The Riemann Hypothesis

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Abstract. Let's define $\delta(x) = (\sum_{q \leq x} \frac{1}{q} - \log\log x - B)$, where $B \approx 0.2614972128$ is the Meissel-Mertens constant. The Robin theorem states that $\delta(x)$ changes sign infinitely often. Let's also define $S(x) = \theta(x) - x$, where $\theta(x)$ is the Chebyshev function. A theorem due to Erhard Schmidt implies that $S(x)$ changes sign infinitely often. Using the Nicolas theorem, we prove that when the inequalities $\delta(p) \leq 0$ and $S(p) \geq 0$ are satisfied for a prime $p \geq 127$, then the Riemann Hypothesis should be false. However, we could restate the Mertens second theorem as $\lim_{n\to\infty} \delta(p_n) = 0$ where p_n is the n^{th} prime number. In addition, we could modify the well-known formula $\lim_{n\to\infty} \frac{\theta(p_n)}{p_n} = 1$ as $\lim_{n\to\infty} S(p_n) = 0$. In this way, this work could mean a new step forward in the direction for finally solving the Riemann Hypothesis.

1 Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. Let $N_n = 2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times p_n$ denotes a primorial number of order n such that p_n is the n^{th} prime number. Say $\mathsf{Nicolas}(p_n)$ holds provided

$$\prod_{q \mid N_n} \frac{q}{q-1} > e^{\gamma} \times \log\log N_n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, log is the natural logarithm, and $q \mid N_n$ means the prime q divides to N_n. The importance of this property is:

Theorem 1.1 *[6], [7].* $\mathsf{Nicolas}(p_n)$ *holds for all prime* $p_n > 2$ *if and only if the Riemann Hypothesis is true.*

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where $p \leq x$ means all the prime numbers p that are less than or equal to x. We use the following property of the Chebyshev function:

Theorem 1.2 *[9]. For* $x \geq 41$*:*

$$\theta(x) = (1 + \varepsilon(x)) \times x$$

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where $-\frac{1}{\log x} < \varepsilon(x) < \frac{1}{2 \times \log x}$.

Besides, in the Grönwall paper appears this:

Theorem 1.3 *[3].*

$$\lim_{x \to \infty} \frac{\theta(x)}{x} = 1.$$

Let's define $S(x) = \theta(x) - x$. Nicolas also proves that

Theorem 1.4 *[7]. For* $x \geq 121$*:*

$$\log \log \theta(x) \geq \log \log x + \frac{S(x)}{x \times \log x} - \frac{S(x)^2}{x^2 \times \log x}.$$

From the paper of Schmidt, then we can deduce that:

Theorem 1.5 *[10].* $S(x)$ *changes sign infinitely often.*

The famous Mertens paper provides the statement:

Theorem 1.6 *[5].*

$$\log \left(\prod_{q \leq x} \frac{q}{q-1} \right) = \sum_{q \leq x} \frac{1}{q} + \gamma - B - \frac{1}{2} \times \sum_{q > x} \frac{1}{q^2} - \frac{1}{3} \times \sum_{q > x} \frac{1}{q^3} - \cdots$$

where $B \approx 0.2614972128$ *is the Meissel-Mertens constant.*

Let's define:

$$\delta(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log x - B \right),$$

Robin theorem states the following result:

Theorem 1.7 *[8].* $\delta(x)$ *changes sign infinitely often.*

In addition, the Mertens second theorem states that:

Theorem 1.8 *[5].*

$$\lim_{x \to \infty} \delta(x) = 0.$$

Putting all together yields the proof that when the inequalities $\delta(p) \leq 0$ and $S(p) \geq 0$ are satisfied for a prime $p \geq 127$, then the Riemann Hypothesis should be false.

2 Central Lemma

Lemma 2.1 *For a prime $p \geq 127$:*

$$\frac{S(p)}{p} < 1.$$

Proof By the theorem 1.2, for all prime $p \geq 127$:

$$\begin{aligned}\frac{S(p)}{p} &= \frac{\theta(p) - p}{p}\\ &= \frac{(1+\varepsilon(p)) \times p - p}{p}\\ &= \frac{p \times ((1+\varepsilon(p)) - 1)}{p}\\ &= (1+\varepsilon(p) - 1)\\ &= \varepsilon(p)\\ &< \frac{1}{2 \times \log p}\\ &< 1.\end{aligned}$$

■

3 Main Theorem

Theorem 3.1 *If the inequalities $\delta(p) \leq 0$ and $S(p) \geq 0$ are satisfied for a prime $p \geq 127$, then the Riemann Hypothesis should be false.*

Proof For a prime $p \geq 127$, suppose that simultaneously the inequalities $\mathsf{Nicolas}(p)$, $\delta(p) \leq 0$ and $S(p) \geq 0$ are satisfied. If $\mathsf{Nicolas}(p)$ holds, then

$$\prod_{q \leq p} \frac{q}{q-1} > e^{\gamma} \times \log \theta(p).$$

We apply the logarithm to the both sides of the inequality:

$$\log \left(\prod_{q \leq p} \frac{q}{q-1} \right) > \gamma + \log \log \theta(p).$$

We use that theorem 1.6:

$$\log \left(\prod_{q \leq p} \frac{q}{q-1} \right) = \sum_{q \leq p} \frac{1}{q} + \gamma - B - \frac{1}{2} \times \sum_{q > p} \frac{1}{q^2} - \frac{1}{3} \times \sum_{q > p} \frac{1}{q^3} - \cdots.$$

