The ‘Exclusive Or’ Logical Operator

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One of the logical operators or logical connectives in propositional logic is disjunction, which is used to symbolize an ‘or’ statement. In propositional logic (and predicate logic) ‘or’ is interpreted to mean that one or both of the simple statements is/are true. Here is the truth table:

	p	∨	q
1.	T	**T**	T
2.	T	**T**	F
3.	F	**T**	T
4.	F	**F**	F

In this table ‘p’ and ‘q’ represent statement variables, or any two simple statements which could themselves be true or false. The table shows the various possibilities for the truth values of the simple statements and the truth value for the whole statement under those given conditions. In this truth table the compound statement is true when both simple statements are true (line 1) and when one is true and the other is false (lines 2 and 3). The only instance in which the compound statement is false is when both simple statements are false (line 4).

In some ways this truth table fits the ordinary usage and understanding of ‘or’ but in other ways it does not. The part that seems unnatural to me is the first line. If both simple statements are true it seems strange to connect them with an ‘or’. But as I have thought about it I have noticed that ‘or’ is used that way sometimes in ordinary language. For example, I was eating lunch at a local restaurant one time and when my order was ready I picked it up at the counter and the guy asked me: ‘ketchup or salt?’ I replied, without even thinking about it: ‘Yeah, I’ll have both.’ He gave me salt packets and ketchup packets and did not charge me extra. Compare that with the option of ‘soup or salad’ that comes with the entree at some restaurants. You have to choose either soup or salad. If you wanted both of them you would be charged extra. In that case ‘or’ means ‘one but not both’.

There are thus two types of ‘or’ statements: When both disjuncts (the simple statements) can be, or are true, the compound statement is known as an ‘inclusive or’, and when only one is true it is known as an ‘exclusive or’. You may wonder, as I have, why an ‘inclusive or’ is ever used at all. For example, why didn’t the guy behind the counter just use a conjunction and say that I could have ketchup *and* salt if he was offering both for free? Well, the problem with that would be that then it would seem like I was required to take both of them or I could not have either one, or at least that would have been a possible way of interpreting it. Here is the truth table for a conjunction, or ‘and’ statement:

	p	^	q
1.	T	**T**	T
2.	T	**F**	F
3.	F	**F**	T
4.	F	**F**	F

As before, the lowercase letters represent statement variables and the caret symbol ^ stands for 'and'. In this truth table the only instance in which the compound statement is true is line 1 when both simple statements are true. So, if he had offered me 'ketchup *and* salt' then it might seem like I had to take both or I could not get either one. To avoid possible confusion he used 'or'. The message that he wanted to convey was that I had three options, I could have ketchup, I could have salt, or I could have both ketchup and salt. That fits the 'inclusive or' truth table very well. (I am not saying that he was thinking about the truth table when he said this, but that is the reason that standard usage is to say 'or' and why we have all become used to that.) I knew from the context what he meant, but it would have been more clear and precise if he had said 'ketchup and/or salt'. Perhaps that seems too clunky to some people but it clarifies the meaning.

Here is a truth table for 'exclusive or':

	p	∨	q
1.	T	**F**	T
2.	T	**T**	F
3.	F	**T**	T
4.	F	**F**	F

The compound statement is true when one and only one of the simple statements is true. To me this is the most natural way of understanding an 'or' statement. 'Conjunction' is defined as: 'the act of joining, or the condition of being joined'. 'Dis' means 'not' or the lack of, as in *dislike* and *disrespect*, so 'disjunction' ought to mean that the simple statements are not joined. Therefore they cannot both true. If in fact they are both true then they are joined and the compound statement asserting that they are not joined is false (line 1).

Some might argue that a separate truth table for 'exclusive or' is not necessary because one could use (p ∨ q) ^ ~(p ^ q), which translates to 'p or q but not both p and q', if the context requires that the disjunction be exclusive. This is what some logic textbooks advocate. The truth table for this is logically equivalent to the one that I provided for 'exclusive or':

	p ∨ q	(p ∨ q) ^ ~(p ^ q)
1.	T **F** T	T T T **F** FT T T
2.	T **T** F	T T F **T** TT F F
3.	F **T** T	F T T **T** TF F T
4.	F **F** F	F F F **F** TF F F

However, because the standard truth table that is currently in use is actually for ‘inclusive or’ the second table (on the right) would really be saying ‘p and/or q, and not p and q’, which is self-contradictory. Part of the claim is asserting (p ^ q) ^ ~(p ^ q). This is not an adequate workaround.

The ‘exclusive or’ is more fundamental than ‘inclusive or’, and it is used often enough that it ought to have its own truth table. One can easily derive the ‘exclusive or’ table just from thinking about what the entire statement’s truth value would be based upon the truth value of the simple statements. It is as immediate and natural as the truth table for conjunction. It would be better to start with the simpler and more fundamental of the two and then build out to ‘and/or’ because the latter is really a hybrid of a conjunction and a disjunction and its truth table reflects that.

