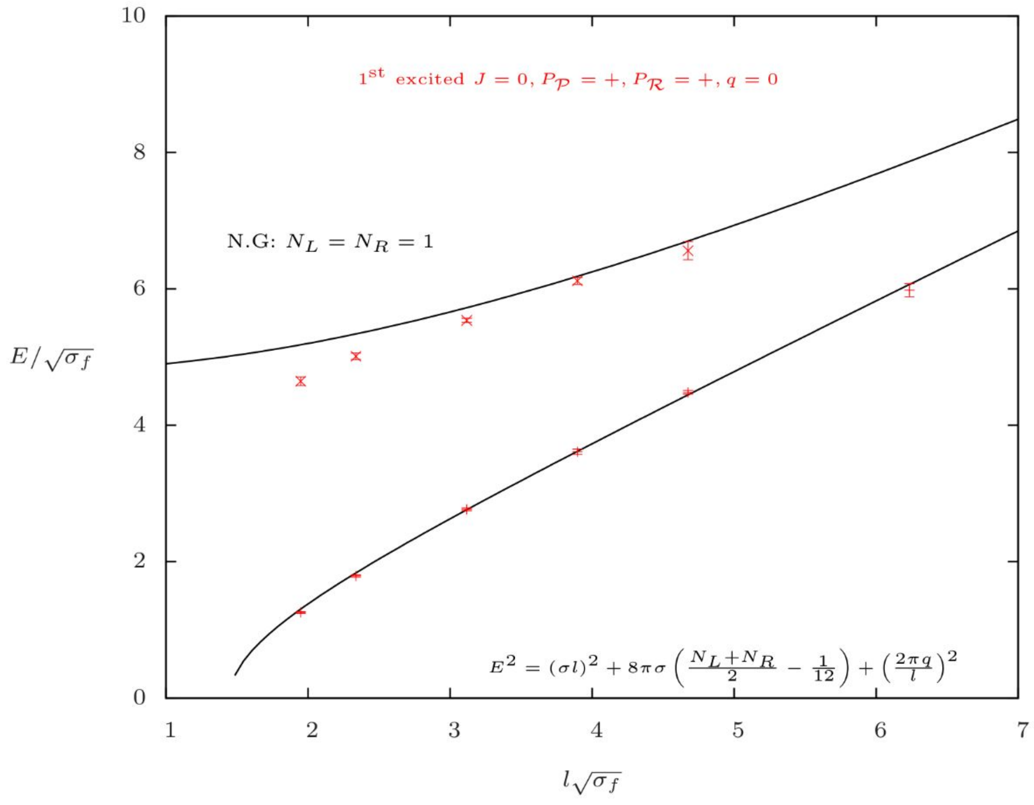
Results: the D=3+1 case

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	P_t	P_l	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	±		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	0	+		$a_1^+ a_1^- \lvert 0\rangle$
$N_L = 2, N_R = 0$	1	±		$\left(a_2^+ \pm a_2^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	+		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	−		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	+		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	−		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	±		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	±		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	+		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	−		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	3	±		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

Results for SU(3) and q=0

First Excited State for $N_L = N_R = 1$.



