EMUE activity A1.2.2
Measurement models involving additive or multiplicative corrections

Alen Bošnjaković[1], Vedran Karahodžić[1], John Greenwood[2], and Maurice Cox[3]

[1]Institute of metrology of Bosnia and Herzegovina, Bosnia and Herzegovina
[2]United Kingdom Accreditation Service, UK
[3]National Physical Laboratory, UK, maurice.cox@npl.co.uk

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1 Summary

A common form of presentation for calibration results involves expressing the result as an additive or multiplicative correction. This is the case for vacuum gauges and is illustrated with data using the models described in [1]. The examples and conclusions do, however, have much wider applicability.

This example demonstrates the effect of model assumptions concerning errors in the reference value. In addition it demonstrates how conformance probability can be affected by these assumptions. The example concludes by demonstrating how correlation can be handled for calibration corrections.

2 Introduction

Calibration measurements are reported in a wide variety of forms. A particularly popular form involves presenting measurement error as a calibration correction.

Often a limit or tolerance is defined for this correction and a conformity test is required. To make such a conformity decision requires knowledge of the measurement uncertainty associated with the correction.

Measurement uncertainty plays a crucial role, both here and in the decision processes found in most activities concerned with product or process conformity assessment. Without some account for measurement uncertainty the risk associated with a decision is *undefinable*.

The evaluation of measurement uncertainty and conformance probability are illustrated here for the calibration of a vacuum gauge; however, the analysis and methods described have more general applicability.

The calibration of vacuum (pressure) gauges is achieved by using a reference standard to establish the calibration pressure value at the inlet port of the unit under calibration (UUC). Often [2] this reference pressure is measured directly by a reference gauge and is obtained from a corrected reading from that gauge. A model for the evaluation of measurement uncertainty in vacuum calibration is described in some detail in ISO 27893 [1], whose principles can readily be transferred to other measurement applications.

Using the same set of calibration data, this example considers an additive ('sum') model, in which the calibration pressure value can be used to determine a reading error Δp for the UUC; it considers a multiplicative 'quotient model' (applicable when the calibration value is used to determine a correction factor, sensitivity coefficient, accommodation coefficient or gauge constant), and a 'combined model'.

Three scenarios are considered, representing different practices, illustrating how these practices can affect the associated conformance probability.

In the first scenario, following best metrological practice, a reference pressure correction (that is, a known systematic bias, due for example to the calibration method, thermal transpiration, height correction, etc.) is applied, and its associated uncertainty is incorporated in the uncertainty evaluation.

In the second scenario, the reference pressure correction is not applied, and instead it is combined with the associated uncertainty to establish a larger 'correction uncertainty'. In metrological terms this way of working represents poor (albeit common) practice when it is adopted for a *known* bias, and can have significant consequences for conformity decisions [3]. In situations where the correction is *not* known, but is perhaps considered to be in a defined range then this approach is more justified [4].

In the final scenario, the correction and its associated uncertainty are simply neglected, representing what might be termed 'bad practice'.

In each case the output PDF is assumed to be normal and conformance probability is calculated using a standard normal cumulative distribution function.

Data for the analyses are taken from a calibration certificate IMT-LMT-80-2019, produced by the Laboratory of Pressure Metrology, Institute of Metals and Technology, Ljubljana, Slovenia. The UUC in this example is a capacitance diaphragm vacuum gauge with full scale range of 11 kPa.

The specification adopted for the UUC in our example requires that calibration errors should be no larger than 0.5 % of the reading.

Finally, a physically different 'sum model with correlation' is presented, demonstrating how correlation might be treated in that case.

3 Measurands

Adopting the nomenclature of [1] the measurand is defined for the various classes of model as one of the following:

Δp – pressure difference (sum model) having standard uncertainty $u(\Delta p)$,

f – correction error (quotient model) having standard uncertainty $u(f)$,

e – error of reading (combined model) having standard uncertainty $u(e)$.

Other useful nomenclature:

$p_{\mathrm{UUC}}, u(p_{\mathrm{UUC}})$ – pressure for unit under calibration and its associated standard uncertainty,

$p_{\mathrm{std}}, u(p_{\mathrm{std}})$ – reference standard pressure and its associated standard uncertainty,

$\Delta p_{\mathrm{m}}, u(\Delta p_{\mathrm{m}})$ – reference pressure correction term and its associated standard uncertainty.

4 Measurement model

The measurement models are of the explicit, univariate type [5]:

In such models, a single real output quantity Y is related to a number of input quantities $\boldsymbol{X} = (X_1, \ldots, X_N)$ by a functional relationship f in the form $Y = \boldsymbol{f}(\boldsymbol{X})$ as stated in the GUM [6]. The estimate of the output quantity is taken as $y = \boldsymbol{f}(\boldsymbol{x})$. The standard uncertainty $u(y)$ is associated with y is evaluated from

$$u^2(y) = \sum_{i=1}^{N} \sum_{j=1}^{N} c_i u(x_{\mathrm{i}}, x_j) c_j,$$

where c_i is the partial derivative $\partial f / \partial X_i$ evaluated at $\boldsymbol{X} = \boldsymbol{x}$ and is known as the ith sensitivity coefficient, $u(x_i)$ is the standard uncertainty associated with x_i, and $u(x_i, x_j)$ the covariance associated with x_i and x_j. For independent input quantities, we would obtain the better-known simplified expression

$$u^2(y) = \sum_{i=1}^{N} [c_i u(x_i)]^2 = \sum_{i=1}^{N} u_i^2(y),$$

where

$$u_i(y) = |c_i| u(x_i).$$

4.1 Sum model

In the sum model, the measurand Δp is defined as the difference between the reading of the unit under calibration (UUC) and the reference value, which is given by the pressure indication of the reference standard corrected by a (possible) reference pressure correction term Δp_{m}:

$$\Delta p = p_{\mathrm{UUC}} - (p_{\mathrm{std}} + \Delta p_{\mathrm{m}}). \tag{1}$$