But if we are going to have separate truth tables we also need to differentiate them symbolically because they are really two different logical operators.

In math they use a circled plus symbol for ‘exclusive or’ (XOR), as in $x \oplus y$; but it seems like that would be a bit random and unconnected to any of the other operators to use that for logic. (They do also use the standard plus symbol + for ‘and/or’, which does connect the operators a bit more, but I do not really like that for logic either. It is best to keep math symbols reserved for math I think.)

I propose that the ‘reversed caret’ symbol ∨, which is widely used to represent ‘or’, be reserved for ‘exclusive or’ statements. (I have also seen this symbol referred to as a ‘wedge’ or a ‘vee’; however it is referred to, it always looks essentially like a lowercase letter ‘v’.)

We now need to decide how to symbolize ‘and/or’. The ‘dot’ symbol • is frequently used for conjunction, so the symbol •/∨, or if it was between simple statements p •/∨ q, could be used. I actually kind of like using the ‘dot’ symbol. It is clear, easy to write, and to recognize. There would not be anything wrong with this symbolization but I think we could still do a little better.

If the caret symbol ^ is used for ‘and’ (it often is already), and the reversed caret is used for ‘exclusive or’, then combining them for ‘and/or’ would be ^/∨, which is a little more aesthetically pleasing because of the symmetry. Either way would be fine, but this allows us to connect three out of the four operators; the only one not included is negation.[1] This would help to unify the logical system. They are all simple and easy to understand conceptually. Here are the truth tables for all of the operators:

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[1] As I have written about elsewhere, a conditional is not a proposition or a statement, it is an inference from antecedent to consequent. Because it is not really a compound proposition its truth value cannot be derived from the truth value of its component parts. There is not a legitimate truth table for conditionals or biconditionals. If the condition has been met then the inference is actual, if not it would be hypothetical. When a conditional is used as a premise in an argument it can be thought of as a subinference within the larger argument.

p and q	p or q (exclusive)	p and/or q	not p
p ^ q	p ∨ q	p^/∨q	~p
T **T** T	T **F** T	T **T** T	**F**T
T **F** F	T **T** F	T **T** F	**T**F
F **F** T	F **T** T	F **T** T	
F **F** F	F **F** F	F **F** F	

The symbolization ^/∨ visually represents the claim that is being made, which is that the compound statement is true if either the conjunction operator or the disjunction operator is applicable. The slash symbol / is used between alternatives in grammar, meaning that it could be either option. This is a hybrid operator, a chimera of sorts, so it is no surprise that the truth table for it is a combination of the truth tables of a conjunction and a disjunction. An ‘and/or’ statement is true when both simple statements are true, just like a conjunction, and it is also true when one and only one simple statement is true, just like a disjunction. There has never been nor ever will be an instance in which the ‘or’ portion of the ‘and/or’ operator is true when both simple statements are true; rather, it is always the conjunction portion that is true in those instances, which is why it is important for the ‘and’ to be stated explicitly rather than having it be merely implied.

In translating a statement from ordinary language you would need to consider whether it is best symbolized as an ‘exclusive or’ or whether it is really ‘and/or’ and the speaker is just not being as precise as he or she should. Sometimes people are lazy. We love acronyms and nicknames and shortening words. Sometimes we just say ‘or’ when what we mean is really ‘and/or’, similar to how we sometimes leave off the ‘then’ in a conditional; but in this case it is more problematic because it confuses the meaning. Some people may also think that it is inelegant to write or say ‘and/or’, but I would argue that it is necessary for clarity. Often an ‘exclusive or’ will be introduced by ‘either’; it would be good to use ‘either’ when it fits; otherwise it should just be ‘or’ when the claim is meant to be exclusive, and ‘and/or’ when it is supposed to be inclusive. Sometimes it is obvious from the context which it should be, but other times it is not. If the author does not clarify his or her meaning then we would just have to do our best to interpret him or her charitably when translating from ordinary language into symbolic form and perhaps evaluate it both ways.

In logic and math the current default usage of ‘or’ is inclusive unless it is otherwise stipulated, but I do not think it is like that in ordinary language. As far as I am concerned the default understanding of ‘or’ should be the exclusive sense. If we mean the inclusive sense that should be stipulated by saying ‘and/or’.