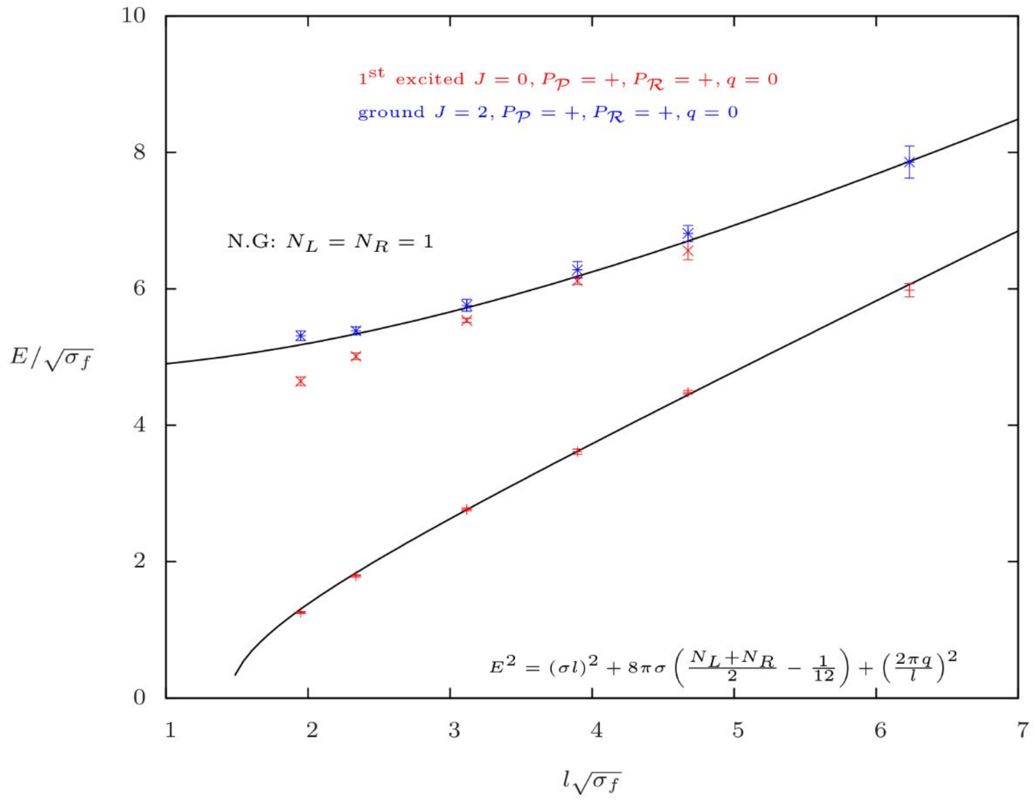
Results: the D=3+1 case

String States & Quantum Numbers

N_L, N_R	$\|J\|$	P_t	P_l	String State
$N_L = N_R = 0$	0	+	+	$\|0\rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right)\|0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\|0\rangle$
$N_L = N_R = 1$	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\|0\rangle$
$N_L = N_R = 1$	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\|0\rangle$
$N_L = N_R = 1$	2	$-$	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\|0\rangle$
$N_L = 2, N_R = 0$	0	+		$a_1^+ a_1^- \|0\rangle$
$N_L = 2, N_R = 0$	1	$\pm$		$\left(a_2^+ \pm a_2^-\right)\|0\rangle$
$N_L = 2, N_R = 0$	2	+		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\|0\rangle$
$N_L = 2, N_R = 0$	2	$-$		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\|0\rangle$
$N_L = 2, N_R = 1$	0	+		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\|0\rangle$
$N_L = 2, N_R = 1$	0	$-$		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\|0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\|0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\|0\rangle$
$N_L = 2, N_R = 1$	2	+		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\|0\rangle$
$N_L = 2, N_R = 1$	2	$-$		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\|0\rangle$
$N_L = 2, N_R = 1$	3	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\|0\rangle$

Results for SU(3) and q=0

First Excited State for $N_L = N_R = 1$.