4.2 Quotient model

The standard ISO 27893 [1] describes how a model can be established in general situations where the UUC output and the reference standard output are not necessarily given in the same units of measurement. For example the UUC output may be measured as a current, voltage or frequency that is relatable to pressure through some functional relationship.

$$r_{\mathrm{UUC}} = \frac{x_{\mathrm{UUC}}}{p_{\mathrm{std}}}.$$

In this example we are only concerned with a simple case in which $x_{\mathrm{UUC}} = p_{\mathrm{UUC}}$ and where $r_{\mathrm{UUC}} = 1/f$ defines a correction factor f as the measurand; hence

$$f = \frac{p_{\mathrm{std}} + \Delta p_{\mathrm{m}}}{p_{\mathrm{UUC}}}, \tag{2}$$

Note that equation (2) is not a pure quotient but is a simple example of a 'combined' model.

4.3 Combined model

In practice, realistic measurement models are seldom a pure sum or product of quantities and a combined model is required. For example, the measurand e, the relative error of reading, can be defined by

$$e = \frac{p_{\mathrm{UUC}} - (p_{\mathrm{std}} + \Delta p_{\mathrm{m}})}{p_{\mathrm{std}} + \Delta p_{\mathrm{m}}} = \frac{p_{\mathrm{UUC}}}{p_{\mathrm{std}} + \Delta p_{\mathrm{m}}} - 1. \tag{3}$$

4.4 Sum model with correlation

An example that demonstrates how to evaluate a sum model with correlation is presented in section 7.

5 Uncertainty propagation

The GUM's law of propagation of uncertainty [6, equation (10)] is applied to establish the standard uncertainty associated with an estimate of the measurand for each of the three measurement models.

5.1 Sum model

The standard uncertainty in the sum model is

$$u(\Delta p) = \left[u^2(p_{\mathrm{UUC}}) + u^2(p_{\mathrm{std}}) + u^2(\Delta p_{\mathrm{m}})\right]^{1/2}. \tag{4}$$

5.2 Quotient model

The standard uncertainty in the quotient model is

$$u(f) = \frac{p_{\mathrm{std}} + \Delta p_{\mathrm{m}}}{p_{\mathrm{UUC}}} \left[\frac{u^2(p_{\mathrm{UUC}})}{p^2_{\mathrm{UUC}}} + \frac{u^2(p_{\mathrm{std}})}{(p_{\mathrm{std}} + \Delta p_{\mathrm{m}})^2} + \frac{u^2(\Delta p_{\mathrm{m}})}{(p_{\mathrm{std}} + \Delta p_{\mathrm{m}})^2}\right]^{1/2}. \tag{5}$$

5.3 Combined model

The standard uncertainty in the combined model is

$$u(e) = \frac{p_{\mathrm{UUC}}}{p_{\mathrm{std}} + \Delta p_{\mathrm{m}}} \left[\frac{u^2(p_{\mathrm{UUC}})}{p_{\mathrm{UUC}}^2} + \frac{u^2(p_{\mathrm{std}})}{(p_{\mathrm{std}} + \Delta p_{\mathrm{m}})^2} + \frac{u^2(\Delta p_{\mathrm{m}})}{(p_{\mathrm{std}} + \Delta p_{\mathrm{m}})^2} \right]^{1/2}. \tag{6}$$

6 Measurand expanded uncertainty and conformance probability for three scenarios

In this section the measurand expanded uncertainty at the 95 % level of confidence ($k = 2$) and the conformance probability are evaluated for three different scenarios, described below.

For the purposes of these examples the reference pressure correction term Δp_{m} is taken to be 0.05 % of p_{std} and the associated expanded uncertainty ($k = 2$) is assumed to be 1 Pa for all pressure values.

The calibration data, calibration corrections and associated expanded measurement uncertainties are summarised in Table 1. This data applies to all scenarios and models.

Table 1: Calibration data used for all scenarios and models.

Point, n	p_{std}/Pa	$U(p_{\mathrm{std}})$/Pa	p_{UUC}/Pa	$U(p_{\mathrm{UUC}})$/Pa
1	10.89	0.050	10.7	0.23
2	17.02	0.090	17.0	0.23
3	26.03	0.13	25.4	0.23
4	40.28	0.20	39.5	0.23
5	63.10	0.32	63.0	0.23
6	96.89	0.48	97.1	0.23
7	161.3	1.0	161.2	0.23
8	256.4	1.5	255.9	0.23
9	403.3	2.4	403.7	0.23
10	647.5	3.9	647.8	0.23
11	978.7	5.9	980.4	0.23
12	1 610.6	5.0	1 613.2	0.23
13	2 505.4	5.0	2 509.2	0.23
14	4 075.9	0.50	4 079.5	0.23
15	6 278.6	0.70	6 282.7	0.23
16	9 069.2	1.0	9 072.9	0.23
17	10 932.8	1.1	10 937.1	0.23

6.1 Scenario 1

In the first scenario, following best metrological practice, the reference pressure correction Δp_{m} is applied to the measured reference pressure p_{std}.

The calibration corrections and expanded measurement uncertainty are summarised in Table 2 where the corrections Δp, f and e are evaluated using equations (1), (2) and (3) respectively and standard uncertainties are evaluated using the corresponding equations (4), (5) and (6).