However, despite the name ‘exclusive or’ it does not have to be the case that the two simple statements are mutually exclusive. It obviously should be used when they are mutually exclusive because the ‘inclusive or’ truth table would make no sense when the simple statements are directly contradictory, such as ‘It is either raining or it is not’. It is also used when the simple statements have a type of opposition because they are antonyms, such as ‘happy or sad’, ‘up or down', 'hot or cold' for the same reason. But I would use 'exclusive or' more broadly than that. It does not have to be the case that the disjuncts cannot be true at once, it may simply be that it happens to factually be the case that in this particular instance one disjunct is true and the other is false; the operator also applies in those cases.

The default understanding and usage of 'or' should be exclusive; 'and/or' would be the exception or specialized case that a writer or speaker would use when wanting to emphasize that one or both simple statements are or could be true. I have actually noticed while writing this paper that I use 'or' a lot between alternatives. I considered whether all of these instances should be 'and/or' but decided against it because while both alternatives could possibly be true, or are actually different names for that thing which are both equally correct, I do not necessarily mean to say that they are both true at once. It is more that either alternative would be acceptable. The 'exclusive or' is also a little more convenient to use, but more importantly, it correctly expresses the idea that the reader should choose one of the alternatives.

I believe that the distinction between 'inclusive or' and 'exclusive or' is an important one in logic. One thing that it allows us to do is to recognize an additional valid form for disjunctive syllogism.[2]

$A \vee B$
$\underline{A \quad\quad}$
$\sim B$ valid

	A B	$A \vee B$	/ A	// $\sim B$
1.	T T	T **F** T	**T**	**F**T
2.	T F	T **T** F	**T**	**T**F
3.	F T	F **T** T	**F**	**F**T
4.	F F	F **F** F	**F**	**T**F

This argument is valid, but this form of disjunctive syllogism is not usually recognized as a valid form because it is not valid with an 'inclusive or':

[2] Speaking of disjunctive syllogism, I used to have a T-shirt that had the following argument on it: Either the moon is made of green cheese or God exists; the moon is not made of green cheese so God exists. People were usually pretty surprised and rather skeptical when I would tell them that it was valid and explained disjunctive syllogism. I would usually keep it up for a few minutes and argue for its validity just to mess with them a little and make them think. But then I would go ahead and explain that the way in which the argument is flawed is that it could, and probably is the case that both disjuncts are false, so this would be the fourth line of the truth table. That means that the first premise is false. Therefore the argument is valid but probably not sound. It was an interesting conversation starter, if nothing else. It was good to get a conversation going in logic and other philosophy classes, but I also had some interesting conversations with random people that I met in public places too. Yeah, I know, I'm a geek, but it was kind of fun. I had another shirt that was white with a small black spot on the back, which was an ink stain from a pen leaking on it, and on the front it had the words (in green) 'This shirt is black'. (That idea actually came from a logic textbook by Stan Baronett.) I got lots of comments on that one. People really liked it even if they did not quite get it, and it started some interesting tongue-in-cheek philosophical conversations about whether it was actually a white shirt with a small black spot or a black shirt with a very big white spot, and of course whether statements must be true, whether you should trust your senses the most or something else, etc. Fun times.

A^/∨B
A
~B   invalid

	A B	A^/∨B / A // ~B
1.	T T	T **T** T **T** **F**T
2.	T F	T **T** F **T** **T**F
3.	F T	F **T** T **F** **F**T
4.	F F	F **F** F **F** **T**F

In this case there are two true premises and a false conclusion on line 1 of the table, which shows that the argument is invalid in the scenario in which both A and B are true. If you think about it carefully you will be able to intuit the same answer. When using an 'inclusive or' we cannot infer that B is false even if we know that A is true because with an 'inclusive or' it is possible for both simple statements to be true. Thus, B could still be true even if both of these premises are true, which would give us all true premises and a false conclusion. With an 'exclusive or' we know that only one of the disjuncts is true, so if we are told which one it is in the premises then we can validly infer that the other disjunct must be false. The standard form of disjunctive syllogism, A ∨ B, ~A, therefore B, is also valid with an 'exclusive or' because in that case we are told in the premises which one of the disjuncts is false so we would then know that the other one has to be true if the first premise is true. There are two more valid forms with 'exclusive or': A ∨ B, ~B, therefore A, and A ∨ B, B, therefore ~A. There are four valid forms of disjunctive syllogism in total when it is an 'exclusive or' and two valid forms when it is an 'inclusive or'. The 'inclusive or' forms are A^/∨B, ~A, therefore B, and A^/∨B, ~B, therefore A.