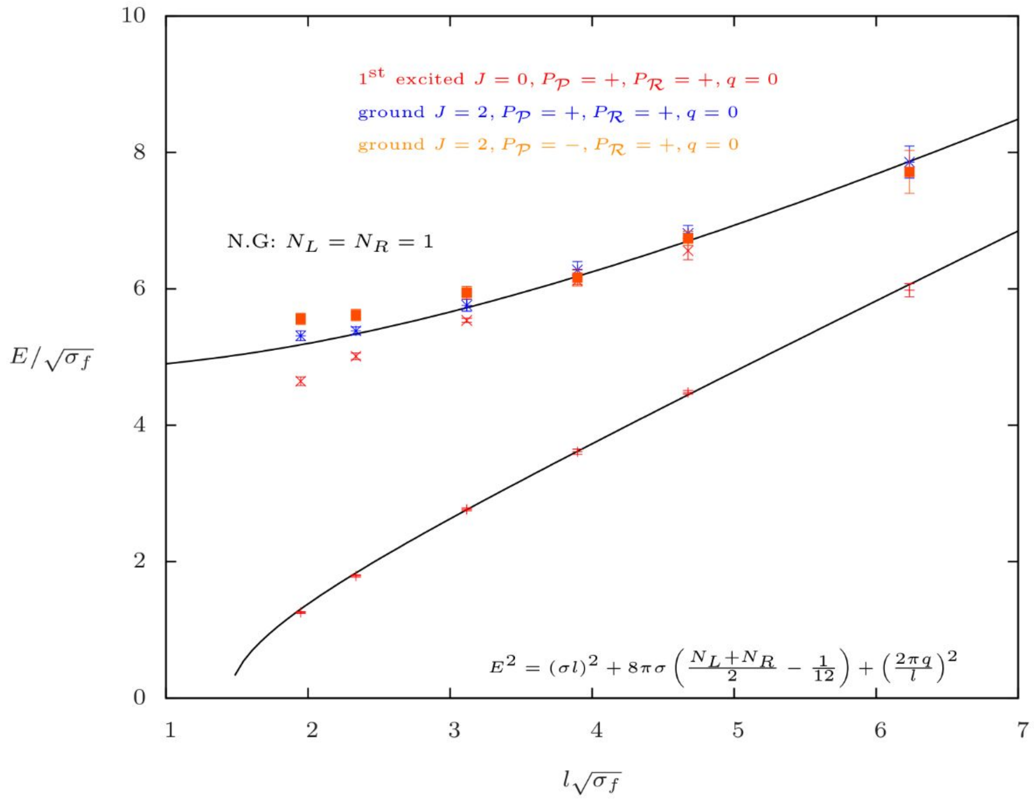
Results: the D=3+1 case

String States & Quantum Numbers

N_L, N_R	$\lvert J\rvert$	P_t	P_l	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	±		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	0	+		$a_1^+ a_1^- \lvert 0\rangle$
$N_L = 2, N_R = 0$	1	±		$\left(a_2^+ \pm a_2^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	+		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	−		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	+		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	−		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	±		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	±		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	+		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	−		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	3	±		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

Results for SU(3) and q=0

First Excited State for $N_L = N_R = 1$.