Table 2: Calibration corrections for Scenario 1. All uncertainties are expanded ($k = 2$)

Point	Δp/Pa	$U(\Delta p)$/Pa	e	$U(e)$	f	$U(f)$
1	−0.20	1.0	−0.017 9	0.093	1.018 3	0.096
2	−0.03	1.0	−0.001 7	0.061	1.001 7	0.061
3	−0.64	1.0	−0.024 7	0.039	1.025 3	0.041
4	−0.80	1.1	−0.019 9	0.002 5	1.020 3	0.026
5	−0.13	1.1	−0.002 1	0.001 7	1.002 1	0.017
6	0.16	1.1	0.001 7	0.001 1	0.998 3	0.011
7	−0.18	1.4	−0.001 1	0.008 8	1.001 1	0.008 9
8	−0.63	1.8	−0.002 4	0.007 1	1.002 5	0.007 1
9	0.20	2.6	0.000 5	0.006 5	0.999 5	0.006 5
10	−0.02	4.0	0.000 0	0.006 2	1.000 0	0.006 2
11	1.21	6.0	0.001 2	0.006 1	0.998 8	0.006 1
12	1.79	5.1	0.001 1	0.003 2	0.998 9	0.003 2
13	2.55	5.1	0.001 0	0.002 0	0.999 0	0.002 0
14	1.56	1.1	0.000 4	0.000 28	0.999 6	0.000 28
15	0.96	1.2	0.000 2	0.000 20	0.999 8	0.000 20
16	−0.83	1.4	−0.000 1	0.000 16	1.000 1	0.000 16
17	−1.17	1.5	−0.000 1	0.000 14	1.000 1	0.000 14

Figures 1 and 2, respectively, depict the correction factor f and the pressure difference Δp as a function of UUC pressure indication.

The red broken lines on these and later figures represent the specification limits defined by the UUC owner. These often, but not necessarily, correspond to limits defined by the equipment manufacturer.

Conformance probability can be calculated regarding Δp, f and e. Since all three quantities are linearly related their PDFs describe the same physical situation and the same conformance probability will be established whichever is evaluated. We therefore arbitrarily choose f for the purpose of these examples.

To calculate the conformance probability we further assume that the probability density function (PDF) for f is normal:

$$p(x) = \frac{1}{\sigma(2\pi)^{1/2}} e^{-[(x-m)^2/(2\sigma^2)]},$$

where m is the mean and σ is the standard deviation.

To demonstrate evaluation of the conformance probability, consider a point 13 where say $m = f = 0.999$ and $\sigma = u(f) = 0.002$. The conformance probability is given by integrating the probability density function over the limits of interest, say $0.995 \leq x \leq 1.005$:

$$p_\mathrm{c} = \int_{0.995}^{1.005} \frac{1}{0.002\ (2\pi)^{1/2}}\ e^{-[(x-0.999)^2/(2\times 0.002^2)]}\,\mathrm{d}x$$

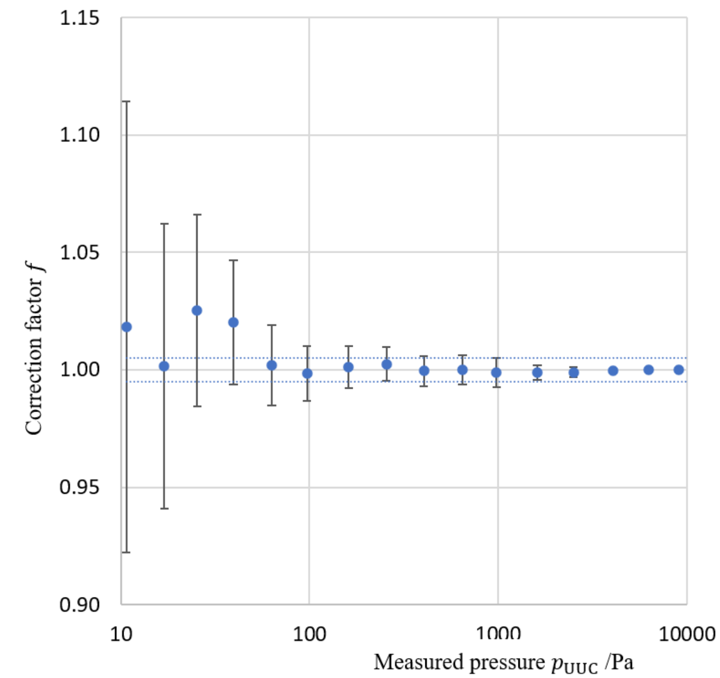

Figure 1: Correction factor as a function of UUC pressure indication (logarithmic scale), scenario 1

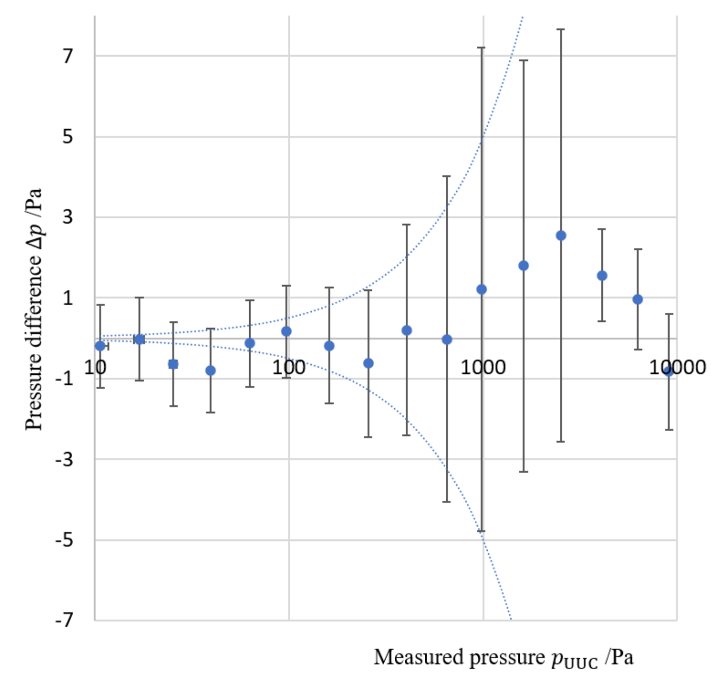

Figure 2: Pressure difference as a function of UUC pressure indication, scenario 1

