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Existence and nonexistence results for p-Laplacian Kirchhoff equation

Kamel Tahri[1*], Safia Benmansour[2] , Khadra Tahri[3]
[1]High School of Management and Abou Bekr Belkaid University,
Tlemcen, Algeria
[2]Ecole Supérieure de Management de Tlemcen and Laboratoire du controle des EDP
Université Djillali Liabes de Sidi Bellabes, Algeria
[3]Abou Bekr Belkaid University,
Tlemcen, Algeria

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Abstract: This paper is dedicated to investigating the following elliptic equation with Kirchhof type involving the p-Laplacian operator:

$$\begin{cases} -\left(a\int_\Omega |\nabla u|^p dx + b\right)\Delta_p u = \mu\, |u|^{p^*-2} u + \lambda\, |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where is Ω a bounded domain in $\mathbb{R}^n\,(n > 3)$, $\Delta_p u := -div\left(|\nabla u|^{p-2}\nabla u\right)$, for $1 < p < n$ denotes the $p-$Laplacian operator, and $\lambda > 0$ is a real parameter and $a, b \geq 0 : a + b > 0$ are parameters.

Using variational methods and critical points theory, we prove that the above problem has a positive solution and multiplicity result in certain cases. Our result is regarded as general which extend the results of related literatures. At the end, we give some real applications.

Key words: Existence, Nonexistence, $p-$Laplacian, Kirchhof equation, Sobolev critical exponent

1. Introduction

In this paper, we consider the $p-$Kirchhoff equation:

$$\begin{cases} -\left(a\int_\Omega |\nabla u|^p dx + b\right)\Delta_p u = \mu\, |u|^{p^*-2} u + \lambda\, |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where is a bounded domain in $\mathbb{R}^n\,(n > 3)$, $\Delta_p u := -div\left(|\nabla u|^{p-2}\nabla u\right)$, for $1 < p < n$ denotes the $p-$Laplacian operator, and $\lambda > 0$ is a real parameter and $a, b \geq 0 : a + b > 0$ are parameters and $p^* := \frac{np}{n-p}$ is the Sobolev critical exponent in the embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$.

In 1883, Kirchhoff introduced a model given in [12], as a generalization of the well known d'Alembert's wave equation,

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

for free vibrations of elastic strings. The parameters in above equation have a physical significant meanings as follow: L is the length of the string, E is the area of the cross section, ρ is the Young modulus of the

*Correspondence: Tahri_kamel@yahoo.fr

material, ρ_0 is the mass density, and is the initial tension. Some interesting studies for Kirchhoff type problem in a bounded domain of $\mathbb{R}^n$ by variational methods can be found in [1], [2],[3], [9], [10], [26],and [27]. More recently, there are several papers having studied (1) with $p=2$, see for example [20] and [21] .

The problem (1) is called nonlocal because of the appearance of the term $\int_\Omega |\nabla u|^p dx$ and which causes some mathematical difficulties. In the same time, this term evokes the problem more interesting and fruitfull in physics. The problem (1) is related to the stationary analogue of the evolution of the equation of Kirchhoff type,

$$u_{tt} - \left(a \int_\Omega |\nabla u|^p dx + b\right) \Delta_p u = f(x,u) \quad \text{in } \Omega, \tag{3}$$

which is used to describe some phenomenon appeared in physics and engineering, due to it is regarded as a good approximation for describing vibrations of beams or plates, see [4] and [5].

We cite here, some brillant works in a particular case when $p=2$ of the problem (1), Mao and Zhang in [18] have considered othe following Kirchhoff type equation:

$$\begin{cases} -\left(a \int_\Omega |\nabla u|^2 dx + b\right) \Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

by using minimax methods, they obtained the existence and multiplicity of sign changing solutions.

In this paper, our main results state as follows

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ $(n>3)$, assume $a,b\geq 0 : a+b>0$ and $p^*=4$ then the following assertions are true:*

(i). Assume that $a>0, b>0, 0<\mu<aK(n,p)^2$ and $0<\lambda<b\lambda_1$, then the equation (1) has no positive nontrivial solution.

(ii). Assume that $a\geq 0, b>0, 0<\mu<aK(n,p)^2$ and $\lambda>b\lambda_1$, then the equation (1) has a positive nontrivial solution.