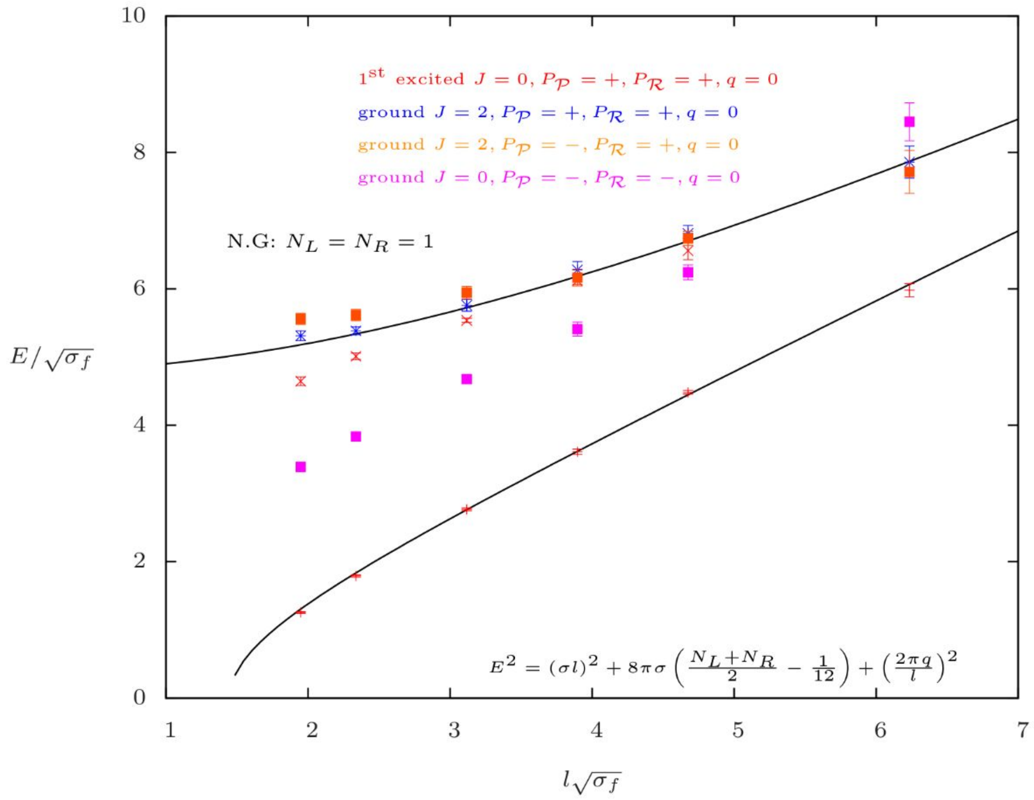
Results: the D=3+1 case

String States & Quantum Numbers

N_L, N_R	$\|J\|$	P_t	P_l	String State
$N_L = N_R = 0$	0	+	+	$\|0\rangle$
$N_L = 1, N_R = 0$	1	±		$\left(a_1^+ \pm a_1^-\right)\|0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\|0\rangle$
$N_L = N_R = 1$	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\|0\rangle$
$N_L = N_R = 1$	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\|0\rangle$
$N_L = N_R = 1$	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\|0\rangle$
$N_L = 2, N_R = 0$	0	+		$a_1^+ a_1^- \|0\rangle$
$N_L = 2, N_R = 0$	1	±		$\left(a_2^+ \pm a_2^-\right)\|0\rangle$
$N_L = 2, N_R = 0$	2	+		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\|0\rangle$
$N_L = 2, N_R = 0$	2	−		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\|0\rangle$
$N_L = 2, N_R = 1$	0	+		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\|0\rangle$
$N_L = 2, N_R = 1$	0	−		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\|0\rangle$
$N_L = 2, N_R = 1$	1	±		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\|0\rangle$
$N_L = 2, N_R = 1$	1	±		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\|0\rangle$
$N_L = 2, N_R = 1$	2	+		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\|0\rangle$
$N_L = 2, N_R = 1$	2	−		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\|0\rangle$
$N_L = 2, N_R = 1$	3	±		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\|0\rangle$

Results for SU(3) and q=0

First Excited State for $N_L = N_R = 1$.

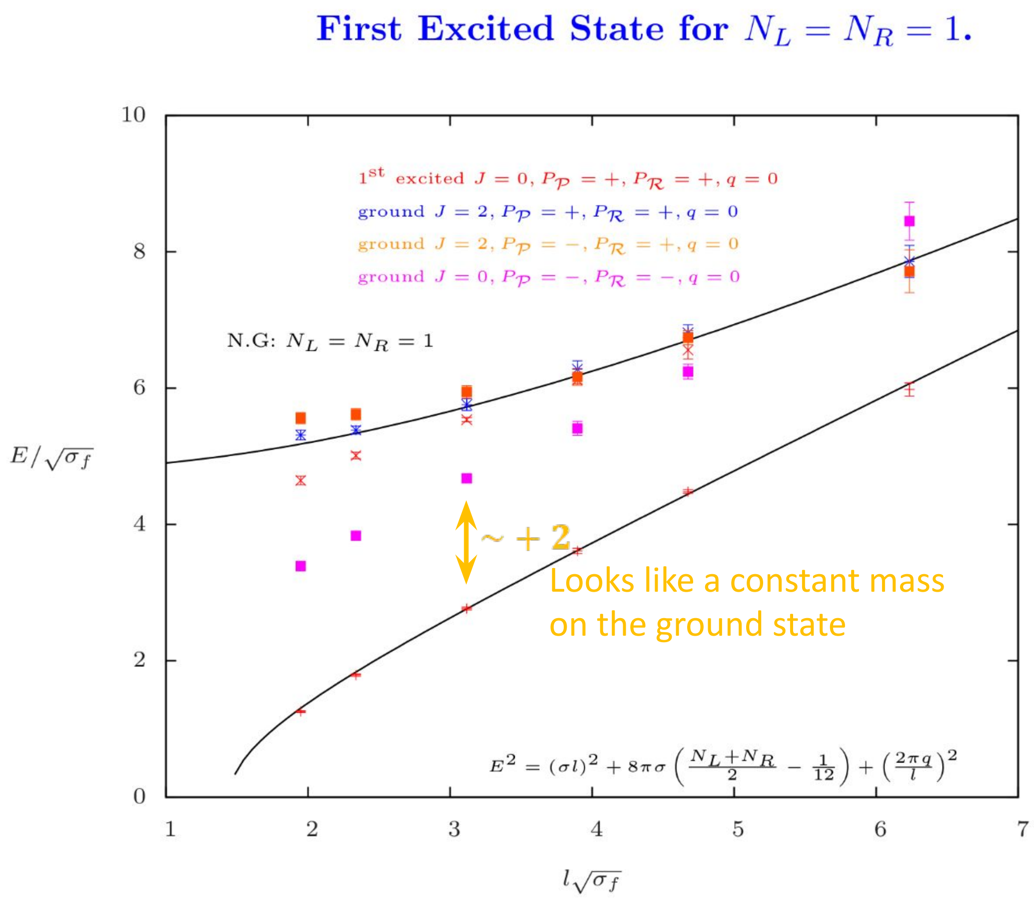


Results: the D=3+1 case

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	P_t	P_l	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	±		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	0	+		$a_1^+ a_1^- \lvert 0\rangle$
$N_L = 2, N_R = 0$	1	±		$\left(a_2^+ \pm a_2^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	+		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	−		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	+		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	−		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	±		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	±		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	+		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	−		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	3	±		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