Another reason that this distinction is important is that De Morgan's rule only works for 'inclusive or'. According to De Morgan's rule 'not both p and q' is logically equivalent to 'not p or not q', and 'not p or q' is logically equivalent to 'not p and not q'. But these are not logically equivalent if the 'or' is an 'exclusive or'. Here are the truth tables to prove it:

			and/or	or
	p q	~(p ^ q)	~p ^/∨~q	~p ∨ ~q
1.	T T	**F** TTT	FT **F** FT	FT **F** FT
2.	T F	**T** TFF	FT **T** TF	FT **T** TF
3.	F T	**T** FFT	TF **T** FT	TF **T** FT
4.	F F	**T** FFF	TF **T** TF	TF **F** TF

Truth values under main operator in **bold.**

	and/or		or
p q	~(p^/∨q)	~p ^ ~q	~(p ∨ q)
T T	**F** T T T	FT **F** FT	**T** T F T
T F	**F** T T F	FT **F** TF	**F** T T F
F T	**F** F T T	TF **F** FT	**F** F T T
F F	**T** F F F	TF **T** TF	**T** F F F

Truth values under main operator in **bold.**

These truth tables also reveal some other interesting things. Notice that ~p ∨ ~q is the contradictory of ~(p ∨ q), or at least that is what the truth tables indicate. But that does not seem right. Obviously p ∨ q and ~(p ∨ q) have to be contradictories, and the truth tables show that they are. (This is demonstrated by the fact that they have opposite truth values under the main operator for every line of the truth table.) But p ∨ q and ~p ∨ ~q are not equivalent, and they should be if both are the contradictory of ~(p ∨ q), yet they do have the same truth values on each line of the truth table, which indicates that they are logically equivalent. What is going on here?

	p ∨ q	~p ∨ ~q
1.	T **F** T	FT **F** FT
2.	T **T** F	FT **T** TF
3.	F **T** T	TF **T** FT
4.	F **F** F	TF **F** TF

These statements are not equivalent, at least not in terms of meaning; in fact it seems like they are saying the exact opposite thing. Because of the negations lines 2 and 3 and lines 1 and 4 are switched in the second table from how it is in the first one, but that still gives us the same result under the main operator on every line. This is a different type of opposition than a contradictory. In this case the two propositions have the same truth value but are making opposite claims. I actually think of it as making the same claim in the exact opposite way. This is one way of explaining why ~(p ∨ q) is not equivalent to ~p ∨ ~q: the latter is asserting a claim that in a sense is equivalent to p ∨ q whereas the former is asserting the contradictory of it; the latter has negated simple statements whereas the former negates the disjunct relationship, asserting that it does not exist, which is why it is true when both simple statements are true and when both are false: there is no true disjunctive relationship in those instances.

In *An Alternative Version of Categorical Logic* I referred to p ∨ q and ~p ∨ ~q as inverse statements and provided some examples of the relation from categorical logic. Here I will give some more examples from propositional logic.

The most basic would be double negation, p and ~~p. Obviously 'not p' would be the contradictory of 'p' so the negation of the contradictory would have a truth value equivalent to the original. But in another sense, asserting that the contradictory is false is actually the exact opposite way of making the claim to asserting that the original is true. It is a more roundabout way, but still has the same truth value. Obviously a triple negation of p would be the inverse of 'not p', and so on.

Here are two more:

	~p ∨ q	p ∨ ~q
1.	FT **T** T	T **T** FT
2.	FT **F** F	T **F** TF
3.	TF **F** T	F **F** FT
4.	TF **T** F	F **T** TF

Both of these have a truth table that is logically equivalent to ~(p ∨ q) as well. However I would not consider ~(p ∨ q) to be the inverse of either one of these. It is actually more like the obverse.

For another set of statements shown in the table above there is also an obverse relationship:

	p q |	~(p^/∨q)	~p ^ ~q
1.	T T	**F** T T T	FT **F** FT
2.	T F	**F** T T F	FT **F** TF
3.	F T	**F** F T T	TF **F** FT
4.	F F	**T** F F F	TF **T** TF

These statements are in one sense opposites and in another sense equivalent. The first is the negation of 'and/or', or denying that p and q are connected by a conjunction or a disjunction. The only instance in which that statement is true is when both p and q are false. The second is asserting a conjunction, but a conjunction of the contradictories of p and of q. Once again the only instance in which this statement is true is when both p and q are false.[3]

I think that the relationship that both of these statements have to p ^ q is an interesting one. Neither is equivalent to ~(p ^ q); as a reminder, here is the truth table for that:

	p q |	~(p ^ q)
1.	T T	**F** TTT
2.	T F	**T** TFF
3.	F T	**T** FFT
4.	F F	**T** FFF

Notice how under the main operator the truth values are exactly opposite from p ^ q on every line of the table which shows that this is the contradictory of p ^ q. But the truth values are also inverted under the main operator from how they are in ~(p^/∨q) and ~p ^ ~q, which makes me wonder about the relationship of those statements to p ^ q. Let's compare them:

	p ^ q	~(p ^ q)	~(p^/∨q)	~p ^ ~q
1.	T **T** T	**F** TTT	**F** T T T	FT **F** FT
2.	T **F** F	**T** TFF	**F** T T F	FT **F** TF
3.	F **F** T	**T** FFT	**F** F T T	TF **F** FT
4.	F **F** F	**T** FFF	**T** F F F	TF **T** TF