(iii). Assume that $a\geq 0, b>0, 0<\mu<aK(n,p)^2$, then for any $k\in\mathbb{Z}^+$, there exists $\Lambda_k>0$ such that the equation (1) has at least k pairs nontrivial solutions for $\lambda>\Lambda_k>0$.

Our paper is organized as follows: in the first section we give a literature review of our problem and a significant applition in physics, in the second one, we establish a the used technique and remind some mathematical materials to prove our main results. The third section deals with the proof of useful and fruitful lemmas for the equation (1). In the forth section is devoted to prove the main result in three part nonexistence, existence and multiplicity results.The last section is devoted to evoke what does mean the equation (1) is a certain dimensions and under some assumptions.

2. Variational Setting

Let $W_0^{1,p}(\Omega)$ be the usual Sobolev space, equipped with the norm

$$|u|_{W_0^{1,p}(\Omega)} := \left(\int_\Omega |\nabla u|^p \, dx\right)^{\frac{1}{p}}.$$

For $1\leq p<\infty$, denote by $L^p(\Omega)$ the Lebesgue spaces and its usual norm is given by

$$||u||_p := \left(\int_\Omega |u|^p dx\right)^{1/p}.$$

We denote by $K(n,p)$ the best constant in the Sobolev's continuous embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$:

$$K(n,p) := \inf_{u \in W_0^{1,p}(\Omega)-\{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\left(\int_\Omega |u|^{p^*} \, dx\right)^{\frac{2}{p^*}}} = \inf_{u \in A} \int_\Omega |\nabla u|^p \, dx$$

where,

$$A := \left\{ W_0^{1,p}(\Omega) - \{0\} : \int_\Omega |u|^{p^*} \, dx = 1 \right\},$$

such that $p^* := \frac{np}{n-p}$ is the critical exponent of the Sobolev's continuous embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$.

Thanks to the pertinent computations and serious works of Aubin in [28], that the extremal functionss for the Sobolev's continuous embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ are given by the radial functions:

$$u_\epsilon(x) := \left(\epsilon^2 + r^{\frac{p}{p-1}}\right)^{-\frac{n-p}{p}} \eta(r),$$

where η is a C^∞ radial function and $\epsilon > 0$.

Consider λ_1 the first eigenvalue of the eigenvalue problem,

$$\begin{cases} -\Delta_p u = \lambda_1 |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

According to the work developed by Peral in [23], has shown that

$$\lambda_1 := \inf_{u \in W_0^{1,p}(\Omega)-\{0\}} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx}. \tag{5}$$

The first eigenvalue λ_1 is isolated and simple and its corresponding first eigenfunction named ϕ_1 is positive.

3. Some useful Lemmas

We define the energy functional corresponding to the problem (1) by

$$J_{\lambda,\mu}(u) = \frac{a}{2p}\left(\int_\Omega |\nabla u|^p dx\right)^2 + \frac{b}{p}\int_\Omega |\nabla u|^p dx - \frac{\mu}{4}\int_\Omega |u|^4 \, dx - \frac{\lambda}{p}\int_\Omega |u|^p \, dx, \tag{6}$$

for all $u \in W_0^{1,p}(\Omega)$.

It is clear that $J_{\lambda,\mu}$ is well defined and of C^1 on $W_0^{1,p}(\Omega)$ and its critical points are weak solutions of problem (1) i.e. they satisfy: for all $\varphi \in W_0^{1,p}(\Omega)$:

$$\left(a\int_\Omega |\nabla u|^p dx + b\right)\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \int_\Omega \left(\mu |u|^2 + \lambda |u|^{p-2}\right) u\varphi dx. \tag{7}$$

Lemma 3.1. *The following assertions are true:*

(i). If $a, b > 0$, and $0 < \mu < aK(n,p)^2$, then the functional $J_{\lambda,\mu}$ is bounded from below and is coercive in $W_0^{1,p}(\Omega)$.

(ii). If $a, b > 0$, $0 < \mu < aK(n,p)^2$ and $0 < \lambda < \frac{b}{2}p\lambda_1$, then the functional $J_{\lambda,\mu}$ is positive.