This integral is not analytically solvable, but it can be expressed through tabulated functions such as Q,

ϕ or erfc. In this example the use of the Q function will be demonstrated where

$$Q(x) = \int_x^{+\infty} \frac{1}{2\pi^{1/2}} e^{-t^2/2}\,\mathrm{d}t.$$

Letting $t = (x - m)/\sigma$, we find new limits for the integral

$$(1.005 - 0.999)/0.002 = 3,$$
$$(0.995 - 0.999)/0.002 = -2,$$

and (6.1) becomes

$$\begin{aligned} p_\mathrm{c} &= \int_{-2}^{3} \frac{1}{2\pi^{1/2}} e^{-t^2/2}\,\mathrm{d}t \\ &= \int_{-2}^{+\infty} \frac{1}{2\pi^{1/2}} e^{-t^2/2}\mathrm{d}t - \int_{3}^{+\infty} \frac{1}{2\pi^{1/2}} e^{-t^2/2}\,\mathrm{d}t \\ &= Q(-2) - Q(3) \\ &= 0.977 - 0.001 \\ &= 97.6\,\%. \end{aligned}$$

Note: To implement the calculations in Microsoft Excel, the $Q(x)$ function can be evaluated using the relation $Q(x) = 0.5\,\mathrm{erfc}(x/\sqrt{2})$ and the Excel function ERFC.PRECISE().

The conformance probability for this scenario using the given data and the stated specification is summarised in Table 3 (restricted to data in the top two full decades for sake of clarity).

Table 3: Conformance probability – Scenario 1

Point	p_UUC/Pa	Δp/Pa	e	f	p_c/%
7	161.30	−0.18	−0.001 12	1.001 1	72.4
8	256.40	−0.63	−0.002 45	1.002 5	74.5
9	403.30	0.20	0.000 492	0.999 5	87.3
10	647.50	−0.02	−0.000 037	1.000 0	89.2
11	978.70	1.21	0.001 236	0.998 8	87.1
12	1 610.60	1.79	0.001 114	0.998 9	99.3
13	2 505.40	2.55	0.001 016	0.999 0	100.0
14	4 075.90	1.56	0.000 383	0.999 6	100.0
15	6 278.60	0.96	0.000 153	0.999 8	100.0
16	9 069.20	−0.83	−0.000 092	1.000 1	100.0

No *general* conclusions should be drawn from the values in this table. The results are however informative for this specific calibration where, as might be expected for a well behaved instrument of this type, conformance probability tends to be highest at higher pressures. The acceptability or otherwise of the result can be based on a straightforward consideration of conformance probability.

6.2 Scenario 2

In the second scenario no reference pressure correction Δp_{m} is applied, that is, the model equations are

$$f = \frac{p_{\mathrm{std}}}{p_{\mathrm{UUC}}},$$
$$\Delta p = p_{\mathrm{UUC}} - p_{\mathrm{std}},$$
$$e = \frac{p_{\mathrm{UUC}}}{p_{\mathrm{std}}} - 1.$$

This situation might (correctly) arise because little is known about Δp_{m}; hence the best estimate of its value is taken to be zero, albeit the uncertainty remains finite. It might also be the case that the value of the correction *is* known but following common (albeit poor) practice it is instead somehow combined with its uncertainty to establish a bigger uncertainty estimate which, it is argued, accounts for the failure to apply the correction. (See [3] and [7] for explanations of why this is considered to be poor practice.) This latter situation is evaluated in this scenario.

In this case, the standard uncertainty $u(\Delta p_{\mathrm{m}})$ associated with the unused correction term is calculated using

$$u(\Delta p_{\mathrm{m}}) = \left[0.5^2 + \frac{(0.05\,\%\, p_{\mathrm{std}})^2}{3}\right]^{1/2}$$

in which $\Delta p_{\mathrm{m}} = 0.05\,\%\, p_{\mathrm{std}}$ is assumed to be the semi-range of a rectangular distribution.

The quantities Δp, f and e and their uncertainties are again recalculated and given in Table 4

Table 4: Data for scenario 2. All uncertainties are expanded ($k = 2$)

Point	Δp/Pa	$U(\Delta p)$/Pa	e	$U(e)$	f	$U(f)$
1	−0.19	1.0	−0.017 4	0.093	1.017 8	0.096
2	−0.02	1.0	−0.001 2	0.061	1.001 2	0.061
3	−0.63	1.0	−0.024 2	0.039	1.024 8	0.041
4	−0.78	1.1	−0.019 4	0.026	1.019 7	0.027
5	−0.10	1.1	−0.001 6	0.017	1.001 6	0.017
6	0.21	1.1	0.002 2	0.012	0.997 8	0.012
7	−0.10	1.4	−0.000 6	0.008 9	1.000 6	0.008 9
8	−0.50	1.8	−0.002 0	0.007 1	1.002 0	0.007 1
9	0.40	2.6	0.001 0	0.006 5	0.999 0	0.006 5
10	0.30	4.0	0.000 5	0.006 2	0.999 5	0.006 2
11	1.70	6.0	0.001 7	0.006 1	0.998 3	0.006 1
12	2.60	5.1	0.001 6	0.003 2	0.998 4	0.003 2
13	3.80	5.3	0.001 5	0.002 1	0.998 5	0.002 1
14	3.60	2.6	0.000 9	0.000 64	0.999 1	0.000 64
15	4.10	3.8	0.000 7	0.000 61	0.999 3	0.000 61
16	3.70	5.4	0.000 4	0.000 60	0.999 6	0.000 60
17	4.30	6.5	0.000 4	0.000 59	0.999 6	0.000 59

Figures 3 and 4, respectively, depict the correction factor f and the pressure difference Δp as a function of UUC pressure indication.