What is the relationship here? It is not the contradictory relation, one can see that. It is also not the inverse or obverse relation. So what is it? It clearly seems to be something; it would not have inverted truth values under the main operator unless there was some sort of opposition. Intuitively, it seems likely that there is some sort of opposition between p ^ q and ~p ^ ~q. So it

[3] The second proposition could be stated as 'not p and not q' or 'neither p nor q'. This is a bit tricky because 'neither . . . nor' sounds a lot like 'either . . . or' but it is not the same logical connective. 'Neither . . . nor' is actually a conjunction of two negative statements because they both have to be false in order for the whole proposition to be true, or for the claim that the whole proposition is making to be correct.

occurred to me that perhaps this is a contrary relationship, similar to what we see in categorical logic. For the contrary relation both propositions cannot be true at once although they can both be false. To test this I tried to think of an instance in which both would be false, and as the truth tables indicate, that would occur on line 2 or line 3. Suppose one were to assert the following: 'We went to dinner and a movie'. This could be symbolized as D ^ M. If either D or M was false, or in other words if in fact they only went to dinner and no movie, or they went to the movie but no dinner, then D ^ M as a compound statement is false. But ~D ^ ~M would also be false. Thus one can easily conceive of an instance in which both p ^ q and ~p ^ ~q are false. That is not possible with contradictory statements. I cannot think of an instance in which both propositions would be true though. I do not think that one exists. The truth tables show that there is no line in which both statements are true. The contrary relation seems to fit. I cannot think of any better alternative, at least at the present time, so I will consider them to be contraries.

The contradictory of ~p ^ ~q is this (left):

~(~p ^ ~q)	~(p ^ q)
TFT F FT	**F** TTT
TFT F TF	**T** TFF
TTF F FT	**T** FFT
FTF T TF	**T** FFF

This has inverted truth values under the main operator to ~(p ^ q). But one can see from the truth tables that ~(p ^ q) and ~(~p ^ ~q) are not contraries because in lines 2 and 3 they are both true. They seem to be more equivalent to subcontraries. Subcontraries can both be true at once but they cannot both be false. The truth tables show that this is the case. There is no instance in which both are false. One could create an Aristotelian Square of Opposition with these four statements:

p ^ q ~p ^ ~q

~(~p ^ ~q) ~(p ^ q)

All of the same relationships hold, even subalternation. (Truth goes down, false goes up.) If we know that p ^ q is true then it must be the first line of the truth table and ~(~p ^ ~q) is also true on the first line. The only instance in which ~p ^ ~q is true is when both p and q are false (line 4), so p and q could not be conjoined, which is exactly what ~(p ^ q) is asserting, so that is also true.

(The truth table confirms that ~(p ^ q) is true on line 4.) On the other hand, the only instance in which ~(~p ^ ~q) is false is on line 4, when both p and q are false, so assuming that to be the case, there is no way p ^ q would be true. (It is false on line 4.) The only instance in which ~(p ^ q) is false is when both p and q are true, which is line 1 of the table, and on line 1 of the table for ~p ^ ~q it is false.

If we connect ~(~p ^ ~q) and ~(p ^ q) we get the following compound proposition:

pq	~(~p ^ ~q) ^ ~(p ^ q)
1. TT	T FT F FT **F** FT T T
2. TF	T FT F TF **T** T T F F
3. FT	T TF F FT **T** T F F T
4. FF	F TF T TF **F** T F F F

Using this we can create something similar to the Triangle of Opposition that I spoke of in *An Alternative Version of Categorical Logic* with essentially the same relationships.

p ^ q   ~p ^ ~q

~(~p ^ ~q) ^ ~(p ^ q)

In the Triangle one and only one of the propositions can be true. The truth tables show that is the case here as well. For p ^ q it is only line 1 of the table, or when both p and q are true. The only instance in which ~p ^ ~q is true is line 4, when both p and q are false, and ~(~p ^ ~q) ^ ~(p ^ q) is true on lines 2 and 3, when either p is true and q is false or vice versa. Similar to the Triangle, if you knew that one of these three propositions was false you could deduce that one of the other two must be true, but you would not know which one.

