A metric relation on the triangle and the circle

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Abstract. In this short note we study a metric relation between the triangle and the circle.

Keywords. triangle, circle, tangent, metric relation.

Introduction. As the Pythagorean theorem is a metric relation in a right triangle, the following theorem is a metric relation between a circle and a triangle of which one of the sides is tangent to the circle and the opposite vertex belonging to the circle. We use the coordinate method to prove this relation.

Theorem. *Let ABC be a triangle in the plane such that the line passing through the point A and the point C is tangent to a circle with center O, and the point B and the point C belonging to this same circle. Let a, b, c, $r \in R_+^*$ such that $AB = c$, $AC = b$, $BC = a$ and r the radius of the center circle O. Then, the following relation hold,*

$$r = \sqrt{\frac{a^4b^2}{2\left(a^2b^2 + a^2c^2 + b^2c^2\right) - a^4 - b^4 - c^4}}$$

Proof. Let be the Cartesian coordinate system of the plane P given by $(G, \vec{i}, \vec{j})$. The point G is the origin of the coordinate system, the two vectors $\vec{i}, \vec{j}$ are orthogonal and unitary.
Let d, f, p, $q \in R$ such that, $d \neq 0$, $f \neq 0$. Let C_1 be the circle with center O. The point B and C belonging to the circle C_1. By rotation and translation, which are isometries, we have the point $O(0, 0)$, the point $C(d, 0)$ and the point $B(p, q)$.
Let d_1 be the line tangent to the circle C_1 passing through the point C, the equation of the line d_1 is,

$$x = d$$

The point A belong to the line d_1, therefore $A(d, f)$.
Let a, b, c, $r \in R_+^*$ such that $AB = c$, $AC = b$, $BC = a$ and $OB = OC = r$.

$$a^2 = (d - p)^2 + q^2$$

$$b^2 = f^2$$

$$c^2 = (d - p)^2 + (f - q)^2$$

$$r^2 = d^2$$

Then,

$$c^2 = a^2 + b^2 - 2fq$$

$$\frac{c^2 - a^2 - b^2}{-2f} = q$$

$$\frac{\left(c^2 - a^2 - b^2\right)^2}{4b^2} = q^2$$

And $|q|$ is the height from B in the triangle OCB, as shown in [1], by Heron's formula,

$$q^2 = \left(\frac{2\sqrt{\left(\frac{2r+a}{2}\right)\left(\frac{2r+a}{2} - a\right)\left(\frac{2r+a}{2} - r\right)\left(\frac{2r+a}{2} - r\right)}}{r} \right)^2$$

$$q^2 = a^2 - \frac{a^4}{4r^2}$$

Then,

$$\frac{\left(c^2 - a^2 - b^2\right)^2}{4b^2} = a^2 - \frac{a^4}{4r^2}$$

Then,

$$r = \sqrt{\frac{a^4 b^2}{2\left(a^2b^2 + a^2c^2 + b^2c^2\right) - a^4 - b^4 - c^4}}$$

□

References

[1] John Roe. *Elementary geometry.* Clarendon Press, 1993.