As would be expected from equations (1), (2) and (3) the uncertainty for scenario 2 is always larger than the corresponding uncertainty for scenario 1. In this case the difference is not large, but this is dictated by the data and may be more (or less) significant for other data.

The conformance probability for this scenario using the given data and the stated specification is summarised in Table 5 (restricted to data in the top two full decades for sake of clarity).

Table 5: Conformance probability – scenario 2

Point	p_{UUC}/Pa	Δp/Pa	e	f	p_{c}/%
7	161.30	−0.10	−0.000 6	1.000 6	73.4
8	256.40	−0.50	−0.002 0	1.002 0	77.8
9	403.30	0.40	0.001 0	0.999 0	85.9
10	647.50	0.30	0.000 5	0.999 5	88.6
11	978.70	1.70	0.001 7	0.998 3	84.2
12	1 610.60	2.60	0.001 6	0.998 4	98.3
13	2 505.40	3.80	0.001 5	0.998 5	100.0
14	4 075.90	3.60	0.000 9	0.999 1	100.0
15	6 278.60	4.10	0.000 7	0.999 3	100.0
16	9 069.20	3.70	0.000 4	0.999 6	100.0

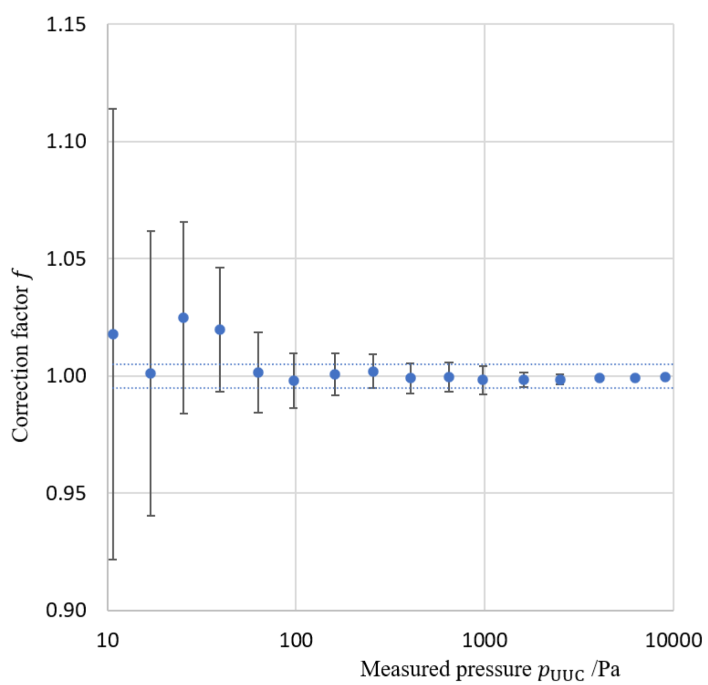

Figure 3: Correction factor as a function of UUC pressure indication (logarithmic scale), scenario 2

The behaviour for conformance probability is generally more complex than is the case for measurement uncertainty. In this example the conformance probability for scenario 2 is generally lower than for

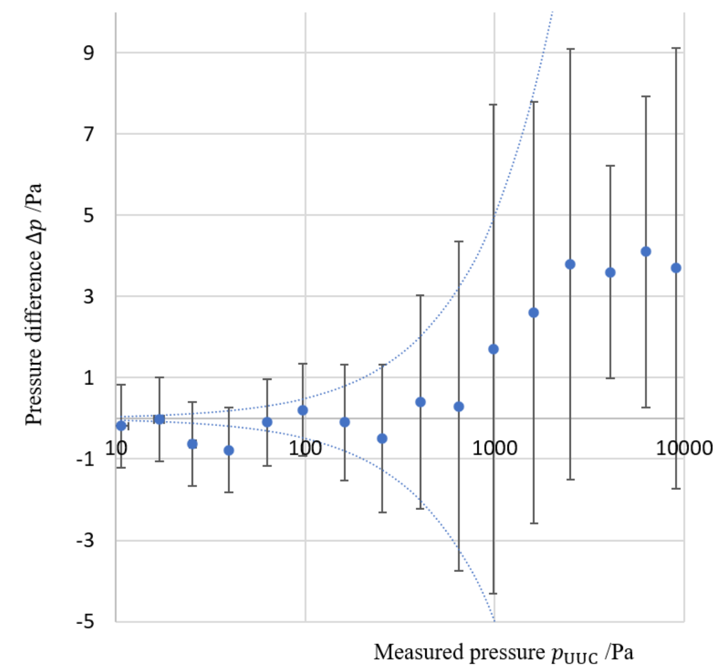

Figure 4: Pressure difference as a function of UUC pressure indication, scenario 2

scenario 1, which can be explained by one or both of the measurand (based upon uncorrected reference pressure) being closer to a tolerance limit, and the uncertainty being larger, and hence the PDF being 'wider' and extending more beyond the tolerance limits. It is however quite possible for a value to be closer to the centre of the tolerance interval when no correction is applied, which may have a larger influence on the calculation of conformance probability than the increase in uncertainty, as is seen for example for our point no. 8.

6.3 Scenario 3

The third scenario is obtained when the correction term and associated uncertainty are excluded, which is equivalent to setting $\Delta p_{\mathrm{m}} = u(\Delta p_{\mathrm{m}}) = 0$ in our three models, yielding the results shown in Table 6 in which the conformance probability p_{c} is again given for all points in the second and third decade.

Figures 5 and 6, respectively, depict the correction factor f and the pressure difference Δp as a function of UUC pressure indication.