There is another obverse-like relationship shown above that I have not talked about yet. That is the one that exists between ~(p ^ q) and ~p ^/∨~q. Here are the truth tables:

p q	~(p ^ q)	~p ^/∨~q
1. T T	**F** TTT	FT **F** FT
2. T F	**T** TFF	FT **T** TF
3. F T	**T** FFT	TF **T** FT
4. F F	**T** FFF	TF **T** TF

This is of course part of De Morgan’s rule, as is the logical equivalence of ~(p^/∨q) and ~p ^ ~q, which has already been discussed. Though the two claims have the same truth values under the main operator, which technically makes them logically equivalent, there is also a type of opposition. The first proposition, ~(p ^ q), negates or denies the claim that p and q are conjoined, while the second asserts that ~p is either conjoined or not joined with ~q. The first instance is making a claim about p and q and the second is making a claim about ~p and ~q, and a different operator is used.

De Morgan’s rule can assist us in creating another Triangle that this time employs an ‘inclusive or’ by substituting ~(p^/∨q) for ~p ^ ~q in the last Triangle. Here are the results:

p ^ q   ~(p^/∨q)

~[~(p^/∨q)] ^ ~(p ^ q)

We could also substitute ~p ^/∨~q for ~(p ^ q) in the third statement. Either way the truth table will be equivalent. Here it is for ~[~(p^/∨q)] ^ ~(p ^ q):

pq	~[~(p^/∨q)] ^ ~(p ^ q)
TT	T F T T T **F** F T T T
TF	T F T T F **T** T T F F
FT	T F F T T **T** T F F T
FF	F T F F F **F** T F F F

Once again there are no lines on the respective truth tables of these three propositions in which they are true at the same time. The first, p ^ q, is only true on line 1, or when both p and q are true; ~(p^/∨q) is only true on the fourth line, when p and q are both false; ~[~(p^/∨q)] ^ ~(p ^ q) is true for lines 2 and 3, or when p is true and q is false (2) or vice versa (3). This is consistent with prior results and also with the Triangle in categorical logic. One could think of all three as contraries since it is always the case that two of them are false (meaning that they can both be false at the same time) and one is true.

Let’s now see if we can do the same thing for an ‘exclusive or’ disjunction. This is not as straightforward, but we know that the contradictory is ~(p ∨ q). The real question is what the contrary would be, if there even is one. I think that all contingent propositions would have a contradictory which asserts the exact opposite and has the opposite truth value but I am not sure that all propositions would have a contrary; if they do sometimes it is not easy to identify what it is. In this case two leading candidates for the contrary would be ~p ∨ q or p ∨ ~q, but it is not clear which it should be. I guess since they are inverse statements and have logically equivalent truth tables perhaps either one could be considered the contrary of p ∨ q. Here is the truth table for ~p ∨ q and p ∨ ~q compared to the one for p ∨ q:

p ∨ q	~p ∨ q	p ∨ ~q
T **F** T	FT **T** T	T **T** FT
T **T** F	FT **F** F	T **F** TF
F **T** T	TF **F** T	F **F** FT
F **F** F	TF **T** F	F **T** TF

It is the case that neither of these statements is true on the same line as p ∨ q, which fits with being contrary statements, but they actually have opposite truth values to p ∨ q on every line of the table, which is characteristic of contradictory statements. As a reminder, here is the truth table for ~(p ∨ q):

~(p ∨ q)
1. **T**T F T
2. **F**T T F
3. **F**F T T
4. **T**F F F

This truth table is as one might expect: the contradictory of a disjunction is true when in fact both p and q are true or when both are false. But this has the same result under the main operator as ~p ∨ q, and p ∨ ~q, which means that technically they are all logically equivalent. So are they all contradictory statements to p ∨ q and inverse statements to each other? That is not quite right, but I think it is close: ~p ∨ q, and p ∨ ~q would be inverse statements and ~(p ∨ q) could be considered the obverse of either of them. To me ~(p ∨ q) seems the most natural and obvious as the contradictory of p ∨ q. But it is still unclear what the contrary of p ∨ q would be, if there even is one. If there is it would need to be false on lines 2 and 3, and true on either line 1 or line 4, but not both. Here are two more possibilities:

~(p^/∨q)	~p ^ ~q
F T T T	FT **F** FT
F T T F	FT **F** TF
F F T T	TF **F** FT
T F F F	TF **T** TF

Either of these could work, at least potentially, because there is no line in which they are true when p ∨ q is also true, but there is a line (the first one) in which they are false when p ∨ q is also false.[4] Let's now attempt to create our Triangle of Opposition:

p ∨ q   ~(p^/∨q)

~[~(p^/∨q)] ^ ~(p ∨ q)

Here is the truth table for the third proposition:

	pq	~[~(p^/∨q)] ^ ~(p ∨ q)
1.	TT	T FT T T **T** TT F T
2.	TF	T FT T F **F** FT T F
3.	FT	T FF T T **F** FF T T
4.	FF	F TF F F **F** T F F F

I believe that this Triangle also works because the first proposition, p ∨ q, is only true on lines 2 and 3; ~(p^/∨q) is only true on line 4, and ~[~(p^/∨q)] ^ ~(p ∨ q) is only true on line 1. Therefore the statements are not consistent. If one is true then we know that the other two must be false, which is the same result as the Triangle in categorical logic.[5]

