Quasi-phasematching in a poled Josephson traveling-wave parametric amplifier with three-wave mixing
(ADDITIONAL INFORMATION)

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1. Wave equation

The wave equation describing a ladder-type LC transmission line with non-centrosymmetric (β) and Kerr (γ) nonlinearities stemming from the current-dependent Josephson inductance is

$$\frac{\partial^2\phi}{\partial x^2} - \omega_0^{-2}\frac{\partial^2\phi}{\partial t^2} + \omega_J^{-2}\frac{\partial^4\phi}{\partial x^2\partial t^2} - \beta\frac{\partial}{\partial x}\left[\left(\frac{\partial\phi}{\partial x}\right)^2\right] - \gamma\frac{\partial}{\partial x}\left[\left(\frac{\partial\phi}{\partial x}\right)^3\right] = 0, \tag{1}$$

where $\phi(x,t) = \Phi(x,t)/\varphi_0$ is the ac phase (dimensionless magnetic flux) on the node x (a continuous coordinate that is normalized on the cell size d) and $\varphi_0 = \Phi_0/2\pi$ is normalized flux quantum. The frequencies $\omega_J = 1/\sqrt{LC_J}$ and $\omega_0 = 1/\sqrt{LC_0}$ (here L is inductance, C_J is capacitance of the nonlinear element, and C_0 is the ground capacitance) are the plasma frequency and the transmission-line cutoff frequency, respectively. Equation (1) was first derived in the case of pure Kerr nonlinearity ($\beta = 0$ and $\gamma \neq 0$) for the Josephson traveling-wave parametric amplifier (JTWPA) with four-wave mixing (4WM) by Yaakobi et al. [1]. This derivation is easily generalized on the case of JTWPA exploiting two types of nonlinearities ($\beta \neq 0$ and $\gamma \neq 0$). This JTWPA can operate in either 4WM or three-wave mixing (3WM) regime. It can be implemented using flux-controlled non-hysteretic rf-SQUIDs [2] or superconducting nonlinear asymmetric inductive elements (SNAILs) [3, 4].

In the case of transmission line with pure non-centrosymmetric nonlinearity described by a slowly varying function $\beta(x)$, the wave equation takes the form

$$\frac{\partial^2\phi}{\partial x^2} - \omega_0^{-2}\frac{\partial^2\phi}{\partial t^2} + \omega_J^{-2}\frac{\partial^4\phi}{\partial x^2\partial t^2} - \beta(x)\frac{\partial}{\partial x}\left[\left(\frac{\partial\phi}{\partial x}\right)^2\right] = 0, \tag{2}$$

where $|d\beta(x)/dx| \ll k|\beta|$ and $k \approx (\omega/\omega_0)/\sqrt{1-\omega^2/\omega_J^2}$ is dimensionless wave vector of the mode $\omega \ll \omega_0$. If $\beta(x)$ is a periodic function changing its sign every half a period, equation (2) can describe quasi-phasematching in a JTWPA with 3WM [5]. Suppression of Kerr nonlinearity ($\gamma = 0$) in this transmission line is achieved by optimal setting of the constant magnetic flux in the rf-SQUIDs ($\phi_{\rm dc} = \pm\pi/2$) [2].

2. Coupled mode equations

The solution of the wave equation Eq. (2) is found using the coupled-mode equations (CMEs) method [6, 7] in the form

$$\phi(x,t) = \frac{1}{2}\sum_{j=\{s,i,p,+,-,2p\}}\left[A_j(x)e^{i(k_jx-\omega_jt)} + A_j^*(x)e^{-i(k_jx-\omega_jt)}\right], \tag{3}$$

where $k_{s,i,p,+,-,2p}$ are wave vectors and $A_{s,i,p,+,-,2p}(x)$ are slowly varying amplitudes of the signal (ω_s), idler ($\omega_i = \omega_p - \omega_s$), pump ($\omega_p$), combination frequencies ($\omega_\pm = \omega_p + \omega_{s,i}$), and the second harmonic of the pump ($\omega_{2p} = 2\omega_p$), respectively. In our case, all these frequencies are below ω_J . Moreover, they are assumed to be much less than the cutoff frequency ω_0. The slowly varying amplitudes of the waves obey the relations

$$\left|\frac{\partial^2 A_j}{\partial x^2}\right| \ll k_j\left|\frac{\partial A_j}{\partial x}\right| \ll k_j^2\,|A_j|\,, \quad \text{where } j = \{s,i,p,+,-,2p\}. \tag{4}$$

The wave vectors of the waves traveling in the ladder-type transmission line with chromatic dispersion due to finite plasma frequency ω_J are [2, 8]

$$k_j = 2\arcsin\left(\frac{\omega_j/2\omega_0}{\sqrt{1-\omega_j^2/\omega_J^2}}\right) \approx \frac{\omega_j/\omega_0}{\sqrt{1-\omega_j^2/\omega_J^2}}, \quad \text{where } j = \{s,i,p,+,-,2p\}. \tag{5}$$

These wave vectors are normalized on the reverse physical size d^{-1} of the elementary cell.