Proof. (i). Let $u \in W_0^{1,p}(\Omega)$, the enery functional writes:

$$J_{\lambda,\mu}(u) = \frac{a}{2p} |u|^{2p}_{W_0^{1,p}(\Omega)} + \frac{b}{p} |u|^{p}_{W_0^{1,p}(\Omega)} - \frac{\mu}{4} \|u\|_4^4 - \frac{\lambda}{p} \|u\|_p^p .$$

Using Sobolev's embedding and the fact that $0 < \mu < aK(n,p)^2$, we have

$$J_{\lambda,\mu}(u) \geq \frac{a}{2p} |u|^{2p}_{W_0^{1,p}(\Omega)} \left(a - \mu K(n,p)^{-2}\right) + \frac{b}{2} |u|^{p}_{W_0^{1,p}(\Omega)} - \frac{\lambda}{p} \|u\|_p^p .$$

Then, we deduce that the energy functional $J_{\lambda,\mu}$ is coercive in $W_0^{1,p}(\Omega)$.

To show that $J_{\lambda,\mu}$ is bounded below in $W_0^{1,p}(\Omega)$, we divided thestudy into two cases:
(1). If $|u|_{W_0^{1,p}(\Omega)} < 1$, then we have

$$J_{\lambda,\mu}(u) \geq -\frac{\lambda}{p}.$$

(2). If $|u|_{W_0^{1,p}(\Omega)} > 1$, then there exists a $\Lambda \in \mathbb{R}$:

$$J_{\lambda,\mu}(u) \geq \Lambda,$$

with

$$\Lambda := \frac{1}{4}\left(a - \mu K(n,p)^{-2}\right) + \frac{b}{2} - \frac{\lambda}{p}.$$

This implies that J_{λ} is bounded from below in $W_0^{1,p}(\Omega)$.

(ii). Combining the two conditions $0 < \mu < aK(n,p)^2$, $0 < \lambda < \frac{b}{2}p\lambda_1$ and using the definition of λ_1 we have:

$$J_{\lambda,\mu}(u) \geq \frac{1}{4} |u|^{2p}_{W_0^{1,p}(\Omega)} \left(a - \mu K(n,p)^{-2}\right) + |u|^{p}_{W_0^{1,p}(\Omega)} \left(\frac{b}{2} - \frac{\lambda\lambda_1^{-1}}{p}\right).$$

Then, for all $u \in W_0^{1,p}(\Omega)$:

$$J_{\lambda,\mu}(u) > 0.$$

The proof of lemma is complete. □

Definition 3.1. A sequence $(u_m)_m$ is said to be a Palais-Smale sequence at level c ((P-S)$_c$ in short) for $J_{\lambda,\mu}$ in $W_0^{1,p}(\Omega)$ if

$$\begin{cases} J_{\lambda,\mu}(u_m) = c + o_m(1), \\ J'_{\lambda,\mu}(u_m) = o_m(1). \end{cases}$$

Definition 3.2. We say that $J_{\lambda,\mu}$ verifies the Palais-Smale condition at level c if any (P-S)$_c$ sequence for $J_{\lambda,\mu}$ has a convergent subsequence in $W_0^{1,p}(\Omega)$.

Lemma 3.2. *Assume that $a \geq 0, b > 0, 0 < \mu < aK(n,p)^2$ and $0 < \lambda < b\lambda_1$ are fulfilled, then the functional $J_{\lambda,\mu}$ satisfies the (P-S)$_c$.*

Proof. Let $(u_m)_m \subset W_0^{1,p}(\Omega)$ be a (P-S)$_c$ sequence for any $c \in \mathbb{R}$ then

$$J_{\lambda,\mu}(u_m) = c + o_m(1), \text{ and } J'_{\lambda,\mu}(u_m) = o_m(1).$$

Suppose that $(u_m)_m$ is not bounded, so

$$|u_m|_{W_0^{1,p}(\Omega)} \to +\infty \text{ as } m \to +\infty.$$

Since $J_{\lambda,\mu}$ is coercive, we have

$$J_{\lambda,\mu}(u_m) \to +\infty \text{ as } |u_m|_{W_0^{1,p}(\Omega)} \to +\infty.$$