One thing that you have probably noticed is that ~(p^/∨q) is the contrary for both p ∨ q and p ^ q, and it is logically equivalent to ~p ^ ~q, which was the contrary that was used initially with p ^ q. That may seem surprising, because p ∨ q and p ^ q are obviously not equivalent to each other. But it is actually not all that surprising if you ponder on it for a bit. The 'inclusive or' operator is a hybrid of both, so if it is negated that is like negating both an 'and' and an 'or'

[4] The definition of consistent statements in propositional logic is that there is a line in which both (or all that are being compared) are true under the main operator. This makes me think that the contrary relationship is connected to consistency. Contraries are inconsistent statements but not contradictory statements because they can both be false on the same line, which cannot happen with contradictories. However I would not want to overstate this relationship. While they are related, I do not think that they are exactly the same thing. Contraries would probably be a particular type of inconsistent statements but not all inconsistent statements would be contraries.

[5] For this and the prior Triangle one could also keep the third statement from being conjoined, which would mean having two separate statements that are analogous to the subalterns, and then the relationships of the statements to one another would be similar to the ones in Aristotle's Square of Opposition, as I discussed with the first example.

simultaneously, which would be correct for a contrary to either one. The negation of ‘and/or’ is the contrary of both ‘and’ and ‘or’(exclusive). But suppose that we started with an affirmative ‘and/or’, p^/∨q, what would be the contrary of that? Because it is true on all lines except the fourth one its contrary could only be true on the fourth line. However one will notice that this is the case for the contradictory, ~(p^/∨q), as well, so is there a contrary?

Perhaps we could try ~(p ^ q) ^ ~(p ∨ q).

pq | ~(p ^ q) ^ ~(p ∨ q)
TT   F T T T **F** TT F T
TF   T T F F **F** F T T F
FT   T F F T **F** F F T T
FF   T F F F **T** T F F F

This has the same results under the main operator as ~(p^/∨q). However it is actually more like ~p ^ ~q, which also has the same results under the main operator. But none of these could be the contrary because there is no line in which both are false.

I do not think that there is a contrary for p^/∨q. The contrary would need to have a truth table in which it is false on the fourth line, because that is the only possibility for them to both be false at once, but since they cannot both be true at once it would have to be false under the main operator on the other three lines as well. If that is the case then the contrary would be a self-contradictory statement itself because it would have all false truth values under the main operator on every line of its table. Some interpretations might consider any self-contradictory statement using the same variables to therefore be its contrary, but I would resist that idea, as that seems very trivial. I would just say that it does not have a contrary.

There is another obverse relation that I would like to mention. Suppose that R stands for ‘It is raining’. Is ~(R ^ ~R) equivalent to R ∨ ~R ? Let’s look at the truth table.

R | ~(R ^ ~R) | R ∨ ~R
T   **T**T F FT   T **T** FT
F   **T**F F TF   F **T** TF

The negation of a self-contradictory statement is a tautology. R ^ ~R is a self-contradiction while R ∨ ~R is a tautology. (R ^/∨~R would be a tautology as well.) I consider these to be obverse statements because in a way they are both saying that it has to be either R or ~R, it cannot be both, and in this particular case it cannot be neither because R and ~R are contradictories. This is important to note because ~(p ^ q) would not be the obverse of p ∨ q when other terms are used and p and q are not contradictories.

Another point that is somewhat related is that the opposite of a self-contradictory conjunction is an ‘exclusive or’. For example, ‘It is raining and it is not raining’, ‘God exists and God does not exist’; in both cases what is actually true is an ‘exclusive or’ involving the same simple statements. We tend to focus on the fact that the disjuncts cannot both be true at once, but it is also the case that they cannot both be false.

Here are some additional statements that are logically equivalent:

pq	~(p ^ q)	(~p ∨ ~q) ∨ (~p ^ ~q)
TT	**F** T T T	FT F FT **F** FT F FT
TF	**T** T F F	FT T TF **T** FT F TF
FT	**T** F F T	TF T FT **T** TF F FT
FF	**T** F F F	TF F TF **T** TF T TF

We do not even need the truth tables to figure these out, although the truth tables are an effective way to double check. For this first one, if we know that it is not the case that p and q are conjoined then we can infer that it must be the case that at least one of them is false, hence it must be that either p is not true or q is not true, or neither one of them are true, which is equivalent to the second statement. (Note: I consider ‘p is false’ to be equivalent to ‘not-p’ or asserting that ‘not-p’ is true.)