The conformance probability for this scenario using the given data and the stated specification is summarised in Table 7 (restricted to data in the top two full decades for sake of clarity).

As was the case for scenario 2, the behaviour for conformance probability is complex. In *this* case the conformance probability is generally higher when compared to scenario 1. The difference is entirely due to the nature of the data and the unused reference pressure correction and will vary depending upon the data and corrections in question. Conformity decisions based upon these (scenario 3) conformance probabilities would therefore likely be unreliable.

Table 6: Data for scenario 3. All uncertainties are expanded ($k = 2$)

Point	Δp/Pa	$U(\Delta p)$/Pa	e	$U(e)$	f	$U(f)$
1	−0.19	0.24	−0.017 4	0.022	1.017 8	0.022
2	−0.02	0.25	−0.001 2	0.015	1.001 2	0.015
3	−0.63	0.26	−0.024 2	0.010	1.024 8	0.011
4	−0.78	0.30	−0.019 4	0.007 5	1.019 7	0.007 8
5	−0.10	0.39	−0.001 6	0.006 2	1.001 6	0.006 3
6	0.21	0.53	0.002 2	0.005 5	0.997 8	0.005 5
7	−0.10	1.0	−0.000 6	0.006 4	1.000 6	0.006 4
8	−0.50	1.5	−0.002 0	0.005 9	1.002 0	0.005 9
9	0.40	2.4	0.001 0	0.006 0	0.999 0	0.006 0
10	0.30	3.9	0.000 5	0.006 0	0.999 5	0.006 0
11	1.70	5.9	0.001 7	0.006 0	0.998 3	0.006 0
12	2.60	5.0	0.001 6	0.003 1	0.998 4	0.003 1
13	3.80	5.0	0.001 5	0.002 0	0.998 5	0.002 0
14	3.60	0.55	0.000 9	0.000 14	0.999 1	0.000 13
15	4.10	0.74	0.000 7	0.000 12	0.999 3	0.000 12
16	3.70	1.0	0.000 4	0.000 11	0.999 6	0.000 11
17	4.30	1.1	0.000 4	0.000 10	0.999 6	0.000 10

Table 7: Conformance probability – scenario 3

Point	p_{UUC}/Pa	Δp/Pa	e	f	p_{c}/%
7	161.30	−0.10	−0.000 6	1.000 6	87.7
8	256.40	−0.50	−0.002 0	1.002 0	83.8
9	403.30	0.40	0.001 0	0.999 0	88.8
10	647.50	0.30	0.000 5	0.999 5	89.9
11	978.70	1.70	0.001 7	0.998 3	84.8
12	1 610.60	2.60	0.001 6	0.998 4	98.6
13	2 505.40	3.80	0.001 5	0.998 5	100.0
14	4 075.90	3.60	0.000 9	0.999 1	100.0
15	6 278.60	4.10	0.000 7	0.999 3	100.0
16	9 069.20	3.70	0.000 4	0.999 6	100.0

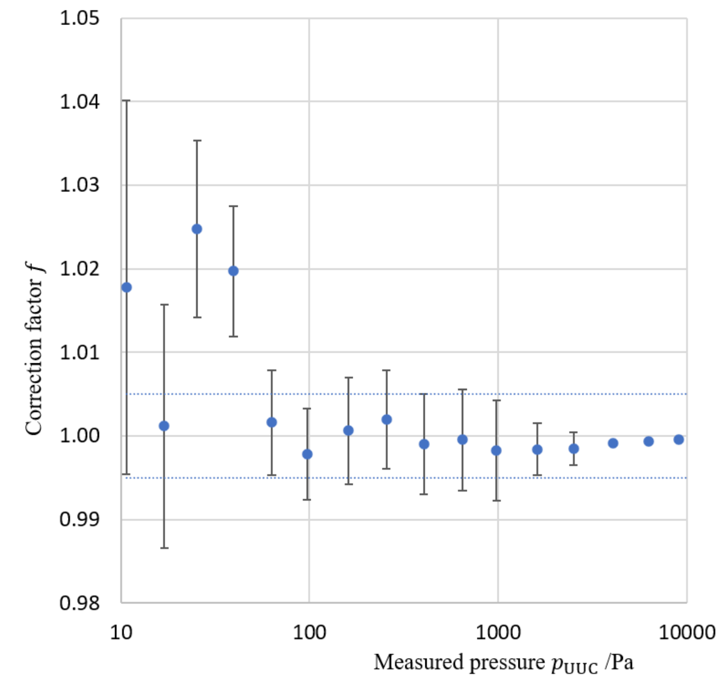

Figure 5: Correction factor as a function of UUC pressure indication (logarithmic scale), scenario 3

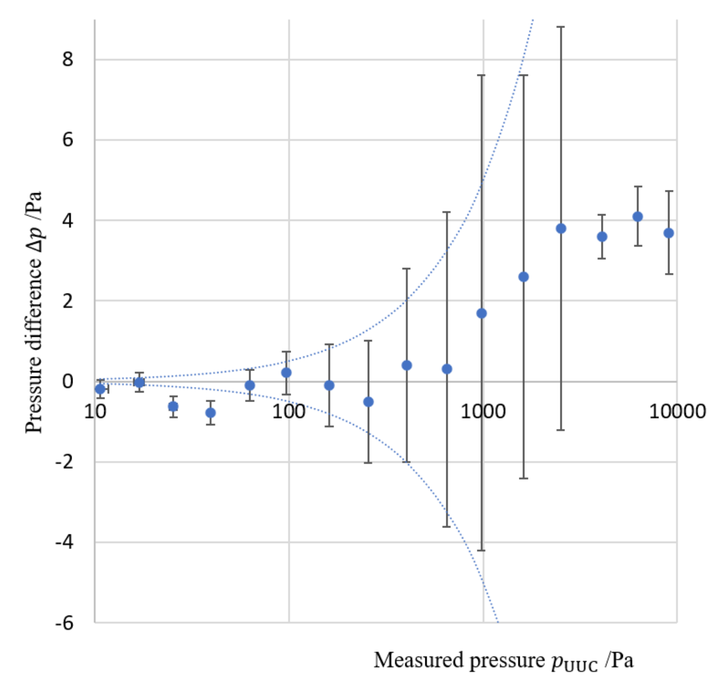

Figure 6: Pressure difference as a function of UUC pressure indication, scenario 3