This contradicts the fact that $(u_m)_m$ is a minimizing sequence, so $(u_m)_m$ is bounded in $W_0^{1,p}(\Omega)$, and therefore, up to a subsequence, there exists $u_{\lambda,\mu} \in W_0^{1,p}(\Omega)$ such that
(i). $u_m \rightharpoonup u_{\lambda,\mu}$ weakly in $W_0^{1,p}(\Omega)$,
(ii). $u_m \longrightarrow u_{\lambda,\mu}$ strongly in $L^s(\Omega)$ for all $1 \leq s < 4$,
(iii). $u_m \longrightarrow u_{\lambda,\mu}$ a.e in Ω.

Put

$$v_m = u_m - u_{\lambda,\mu},$$

and

$$\ell = \lim_{m \to +\infty} |v_m|_{W_0^{1,p}(\Omega)}.$$

We need to prove that $\ell = 0$.

The Brezis-Lieb Lemma, allows us to write

$$|u_m|^{2p}_{W_0^{1,p}(\Omega)} = |v_m|^{2p}_{W_0^{1,p}(\Omega)} + |u_{\lambda,\mu}|^{2p}_{W_0^{1,p}(\Omega)} + 2|v_m|^{p}_{W_0^{1,p}(\Omega)} |u_{\lambda,\mu}|^{p}_{W_0^{1,p}(\Omega)} + o_m(1), \tag{8}$$

$$|u_m|^{p}_{W_0^{1,p}(\Omega)} = |v_m|^{p}_{W_0^{1,p}(\Omega)} + |u_{\lambda,\mu}|^{p}_{W_0^{1,p}(\Omega)} + o_m(1), \tag{9}$$

$$\int_\Omega |u_m|^4 dx = \int_\Omega |v_m|^4 dx + \int_\Omega |u_{\lambda,\mu}|^4 dx + o_m(1), \tag{10}$$

and

$$\int_\Omega |u_m|^p dx = \int_\Omega |v_m|^p dx + \int_\Omega |u_{\lambda,\mu}|^p dx + o_m(1). \tag{11}$$

Since $J_{\lambda,\mu}(u_m) = c + o_n(1)$, we get

$$c = \lim_{m \to +\infty} J_{\lambda,\mu}(u_m)$$

$$c = \lim_{m \to +\infty} \left[\frac{a}{2p} |u_m|^{2p}_{W_0^{1,p}(\Omega)} + \frac{b}{p} |u_m|^{p}_{W_0^{1,p}(\Omega)} - \frac{\mu}{4} \|u_m\|_4^4 - \frac{\lambda}{p} \|u_m\|_p^p \right].$$

$$= \frac{a}{2p} \lim_{m \to +\infty} \left[|u_m|^{2p}_{W_0^{1,p}(\Omega)} \right] + \frac{b}{p} \lim_{m \to +\infty} \left[|u_m|^{p}_{W_0^{1,p}(\Omega)} \right]$$

$$-\frac{\mu}{4} \lim_{m \to +\infty} \left[\|u_m\|_4^4 \right] + \frac{\lambda}{p} \lim_{m \to +\infty} \left[\|u_m\|_p^p \right].$$

Then,

$$c=\frac{a}{2p}\lim_{m\to+\infty}\left[|v_m|^{2p}_{W_0^{1,p}(\Omega)}+|u_{\lambda,\mu}|^{2p}_{W_0^{1,p}(\Omega)}+2\,|v_m|^{p}_{W_0^{1,p}(\Omega)}\,|u_{\lambda,\mu}|^{p}_{W_0^{1,p}(\Omega)}\right]$$

$$+\frac{b}{p}\lim_{m\to+\infty}\left[|v_m|^{p}_{W_0^{1,p}(\Omega)}+|u_{\lambda,\mu}|^{p}_{W_0^{1,p}(\Omega)}\right]$$

$$-\frac{\mu}{4}\lim_{m\to+\infty}\left[\|v_m\|_4^4+\|u_{\lambda,\mu}\|_4^4\right]+\frac{\lambda}{p}\lim_{m\to+\infty}\left[\|v_m\|_p^p+\|u_{\lambda,\mu}\|_p^p\right].$$

