Noname manuscript No.
(will be inserted by the editor)

The Riemann Hypothesis

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the date of receipt and acceptance should be inserted later

Abstract The Nicolas' theorem states that the Riemann Hypothesis is true if and only if the inequality $\prod_{q|p\#} \frac{q}{q-1} > e^{\gamma} \times \log\log p\#$ is true, where $p\#$ is a primorial for $p > 2$ and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. This means that the Nicolas' inequality is true when

$$\liminf_{p\to\infty}\left(\frac{\prod_{q|p\#} \frac{q}{q-1}}{e^{\gamma} \times \log\log p\#}\right) > 1.$$

This is because of the Nicolas' theorem also states that the Riemann Hypothesis is false if and only if there are infinitely many primorial numbers for which the Nicolas' inequality is false and infinitely many others for which the Nicolas' inequality is true. However, we prove that

$$\lim_{p\to\infty}\left(\frac{\prod_{q|p\#} \frac{q}{q-1}}{e^{\gamma} \times \log\log p\#}\right) = 1.$$

In this way, we show that the Nicolas' inequality is false for many primorial numbers $p\#$ when p tends to the infinity and thus, the Riemann Hypothesis is false too.

Keywords number theory · inequality · primorial · Chebyshev function · prime

Mathematics Subject Classification (2010) MSC 11M26 · MSC 11A41 · MSC 11A25

1 Introduction

Let $p\# = 2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times p$ denotes a primorial. Say $\mathsf{Nicolas}(p)$ holds provided

$$\prod_{q|p\#} \frac{q}{q-1} > e^{\gamma} \times \log\log p\#.$$

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The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, log is the natural logarithm, and $q \mid p\#$ means the prime q divides to $p\#$. The importance of this property is:

Theorem 1.1 *[3], [4].* $\mathsf{Nicolas}(p)$ *holds for all prime* $p > 2$ *if and only if the Riemann Hypothesis is true. Moreover, the Riemann Hypothesis is false if and only if there are infinitely many prime numbers* q *for which* $\mathsf{Nicolas}(q)$ *does not holds and infinitely many prime numbers* r *for which* $\mathsf{Nicolas}(r)$ *holds indeed.*

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x. We use the following property of the Chebyshev function:

Theorem 1.2 *[1].*

$$\lim_{p \to \infty} \frac{\theta(p)}{p} = 1.$$

We also use the Mertens' second theorem which states:

Theorem 1.3 *[2].*

$$\lim_{n \to \infty} (\sum_{q \leq n} \frac{1}{q} - \log \log n - B) = 0,$$

where $B \approx 0.2614972128$ *is the Meissel-Mertens constant.*

Besides, we use the following property of the Meissel-Mertens constant:

Theorem 1.4 *[2].*

$$B = \gamma + \log(\prod_{q} \frac{q-1}{q}) + \sum_{q} \frac{1}{q}.$$

We prove that there is a contradiction just simultaneously assuming that $\mathsf{Nicolas}(p)$ holds for all prime $p > 2$ and the Mertens' second theorem. Since the Mertens' second theorem is a proven result, then $\mathsf{Nicolas}(p)$ does not hold for many sufficiently large prime numbers p. Consequently, the Riemann Hypothesis is false.

2 Main Theorem

Theorem 2.1

$$\lim_{p \to \infty} (\frac{\prod_{q \mid p\#} \frac{q}{q-1}}{e^{\gamma} \times \log \log p\#}) = 1.$$

Proof By the theorem 1.3,

$$\lim_{p \to \infty} (\sum_{q \leq p} \frac{1}{q} - \log \log p - B) = 0,$$

and by the theorem 1.4,

$$B = \gamma + \log(\prod_{q} \frac{q-1}{q}) + \sum_{q} \frac{1}{q}.$$

Putting all this together yields the result,

$$\lim_{p\to\infty} (\sum_{q\leq p} \frac{1}{q} - \log\log p - \gamma - \log(\prod_{q\leq p} \frac{q-1}{q}) - \sum_{q\leq p} \frac{1}{q}) = 0,$$

that is equivalent to

$$\lim_{p\to\infty} (\log(\prod_{q\leq p} \frac{q}{q-1}) - \gamma - \log\log p) = 0.$$

We use that theorem 1.2:

$$\lim_{p\to\infty} (\log(\prod_{q\mid p\#} \frac{q}{q-1}) - \gamma - \log\log\log p\#) = 0.$$

Finally, we can apply the exponentiation to show:

$$\lim_{p\to\infty} (\frac{\prod_{q\mid p\#} \frac{q}{q-1}}{e^{\gamma} \times \log\log p\#}) = 1.$$

3 Result

Theorem 3.1 *The Riemann Hypothesis is false.*

Proof If $\mathsf{Nicolas}(p)$ holds for all prime $p > 2$, then we obtain

$$\frac{\prod_{q\mid p\#} \frac{q}{q-1}}{e^{\gamma} \times \log\log p\#} > 1$$

for every prime $p > 2$. This would happen no matter how large could we pick the prime number p just assuming that $\mathsf{Nicolas}(p)$ holds: This actually means that the limit inferior of the previous fraction is defined by

$$\liminf_{p\to\infty} (\frac{\prod_{q\mid p\#} \frac{q}{q-1}}{e^{\gamma} \times \log\log p\#}) > 1$$

due to the theorem 1.1, which states that could be infinitely many prime numbers p for which $\mathsf{Nicolas}(p)$ does not holds in case of the Nicolas' inequality might be false. However, this is a contradiction with the theorem 2.1. Hence, $\mathsf{Nicolas}(p)$ does not hold for many sufficiently large prime numbers p. By contraposition, we have that the Riemann Hypothesis is false according to the theorem 1.1.

References

1. Grönwall, T.H.: Some asymptotic expressions in the theory of numbers. Transactions of the American Mathematical Society **14**(1), 113–122 (1913). DOI 10.2307/1988773
2. Mertens, F.: Ein Beitrag zur analytischen Zahlentheorie. J. reine angew. Math. **1874**(78), 46–62 (1874). DOI 10.1515/crll.1874.78.46. URL `https://doi.org/10.1515/crll.1874.78.46`
3. Nicolas, J.L.: Petites valeurs de la fonction d'Euler et hypothese de Riemann. Séminaire de Théorie des nombres DPP, Paris **82**, 207–218 (1981)
4. Nicolas, J.L.: Petites valeurs de la fonction d'Euler. Journal of number theory **17**(3), 375–388 (1983). DOI 10.1016/0022-314X(83)90055-0