7 Measurement model: sum model with correlation

Suppose that by some mechanism not already accounted for, both p_{std} and p_{UUC} are both dependent upon another common quantity, say for example a potential systematic error ΔT in measuring gas temperature due to placement of temperature probes. This situation might be modelled as:

$$\begin{aligned} p_{\mathrm{std}} &= p'_{\mathrm{std}}(1+\alpha_{\mathrm{std}}\,\Delta T), \\ p_{\mathrm{UUC}} &= p'_{\mathrm{UUC}}(1+\alpha_{\mathrm{UUC}}\,\Delta T), \end{aligned} \tag{7}$$

where, for example, p'_{std} and p'_{UUC} now represent the observed values and α_{std} and α_{UUC} are corresponding temperature coefficients. Suppose also that a reliable estimate of ΔT is not available; hence the 'best' estimate is taken to be $\Delta T = 0\,\mathrm{K}$ with an associated standard uncertainty $u(\Delta T)$.

For this example we will also assume that the reference pressure correction Δp_{m} is fully independent and is not affected by the possible temperature error; hence the measurands are to be calculated using equations (1), (2) and (3).

The standard uncertainties associated with the quantities p_{std} and p_{UUC} in equation (7) are therefore

$$\begin{aligned} u^2(p_{\mathrm{std}}) &= u^2(p'_{\mathrm{std}})(1+\alpha_{\mathrm{std}}\,\Delta T)^2 + u^2(\alpha_{\mathrm{std}})(p'_{\mathrm{std}}\Delta T)^2 + u^2(\Delta T)(p'_{\mathrm{std}}\alpha_{\mathrm{std}})^2, \\ u^2(p_{\mathrm{UUC}}) &= u^2(p'_{\mathrm{UUC}})(1+\alpha_{\mathrm{UUC}}\,\Delta T)^2 + u^2(\alpha_{\mathrm{UUC}})(p'_{\mathrm{UUC}}\Delta T)^2 + u^2(\Delta T)(p'_{\mathrm{UUC}}\alpha_{\mathrm{UUC}})^2. \end{aligned}$$

To illustrate the situation, let $u(\Delta T) = 0.57\,\mathrm{K}$, $\alpha_{\mathrm{std}} = \alpha_{\mathrm{UUC}} = 1/300\,\mathrm{K}^{-1}$ and $u(\alpha_{\mathrm{std}}) = u(\alpha_{\mathrm{UUC}}) = 1/3\,000\,\mathrm{K}^{-1}$; hence, for example, at calibration point 8, $p'_{\mathrm{std}} = 256.4\,\mathrm{Pa}$ and $p'_{\mathrm{UUC}} = 255.9\,\mathrm{Pa}$. We find from equation (1) that $\Delta p = -0.63\,\mathrm{Pa}$ and

$$\begin{aligned} u(p_{\mathrm{std}}) &= 0.90\,\mathrm{Pa}, \\ u(p_{\mathrm{UUC}}) &= 0.51\,\mathrm{Pa}. \end{aligned}$$

Combining these *without* taking account of the correlation between p_{std} and p_{UUC} gives, using equation (4), a standard uncertainty of $u(\Delta p) = 1.14\,\mathrm{Pa}$.

In this case the estimate is about 25 % larger than is obtained by taking account of the correlation, as can be achieved by following the process described in matrix form in clause 6.2 of GUM Supplement 2 [5] (demonstrated below) and in subscripted summation form in the GUM [6, Annex F.1.2.3].

Equations (1), (2) and (3) are real univariate measurement functions of the form $Y = f(\boldsymbol{X})$ where in the case of equation (1) we have

$$\begin{aligned} Y &= \Delta p, \\ \boldsymbol{X} &= (p_{\mathrm{std}}, p_{\mathrm{UUC}}, \Delta p_{\mathrm{m}})^{\top}. \end{aligned}$$

Applying GUM-LPU, the variance in this case is given by

$$u^2(\Delta p) = \boldsymbol{V}_{\Delta p} = \boldsymbol{C}_{\Delta p}^{\top}\boldsymbol{V}_{in}\boldsymbol{C}_{\Delta p}, \tag{8}$$

where $\boldsymbol{C}_{\Delta p}$ is an array containing sensitivity coefficients, and $\boldsymbol{V}_{in}$ is the corresponding covariance matrix for the input quantities:

$$\boldsymbol{C}_{\Delta p} = \begin{bmatrix} \frac{\partial \Delta p}{\partial p_{\mathrm{std}}} \\ \frac{\partial \Delta p}{\partial p_{\mathrm{UUC}}} \\ \frac{\partial \Delta p}{\partial \Delta p_{\mathrm{m}}} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix},$$

$$
\boldsymbol{V}_{in} = \begin{bmatrix} u^2 p_{\mathrm{std}}) & u(p_{\mathrm{std}}, p_{\mathrm{UUC}}) & u(p_{\mathrm{std}}, \Delta p_{\mathrm{m}}) \\ u(p_{\mathrm{UUC}}, p_{\mathrm{std}}) & u^2(p_{\mathrm{UUC}}) & u(p_{\mathrm{UUC}}, \Delta p_{\mathrm{m}}) \\ u(\Delta p_{\mathrm{m}}, p_{\mathrm{std}}) & u(\Delta p_{\mathrm{m}}, p_{\mathrm{UUC}}) & u^2(\Delta p_{\mathrm{m}}) \end{bmatrix}. \tag{9}
$$

Equation (8) is the matrix representation of the equation obtained by applying GUM equation (13).