We have,

$$c=\frac{a}{2p}\lim_{m\to+\infty}\left[|v_m|^{2p}_{W_0^{1,p}(\Omega)}+2\,|v_m|^{p}_{W_0^{1,p}(\Omega)}\,|u_{\lambda,\mu}|^{p}_{W_0^{1,p}(\Omega)}\right]$$

$$+\lim_{m\to+\infty}\left[\frac{b}{p}\,|v_m|^{p}_{W_0^{1,p}(\Omega)}-\frac{\mu}{4}\,\|v_m\|_4^4+\frac{\lambda}{p}\,\|v_m\|_p^p\right]$$

$$+\frac{a}{2p}\,|u_{\lambda,\mu}|^{2p}_{W_0^{1,p}(\Omega)}+\frac{b}{p}\,|u_{\lambda,\mu}|^{p}_{W_0^{1,p}(\Omega)}-\frac{\mu}{4}\,\|u_{\lambda,\mu}\|_4^4+\frac{\lambda}{p}\,\|u_{\lambda,\mu}\|_p^p$$

Then,

$$c=\lim_{m\to+\infty}J_{\lambda,\mu}\left(u_m\right)=J_{\lambda,\mu}\left(u_{\lambda,\mu}\right)+$$

$$\frac{a}{2p}\lim_{m\to+\infty}\left[|v_m|^{2p}_{W_0^{1,p}(\Omega)}+2\,|v_m|^{p}_{W_0^{1,p}(\Omega)}\,|u_{\lambda,\mu}|^{p}_{W_0^{1,p}(\Omega)}\right]$$

$$+\lim_{m\to+\infty}\left[\frac{b}{p}\,|v_m|^{p}_{W_0^{1,p}(\Omega)}-\frac{\mu}{4}\,\|v_m\|_4^4+\frac{\lambda}{p}\,\|v_m\|_p^p\right].$$

By the fact that $J'_{\lambda,\mu}\left(u_m\right)=o_m(1)$ so we test by u_m

$$\left\langle J'_{\lambda,\mu}\left(u_m\right),u_m\right\rangle=o_m(1).$$

Then we have

$$\lim_{m\to+\infty}\left\langle J'_{\lambda,\mu}\left(u_m\right),u_m\right\rangle=0$$

$$=a\,|u_m|^{2p}_{W_0^{1,p}(\Omega)}+b\,|u_m|^{p}_{W_0^{1,p}(\Omega)}-\mu\,\|u_m\|_4^4-\lambda\,\|u_m\|_p^p, \tag{12}$$

and also testing by $u_{\lambda,\mu}$, we obtain

$$\left\langle J'_{\lambda,\mu}\left(u_m\right),u_{\lambda,\mu}\right\rangle=o_m(1).$$

Then,

$$a\,|u_{\lambda,\mu}|^{2p}_{W_0^{1,p}(\Omega)}+b\,|u_{\lambda,\mu}|^{p}_{W_0^{1,p}(\Omega)}+a\,|u_{\lambda,\mu}|^{p}_{W_0^{1,p}(\Omega)}\,\ell^p=$$

$$\mu\,\|u_{\lambda,\mu}\|_4^4+\lambda\,\|u_{\lambda,\mu}\|_p^p+o(1).$$

From the above equalities, one has

$$a\,|v_m|^{2p}_{W_0^{1,p}(\Omega)}+a\,|v_m|^{p}_{W_0^{1,p}(\Omega)}\,|u_{\lambda,\mu}|^{p}_{W_0^{1,p}(\Omega)}+b\,|v_m|^{p}_{W_0^{1,p}(\Omega)}-\mu\,\|u_{\lambda,\mu}\|_4^4=o_m(1).$$

Combining (8), (9), (10), (11) and (12), we obtain

$$a\left[|v_m|^{2p}_{W_0^{1,p}(\Omega)}+|u_{\lambda,\mu}|^{2p}_{W_0^{1,p}(\Omega)}+2\,|v_m|^{p}_{W_0^{1,p}(\Omega)}\,|u_{\lambda,\mu}|^{p}_{W_0^{1,p}(\Omega)}\right]$$

$$+b\left[|v_m|^{p}_{W_0^{1,p}(\Omega)}+|u_{\lambda,\mu}|^{p}_{W_0^{1,p}(\Omega)}\right]$$

$$-\lambda\left[\|v_m\|_p^p+\|u_{\lambda,\mu}\|_p^p\right]-\mu\left[\|v_m\|_4^4+\|u_{\lambda,\mu}\|_4^4\right]=0.$$