Besides, we use that theorem 1.4:

$$\log \log \theta(p) \geq \log \log p + \frac{S(p)}{p \times \log p} - \frac{S(p)^2}{p^2 \times \log p}.$$

Putting all together yields the result:

$$\sum_{q \leq p} \frac{1}{q} + \gamma - B - \frac{1}{2} \times \sum_{q > p} \frac{1}{q^2} - \frac{1}{3} \times \sum_{q > p} \frac{1}{q^3} - \cdots$$
$$> \gamma + \log\log \theta(p)$$
$$\geq \gamma + \log\log p + \frac{S(p)}{p \times \log p} - \frac{S(p)^2}{p^2 \times \log p}.$$

Let distribute it and remove γ from the both sides:

$$\sum_{q \leq p} \frac{1}{q} - \log\log p - B - \frac{1}{2} \times \sum_{q > p} \frac{1}{q^2} - \frac{1}{3} \times \sum_{q > p} \frac{1}{q^3} - \cdots >$$
$$\frac{1}{\log p} \times \left(\frac{S(p)}{p} - \frac{S(p)^2}{p^2} \right).$$

We know that $\delta(p) = \sum_{q \leq p} \frac{1}{q} - \log\log p - B$. Moreover, we know that

$$\left(\frac{S(p)}{p} - \frac{S(p)^2}{p^2} \right) \geq 0.$$

Certainly, according to the lemma 2.1, we have that $\frac{S(p)}{p} < 1$. Consequently, we obtain that $\frac{S(p)}{p} \geq \frac{S(p)^2}{p^2}$ under the assumption that $S(p) \geq 0$, since for every real number $0 \leq x < 1$, the inequality $x \geq x^2$ is always satisfied. To sum up, we would have that

$$\delta(p) - \frac{1}{2} \times \sum_{q > p} \frac{1}{q^2} - \frac{1}{3} \times \sum_{q > p} \frac{1}{q^3} - \cdots > 0$$

because of

$$\frac{1}{\log p} \times \left(\frac{S(p)}{p} - \frac{S(p)^2}{p^2} \right) \geq 0.$$

However, the inequality

$$\delta(p) - \frac{1}{2} \times \sum_{q > p} \frac{1}{q^2} - \frac{1}{3} \times \sum_{q > p} \frac{1}{q^3} - \cdots > 0$$

is never satisfied when $\delta(p) \leq 0$. By contraposition, $\mathsf{Nicolas}(p)$ does not hold when $\delta(p) \leq 0$ and $S(p) \geq 0$ are satisfied for a prime $p \geq 127$. In conclusion, if $\mathsf{Nicolas}(p)$ does not hold for a prime $p \geq 127$, then the Riemann Hypothesis should be false due to the theorem 1.1. ■

4 Discussion

The Riemann Hypothesis has been qualified as the Holy Grail of Mathematics [4]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution [2]. In the theorem 3.1, we show that if the inequalities $\delta(p) \leq 0$ and $S(p) \geq 0$ are satisfied for a prime $p \geq 127$, then the Riemann Hypothesis should be false. Nevertheless, the well-known theorem 1.8 could be restated as

$$\lim_{n\to\infty} \delta(p_n) = 0$$

because of there are infinitely many prime numbers p_n. At the same time, we can restate the theorem 1.3 as

$$\lim_{n\to\infty} S(p_n) = 0.$$

Indeed, we think this work could help to the scientific community in the global efforts for trying to solve this outstanding and difficult problem.

References

[1] Peter B. Borwein, Peter Borwein, Stephen Choi, Brendan Rooney, and Andrea Weirathmueller. *The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike*, volume 27. Springer Science & Business Media, 2008.

[2] Keith J. Devlin. *The Millennium Problems: The Seven Greatest Unsolved Mathematical Puzzles Of Our Time*, volume 100. Basic Books New York, NY (USA), 2002.

[3] Thomas H. Grönwall. Some asymptotic expressions in the theory of numbers. *Transactions of the American Mathematical Society*, 14(1):113–122, 1913. doi:10.2307/1988773.

[4] Gilles Lachaud. L'hypothese de Riemann: le Graal des mathématiciens. *La Recherche Hors-Série*, (20), 2005.

[5] Franz Mertens. Ein Beitrag zur analytischen Zahlentheorie. *J. reine angew. Math.*, 1874(78):46–62, 1874. URL: https://doi.org/10.1515/crll.1874.78.46, doi:10.1515/crll.1874.78.46.

[6] Jean-Louis Nicolas. Petites valeurs de la fonction d'Euler et hypothese de Riemann. *Séminaire de Théorie des nombres DPP, Paris*, 82:207–218, 1981.

[7] Jean-Louis Nicolas. Petites valeurs de la fonction d'Euler. *Journal of number theory*, 17(3):375–388, 1983. doi:10.1016/0022-314X(83)90055-0.

[8] Guy Robin. Sur l'ordre maximum de la fonction somme des diviseurs. *Séminaire Delange-Pisot-Poitou Paris*, 82:233–242, 1981.

[9] J. Barkley Rosser and Lowell Schoenfeld. Approximate Formulas for Some Functions of Prime Numbers. *Illinois Journal of Mathematics*, 6(1):64–94, 1962. doi:10.1215/ijm/1255631807.

[10] Erhard Schmidt. Über die Anzahl der Primzahlen unter gegebener Grenze. *Mathematische Annalen*, 57(2):195–204, 1903.

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