Substituting solution (3) into equation (2) and using relations (4) and (5) we obtain the set of CMEs for six basic modes (according to classification of Dixon et al. [9], the so-called CME-2 set)

$$\frac{dA_s}{dx} = -\frac{\beta(x)}{2}\left(\frac{k_ik_p(k_p-k_i)}{k_s}A_i^*A_pe^{i\Delta k_0x} + \frac{k_+k_p(k_+-k_p)}{k_s}A_p^*A_+e^{i\Delta k_1x} + \frac{k_{2p}k_-(k_{2p}-k_-)}{k_s}A_-^*A_{2p}e^{i\Delta k_4x}\right), \quad (6)$$

$$\frac{dA_i}{dx} = -\frac{\beta(x)}{2}\left(\frac{k_sk_p(k_p-k_s)}{k_i}A_s^*A_pe^{i\Delta k_0x} + \frac{k_-k_p(k_--k_p)}{k_i}A_p^*A_-e^{i\Delta k_2x} + \frac{k_{2p}k_+(k_{2p}-k_+)}{k_i}A_+^*A_{2p}e^{i\Delta k_3x}\right), \quad (7)$$

$$\begin{aligned}\frac{dA_p}{dx} &= \frac{\beta(x)}{2}\left(\frac{k_sk_i(k_s+k_i)}{k_p}A_sA_ie^{-i\Delta k_0x} - \frac{k_+k_s(k_+-k_s)}{k_p}A_+A_s^*e^{i\Delta k_1x} - \frac{k_-k_i(k_--k_i)}{k_p}A_-A_i^*e^{i\Delta k_2x}\right)\\ &- \frac{\beta(x)}{2}\frac{k_{2p}k_p(k_{2p}-k_p)}{k_p}A_{2p}A_p^*e^{i\Delta k_5x}, \end{aligned} \quad (8)$$

$$\frac{dA_+}{dx} = \frac{\beta(x)}{2}\left(\frac{k_sk_p(k_s+k_p)}{k_+}A_sA_pe^{-i\Delta k_1x} - \frac{k_{2p}k_i(k_{2p}-k_i)}{k_+}A_i^*A_{2p}e^{i\Delta k_3x}\right), \quad (9)$$

$$\frac{dA_-}{dx} = \frac{\beta(x)}{2}\left(\frac{k_ik_p(k_i+k_p)}{k_-}A_iA_pe^{-i\Delta k_2x} - \frac{k_{2p}k_s(k_{2p}-k_s)}{k_-}A_s^*A_{2p}e^{i\Delta k_4x}\right), \quad (10)$$

$$\frac{dA_{2p}}{dx} = \frac{\beta(x)}{2}\left(\frac{k_p^3}{k_{2p}}A_p^2e^{-i\Delta k_5x} + \frac{k_-k_s(k_-+k_s)}{k_{2p}}A_-A_se^{-i\Delta k_4x} + \frac{k_+k_i(k_++k_i)}{k_{2p}}A_+A_ie^{-i\Delta k_3x}\right), \quad (11)$$

where

$$\begin{aligned} \Delta k_0 &= k_p - k_s - k_i, & (12)\\ \Delta k_1 &= k_+ - k_p - k_s, & (13)\\ \Delta k_2 &= k_- - k_p - k_i, & (14)\\ \Delta k_3 &= k_{2p} - k_+ - k_i, & (15)\\ \Delta k_4 &= k_{2p} - k_- - k_s, & (16)\\ &\text{and} & \\ \Delta k_5 &= k_{2p} - 2*k_p & (17)\end{aligned}$$

describe the phase mismatches for corresponding mixing processes.

The set of differential equations (6)-(11) with initial conditions

$$A_p(0) = A_{p0},\ A_s(0) = A_{s0},\ \text{ and }\ A_i(0) = A_+(0) = A_-(0) = A_{2p}(0) = 0 \quad (18)$$

is numerically solved on the interval $0 \leq x \leq N$ for various shapes of periodic function $\beta(x)$ (with the period equal to $m = 2\pi/k_m$) by means of the standard Runge-Kutta method.

3. Poling profiles

Implementation of a slowly varying nonlinear coefficient $\beta(x) = \eta(x)|\beta_0|$ (β_0 is the maximum value of $\beta(x)$) in the rf-SQUID-based transmission line is, in principle, possible by means of modulation of the critical current value $I_c(x)$ (i.e. modulation of the Josephson junction area) yielding modulation of the SQUID screening parameter $\beta_L(x) = LI_c(x)/\varphi_0$ and thus of coefficient $\beta \propto \beta_L$. However, manufacturing identical rf-SQUIDs with fixed β_L is obviously preferred in the circuits consisting of a large number of these elements ($N \sim 1000$). Because the modulation period m (~ 500) greatly exceeds the typical dimensionless wavelength λ ($= 2\pi/k \sim 50 \gg 1$), implementation of slow spatial variation of nonlinear coefficient β is possible using variable density of inverted identical rf-SQUIDs per unit length, or the so-called pulse-width modulation (PWM). In this case, the continuous signal is converted into digital code by means of a 1-bit quantizer with the levels $+1$ and -1.