Then, we get

$$\left[a\,|u_{\lambda,\mu}|^{2p}_{W_0^{1,p}(\Omega)}+b\,|u_{\lambda,\mu}|^{p}_{W_0^{1,p}(\Omega)}-\mu\,\|u_{\lambda,\mu}\|_4^4-\lambda\,\|u_{\lambda,\mu}\|_p^p\right]+$$

$$\left[a\,|v_m|^{2p}_{W_0^{1,p}(\Omega)}+2a\,|v_m|^{p}_{W_0^{1,p}(\Omega)}\,|u_{\lambda,\mu}|^{p}_{W_0^{1,p}(\Omega)}+b\,|v_m|^{p}_{W_0^{1,p}(\Omega)}\right]-$$

$$\left[\mu\,\|v_m\|_4^4+\mu\,\|u_{\lambda,\mu}\|_4^4\right]=0. \tag{13}$$

Consequently, combining (12), and (13), one gets

$$\left(a\ell^p+a\,|u_{\lambda,\mu}|^{p}_{W_0^{1,p}(\Omega)}+b\right)\ell^p=\mu\,\|v_m\|_4^4+\lambda\,\|v_m\|_p^p+o\left(1\right).$$

From the definitions of $K\left(n,p\right)$ and λ_1, we have:

$$\int_\Omega |v_m|^4\,dx\leq K\left(n,p\right)^{-2}\left(\int_\Omega |\nabla v_m|^p\,dx\right)^2,$$

and

$$\int_\Omega |v_m|^p dx\leq \lambda_1^{-1}\int_\Omega |\nabla v_m|^p dx.$$

It follows that

$$\left(a\ell^p+a\,|u_{\lambda,\mu}|^{p}_{W_0^{1,p}(\Omega)}+b\right)\ell^p\leq \mu K\left(n,p\right)^{-2}\ell^{2p}+\lambda\lambda_1^{-1}\ell^p+o\left(1\right).$$

Then,

$$\left(a-\mu K\left(n,p\right)^{-2}\right)\times\ell^{2p}+\left(b-\lambda\lambda_1^{-1}\right)\times\ell^p\leq 0.$$

Consequently, since $a\geq 0,\ b>0,\ 0<\mu<K\left(n,p\right)^2 a$ and $0<\lambda<b\lambda_1,$ we conclude that $\ell=0.$
Therefore $u_n\longrightarrow u$ strongly in $W_o^{1,p}\left(\Omega\right).$ □

4. Proof of the generic theorem:

4.1. Nonexistence Result, proof of the part (i):

To prove the nonexistence result, we argue by contradiction supposing that the problem (1) admits a nontrivial solution $u_{\lambda,\mu}\in W_0^{1,p}(\Omega)$ as in (7), we have

$$0=a\,|u_{\lambda,\mu}|^{2p}_{W_0^{1,p}(\Omega)}+b\,|u_{\lambda,\mu}|^{p}_{W_0^{1,p}(\Omega)}-\mu\,\|u_{\lambda,\mu}\|_4^4-\lambda\,\|u_{\lambda,\mu}\|_p^p.$$

Using the Sobolev's inequality and the definitions of $K(n,p)$ and λ_1, one gets

$$0 \geq a\left|u_{\lambda,\mu}\right|^{2p}_{W_0^{1,p}(\Omega)} + b\left|u_{\lambda,\mu}\right|^{p}_{W_0^{1,p}(\Omega)} - \mu K(n,p)^{-2}\left|u_{\lambda,\mu}\right|^{2p}_{W_0^{1,p}(\Omega)} - \lambda\lambda_1^{-1}\left|u_{\lambda,\mu}\right|^{p}_{W_0^{1,p}(\Omega)}.$$

Then we obtain

$$0 \geq \left(a - \mu K(n,p)^{-2}\right)\left|u_{\lambda,\mu}\right|^{2p}_{W_0^{1,p}(\Omega)} + \left(b - \lambda\lambda_1^{-1}\right)\left|u_{\lambda,\mu}\right|^{p}_{W_0^{1,p}(\Omega)},$$

and since $a+b>0$, $0<\mu<K(n,p)^2 a$ and $0<\lambda<b\lambda_1$, we conclude that $u_{\lambda,\mu}$ identically equals to zero which leads to a contradiction. Then the proof is achieved.