Results for SU(3) and q=0



Results: the D=3+1 case

S. Dubovsky, R. Flauger, V. Gorbenko '13

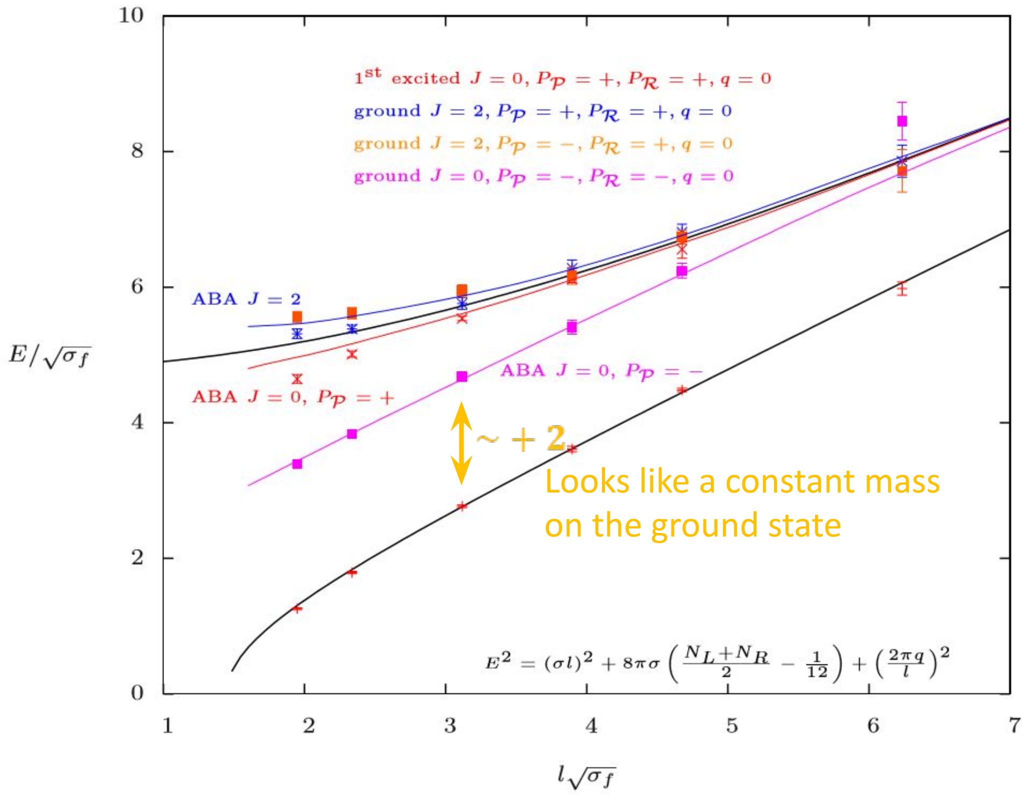
- *Calculating the energy shifts from the S matrix of colliding winding phonons*
- *Using approximate integrability as well as the Thermodynamic Bethe Ansatz*
- *Test what happens when we have both left and right movers – energy levels cannot explain the anomalous state*
- *the most straightforward way to explain this level is the introduction of a massive pseudoscalar particle φ on the worldsheet*

$$S_{int} = \frac{\alpha}{8\pi} \int d^2\sigma \phi K^i_{\alpha\gamma} K^{j\gamma}_{\beta} \epsilon^{\alpha\beta} \epsilon_{ij}$$

- *field φ as the worldsheet axion*
- *By fitting the two free parameters:*

$$m\ell_s = 1.85^{+0.02}_{-0.03} \qquad \alpha = 9.6 \pm 0.1$$

Results for SU(3) and q=0



Outline

- The flux tube looks pretty much like a bosonic string even for short flux tubes
- Looks like there is a massive “axion” particle on the worldsheet of the flux tube in D=3+1
- However we would like to know what happens for higher excitations in D=3+1
- How does the “axion” behaves in the existence of a θ-vacuum?
- What happens for long flux tubes (level crossing)?