The covariance matrix (9) is obtained from

$$
\boldsymbol{V}_{in} = \boldsymbol{C_X} \boldsymbol{V_X} \boldsymbol{C_X}^{\top},
$$

where, in this example we have

$$
\boldsymbol{C}_{\boldsymbol{X}}^{\top} = \begin{bmatrix} \frac{\partial p_{\mathrm{std}}}{\partial p'_{\mathrm{std}}} & \frac{\partial p_{\mathrm{UUC}}}{\partial p'_{\mathrm{std}}} & \frac{\partial \Delta p_{\mathrm{m}}}{\partial p'_{\mathrm{std}}} \\ \frac{\partial p_{\mathrm{std}}}{\partial \alpha_{\mathrm{std}}} & \frac{\partial p_{\mathrm{UUC}}}{\partial \alpha_{\mathrm{std}}} & \frac{\partial \Delta p_{\mathrm{m}}}{\partial \alpha_{\mathrm{std}}} \\ \frac{\partial p_{\mathrm{std}}}{\partial p'_{\mathrm{UUC}}} & \frac{\partial p_{\mathrm{UUC}}}{\partial p'_{\mathrm{UUC}}} & \frac{\partial \Delta p_{\mathrm{m}}}{\partial p'_{\mathrm{UUC}}} \\ \frac{\partial p_{\mathrm{std}}}{\partial \alpha_{\mathrm{UUC}}} & \frac{\partial p_{\mathrm{UUC}}}{\partial \alpha_{\mathrm{UUC}}} & \frac{\partial \Delta p_{\mathrm{m}}}{\partial \alpha_{\mathrm{UUC}}} \\ \frac{\partial p_{\mathrm{std}}}{\partial \Delta T} & \frac{\partial p_{\mathrm{UUC}}}{\partial \Delta T} & \frac{\partial \Delta p_{\mathrm{m}}}{\partial \Delta T} \\ \frac{\partial p_{\mathrm{std}}}{\partial \Delta p_{\mathrm{m}}} & \frac{\partial p_{\mathrm{UUC}}}{\partial \Delta p_{\mathrm{m}}} & \frac{\partial \Delta p_{\mathrm{m}}}{\partial \Delta p_{\mathrm{m}}} \end{bmatrix} = \begin{bmatrix} 1 + \alpha_{\mathrm{std}} \, \Delta T & 0 & 0 \\ p'_{\mathrm{std}} \, \Delta T & 0 & 0 \\ 0 & 1 + \alpha_{\mathrm{UUC}} \, \Delta T & 0 \\ 0 & p'_{\mathrm{UUC}} \, \Delta T & 0 \\ p'_{\mathrm{std}} \, \alpha_{\mathrm{std}} & p'_{\mathrm{UUC}} \, \alpha_{\mathrm{UUC}} & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

and

$$
\boldsymbol{V_X} = \begin{bmatrix} u^2(p'_{\mathrm{std}}) & 0 & 0 & 0 & 0 & 0 \\ 0 & u^2(\alpha_{\mathrm{std}}) & 0 & 0 & 0 & 0 \\ 0 & 0 & u^2(p'_{\mathrm{UUC}}) & 0 & 0 & 0 \\ 0 & 0 & 0 & u^2(\alpha_{\mathrm{UUC}}) & 0 & 0 \\ 0 & 0 & 0 & 0 & u^2(\Delta T) & 0 \\ 0 & 0 & 0 & 0 & 0 & u^2(\Delta p_{\mathrm{m}}) \end{bmatrix}.
$$

Alternatively, the elements of (9) can be calculated in terms of subscripted summations in line with annex F.1.2 of the GUM [6] using GUM equations (F.1) and (F.2). As we have assumed that the reference pressure correction Δp_{m} is fully independent and is not affected by the possible temperature error, off-diagonal covariances involving Δp_{m} are zero. The remaining covariance $u(p_{\mathrm{std}}, p_{\mathrm{UUC}}) = u(p_{\mathrm{UUC}}, p_{\mathrm{std}})$ is non-zero since both p_{std} and p_{UUC} depend upon ΔT. Its value is given by

$$
u(p_{\mathrm{std}}, p_{\mathrm{UUC}}) = (p'_{\mathrm{std}} \, \alpha_{\mathrm{std}})(p'_{\mathrm{UUC}} \, \alpha_{\mathrm{UUC}}) u^2(\Delta T).
$$

Evaluating the uncertainty by either equation (8) or GUM equation (13) yields a value of $u(\Delta p) = 0.91\,\mathrm{Pa}$ and a conformance probability of $p_{\mathrm{c}} = 0.75$ rather than a probability of $p_{\mathrm{c}} = 0.67$ that is obtained when correlation is neglected. The impact of such a difference in conformance probability is dependent upon the particular application of interest, but clearly any such differences have the potential to ~~significantly~~ affect conformity decisions.

8 Interpretation of results

This example has demonstrated the evaluation of measurement uncertainty and conformance probability for various related calibration models under several common scenarios. It has also demonstrated how correlation between quantities (arising from dependency on a common effect) is treated within the GUM uncertainty framework. In each case the consequences for conformity decisions are complex, depending as they do upon the particular data, model, and assumptions. No general rule can easily be established. For some data points the conformance probability decreases when simplifying assumptions are made, in others it increases. Caution is therefore needed unless best practice (scenario 1) is followed to avoid the risk of making poor decisions.

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