4.2. Existence Result, proof of the part (ii):

The following lemma, whose proof is given in [7] plays a crucial role to show the multiplicity result.

Lemma 4.1. *Let $J \in C^1(X,\mathbb{R})$ be a function satisfying the Palais-Smale condition in the real Banach space X and α, β two reals such that $\alpha < \beta$. Assume that either $J(0) < \alpha$ or $J(0) > \beta$. If further we assume that the following assumptions hold:*

(i). There exist a linear m-dimensional subspace E and $\rho > 0$ such that

$$\sup_{t \in E \cap \partial B_\rho(0)} J(t) \leq \beta,$$

where

$$\partial B_\rho(0) = \{t \in X : \|t\|_X = \rho\}.$$

(ii). There exist F a linear j-dimensional subspace and $\rho > 0$ such that

$$\sup_{t \in F^\perp} J(t) > \alpha,$$

where

$$F^\perp \oplus F = X.$$

(iii).If $m > j$.

Then J has $(m-j)$ pairs of distinct critical points.

Since the functional $J_{\lambda,\mu}$ is bounded from below and is coercive in $W_0^{1,p}(\Omega)$, we obtain that

$$m := \inf_{u_{\lambda,\mu} \in W_0^{1,p}(\Omega)} J_{\lambda,\mu}(u_{\lambda,\mu}),$$

is well defined.

Put

$$w = t\phi_1(x),$$

where t is a real parameter and ϕ_1 is the first eigenfunction corresponding to the first eigenvalue λ_1. Then, we have

$$J_{\lambda,\mu}(w) = \frac{a}{2p}\left|\phi_1\right|^{2p}_{W_0^{1,p}(\Omega)} t^{2p} + \frac{b}{p}\left|\phi_1\right|^{p}_{W_0^{1,p}(\Omega)} t^p - \frac{\mu}{4}\|\phi_1\|_4^4 t^4 - \frac{\lambda}{p}\|\phi_1\|_p^p t^p.$$

Then, we get

$$J_{\lambda,\mu}(w) = \frac{a}{2p}\left|\phi_1\right|^{2p}_{W_0^{1,p}(\Omega)} t^{2p} - \frac{\mu}{4}\left\|\phi_1\right\|_4^4 t^4 + \left(b - \lambda_1^{-1}\lambda\right)\frac{t^p}{p}\left|\phi_1\right|^p_{W_0^{1,p}(\Omega)}.$$

Since $\lambda > b\lambda_1$ and $p < 2p < 4$, we have $J_{\lambda,\mu}(w) < 0$ for $t > 0$ small. Then, we obtain

$$m < 0.$$

Since the energy functional $J_{\lambda,\mu}$ verifies a global Palais-Smale condition in Lemma (4.1), a direct application of Theorem 4.4 in [19] certifies the existence of $v_{\lambda,\mu} \in W_0^{1,p}(\Omega)$ such that

$$J_{\lambda,\mu}(v_{\lambda,\mu}) = m < 0.$$

Note that $J_{\lambda,\mu}(|v_{\lambda,\mu}|) = J_{\lambda,\mu}(v_{\lambda,\mu})$, then by the strong maximum principle, we infer that problem (1) has a positive nontrivial solution whose energy is negative.

4.3. Multiplicity Result, proof of the part (iii):

In order to prove the multiplicity result, we rely on Lemma (4.1) so, we have to verify its conditions. First, let $X = W_0^{1,p}(\Omega)$ and suppose that $b > 0$, $\lambda > b\lambda_1$. For all integer $k \geq 1$, consider $(\lambda_k)_k$ the k first eigenvalues of the eigenvalue problem,

$$\begin{cases} -\Delta_p \phi_k = \lambda_k \left|\phi_k\right|^{p-2}\phi_k & \text{in } \Omega, \\ \phi_k = 0 & \text{on } \partial\Omega. \end{cases}$$

and $(\phi_k)_k$ the corresponding eigenfunctions.