To convert a continuous nonlinearity profile $\eta(x)$ into binary code $\xi(x)$, where $\xi(x) = \xi_n$ is a piecewise function taking the values ± 1, we use PWM of delta-sigma type (see, for example, Ref. 10). In this method, the quantization error,

$$\epsilon(x) = \xi(x) - \eta(x), \quad (19)$$

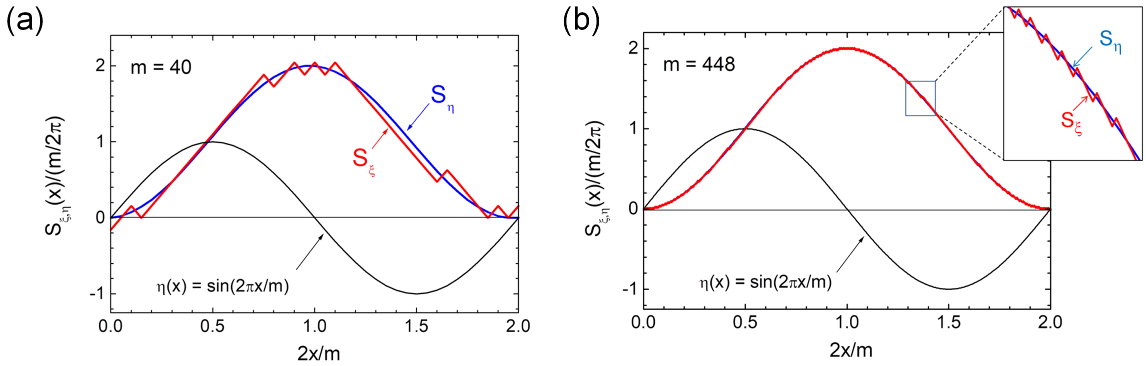

FIG. 1: Functions $S_\eta(x)$ and $S_\eta(x)$ plotted after conversion of the sine profile (21) (black curve) into binary code $\xi(x)$ (not shown) for two different modulation periods (sampling rates): (a) $m = 40$ and (b) $m = 448$.

is integrated, and when the integral

$$\int_0^x \epsilon(x')dx' = \int_0^x \xi(x')dx' - \int_0^x \eta(x')\,dx' \equiv S_\xi(x) - S_\eta(x) \tag{20}$$

exceeds the limits of ± 1, the digital output $\xi(x)$ changes its state.

We used this quantization method for conversion of two possible continuous QPM profiles into binary codes, i.e., a sinusoidal profile

$$\eta(x) = \sin(2\pi x/m) \tag{21}$$

and a tapered meander, whose positive half-period ($0 \leq x \leq m/2$) is described by the formula

$$\eta(x) = -a \ln[e^{-2x/abm} + e^{-1/a} + e^{-(m-2x)/abm}] \tag{22}$$

with fixed parameter values, $a = 0.15$ (describes rounding of the corners) and $b = 0.2$ (describes the slopes). As an illustration of our method, the results of conversion of the sine waveform (21) using two different sampling rates are shown in Fig. 1.

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[1] O. Yaakobi, L. Friedland, C. Macklin, and I. Siddiqi, Phys. Rev. B **87**, 144301 (2013).
[2] A. B. Zorin, Phys. Rev. Applied **6**, 034006 (2016).
[3] N. E. Frattini, U. Vool, S. Shankar, A. Narla, K. M. Sliwa, and M. H. Devoret, Appl. Phys. Lett. **110**, 222603 (2017).
[4] A. B. Zorin, M. Khabipov, J. Dietel, and R. Dolata, 2017 16th International Superconductive Electronics Conference (ISEC), Naples, 2017, pp. 1-3, doi: 10.1109/ISEC.2017.8314196.
[5] R. W. Boyd, *Nonlinear optics* (Academic Press, London, 2008).
[6] A. Yariv, *Quantum electronics* (Wiley, New York, 1967).
[7] G. P. Agrawal, *Nonlinear fiber optics* (Academic press, San Diego, California, 2007).
[8] F. Martín, *Artificial transmission lines for rf and microwave applications*, Wiley Series in Microwave and Optical Engineering (Wiley, 2015).
[9] T. Dixon, J. W. Dunstan, G. B. Long, J. M. Williams, P. J. Meeson, and C. D. Shelly, Phys. Rev. Applied **14**, 034058 (2020).
[10] K. C. Pohlmann, *Principles of Digital Audio* (McGraw-Hill, New York, 2000).