Here is another:

pq	~(p ∨ q)	(p ^ q) ∨ (~p ^ ~q)
TT	**T** T F T	T T T **T** FT F FT
TF	**F** T T F	T F F **F** FT F TF
FT	**F** F T T	F F T **F** TF F FT
FF	**T** F F F	F F F **T** TF T TF

Here we can infer that if ~(p ∨ q) is true then it would have to either be the case that both p and q are true or that both are false, which is equivalent to what the second statement asserts.

And finally:

pq	~(p^/∨q)	~p ^ ~q
TT	**F** T T T	FT **F** FT
TF	**F** T T F	FT **F** TF
FT	**F** F T T	TF **F** FT
FF	**T** F F F	TF **T** TF

This equivalence has already been identified and discussed, but here is the explanation for why they are equivalent: One can reason that if it is not the case that p and q are conjoined or disjoined then it must be the case that both p and q are false, for that is the only way that they would not be connected by either a conjunction or a disjunction.

Now some other equivalences. If p ^ q is true then both p and q are true, so this is equivalent to ~{[(p ^~q) ∨ (~p ^ q)] ∨ (~p ^ ~q)}, or ‘it is not the case that either p or q is true and the other is false or that they are both false’.

pq | p ^ q | ~{[(p ^~q) ∨ (~p ^ q)] ∨ (~p ^ ~q)}
TT T **T** T **T** T F FT F FT F T F FT F FT
TF T **F** F **F** T T TF T FT F F T FT F TF
FT F **F** T **F** F F FT T TF T T T TF F FT
FF F **F** F **F** F F TF F TF F F T TF T TF

If p ∨ q is true then it must be the case that either p or q is true and the other is false, so asserting this is going to be equivalent to ~[(p ^ q) ∨ (~p ^ ~q)]:

pq | p ∨ q | ~[(p ^ q) ∨ (~p ^ ~q)]
TT T **F** T **F** T T T T FT F FT
TF T **T** F **T** T F F F FT F TF
FT F **T** T **T** F F T F TF F FT
FF F **F** F **F** F F F T TF T TF

And finally, there are three ways in which p^/∨q can be true, and only one way in which it can be false. If we know that it is true we therefore know that it cannot be the case that p and q are both false. Thus, p^/∨q is equivalent to ~(~p ^ ~q), or asserting that it is not the case that both p and q are false, or the equivalent of that.

pq | p^/∨q | ~(~p ^ ~q)
TT T **T** T **T** FT F FT
TF T **T** F **T** FT F TF
FT F **T** T **T** TF F FT
FF F **F** F **F** TF T TF

Now just a few additional points.

In natural deduction it is considered a valid step to add any proposition that we choose to a proposition that already exists by itself on a line in the proof by means of the ‘or’ logical operator. This is because only one disjunct needs to be true in order for the whole statement to be true, so even if what you are adding is false we already know that the whole statement will be true because the portion that is already in the proof on a prior line is known to be true, either as given information in a premise or as something that was validly derived. But the ‘or’ that is typically used is inclusive; for an ‘exclusive or’ you would actually need to add something that is known to be false because the other proposition that already appeared on a prior line of the proof would have to be true. If they were both true then an ‘exclusive or’ statement would be false, which of course would make that line of the proof false. So, as in other instances, one would need to be aware of which type of ‘or’ one is using.

In my version of categorical logic the contradictory of the original categorical proposition is an ‘exclusive or’ statement because if that categorical proposition is false then it follows that one and only one of the other two must be true. For instance, ‘It is not the case that All S are P’, if true, entails ‘Either Some S are P or No S are P’.

This means that there can sometimes be multiple valid conclusions that follow from the premises of a syllogism. For example:

Some M are S

<u>All M are P</u>

Either All S are P or Some S are P

This argument is valid. In two diagrams the conclusion that follows would be ‘All S are P’ but in the third the conclusion would be ‘Some S are P’. It is not possible to draw a diagram that is consistent with the premises in which ‘No S are P’. Thus, the conclusion that does follow is: ‘It is not the case that No S are P’. Or, alternatively, one could also validly infer: ‘Either All S are P or Some S are P’.

There are a number of valid forms beyond just the standard ones in which one could validly infer that it is *not* one of the three categorical propositions.

Finally, I suppose I should also address existential import because it is such a frequently discussed topic in logic. Disjunctions that refer to something that does not actually exist could have a hypothetical truth value but they would not have an actual truth value. For instance, ‘Either centaurs are fast or they are not’. This is a tautology and has the form of a good ‘exclusive or’ disjunction. If there were centaurs it would be true because one of the disjuncts would be true. So, the disjunction could be considered hypothetically true but it has no actual truth value for the actual world, as centaurs do not really exist in the actual world. One could also say that it is true relative to the context of mythology. The same analysis would apply to conjunctions as well, such as ‘Hercules is strong and brave’.