Define the $k-$dimensional set

$$E = span\left\{\phi_1, \phi_2, ..., \phi_k\right\}.$$

For $u \in E$, there exists a constant $C > 0$ such that

$$\lambda_k \left\|u\right\|_p^p \leq C\left\|\nabla u\right\|_p^p.$$

It follows that

$$J_{\lambda,\mu}(u) = \frac{a}{2p}\left|u\right|^{2p}_{W_0^{1,p}(\Omega)} + \frac{b}{p}\left|u\right|^p_{W_0^{1,p}(\Omega)} - \frac{\mu}{4}\left\|u\right\|_4^4 - \frac{\lambda}{p}\left\|u\right\|_p^p.$$

Then

$$J_{\lambda,\mu}(u) \leq \frac{a}{2p}\left|u\right|^{2p}_{W_0^{1,p}(\Omega)} + \frac{b}{p}\left|u\right|^p_{W_0^{1,p}(\Omega)} - \frac{\lambda}{p}\left\|u\right\|_p^p.$$

since $\lambda > b\lambda_1$, there exists $\rho > 0$ such that

$$\sup_{t \in E \cap \partial B_\rho(0)} J_{\lambda,\mu}(t) \leq b < 0 = J_{\lambda,\mu}(0).$$

As $0 < \mu < \alpha S^2$, the coercivity of $J_{\lambda,\mu}$ on $X = W_0^{1,p}(\Omega)$, implies that

$$\inf_{u \in W_0^{1,p}(\Omega))} J_{\lambda,\mu}(u) > -\infty.$$

So if we choose $F = \varnothing$ then $F^{\perp} = X$ and so

$$\sup_{t \in E \cap \partial B_{\rho}(0)} J_{\lambda,\mu}(t) > -\infty.$$

As already noticed $J_{\lambda,\mu}$ satisfies a global Palais Smale condition. Consequently, all conditions of Lemma (4.1) are obviously verfied when $\beta = 0$ for all $\lambda > 0$, this ensures the existence of $\Lambda_k > 0$, such that the problem (1) has at least k pairs of nontrivial distinct solutions for all $\lambda > \Lambda_k$.

5. Application

When $p = 2, a = 0$ and $b = 1$, the problem (1) reduces to the elliptic eigenvalue problem,

$$\begin{cases} -\Delta u = |u|^{2^*-2} u + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In 1983 and when $n = 3$, Brézis and Nirenberg in [6] have obtained a positive solution when $n = 3$ if only if $\lambda \in \left(\frac{\lambda_1}{4}, \lambda_1\right)$, where λ_1 is the first eigenvalue of the following eigenvalue problem,

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and when $n \geq 4$, they obtained a positive solution for $\lambda \in (0, \lambda_1)$, and no positive solution if $\lambda \notin (0, \lambda_1)$ and Ω is starshaped.

In 2018, the authors of the paper [14], have considered the following problem,

$$\begin{cases} -\left(a \int_{\Omega} |\nabla u|^2 dx + b\right) \Delta u = \mu u^3 + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is bounded domain of $\mathbb{R}^4$.
They obtained the following result,

(i) For $0 < \mu < bS^2$ and $\lambda \in (0, a\lambda_1)$, the problem has no nontrivial solution.

(ii) For $0 < \mu < bS^2$ and $\lambda > a\lambda_1$, the problem has a unique positive solution.

(iii) For $0 < \mu < bS^2$ and $\lambda > \Lambda_k > 0$, the problem has k-pairs of distinct solutions,

where

$$S := \inf_{u \in W_0^{1,2}(\Omega) - \{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\left(\int_{\Omega} |u|^4 dx\right)^{\frac{1}{2}}} = \inf_{u \in B} \int_{\Omega} |\nabla u|^2 dx,$$

with

$$B = \left\{ W_0^{1,2}(\Omega) - \{0\} : \int_{\Omega} |u|^4 dx = 1 \right\}.$$

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