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SOLS and SODLS

Chapter 1. Basic concepts

I learned about SOLS (10) and SODLS (10) in the discussion group "Magic and Latin squares" (in my home mail).
I will quote two letters.
The first letter was received in March 2017.

"Dear friends,

A self-orthogonal diagonal Latin square of order 10 (SODLS(10)) was found by Bennett, Du, and Zhang by an exhaustive search using a computer [1].
SODLS(10) : orthogonal to its transpose

0 7 6 9 3 2 4 8 5 1
9 1 0 6 8 7 5 3 4 2
4 8 2 7 1 3 9 5 6 0
6 5 8 3 0 9 2 1 7 4
8 9 5 1 4 6 7 0 2 3
1 3 4 0 7 5 8 2 9 6
7 4 3 8 2 1 6 9 0 5
5 2 9 4 6 0 1 7 3 8
2 0 1 5 9 4 3 6 8 7
3 6 7 2 5 8 0 4 1 9

transpose
0 9 4 6 8 1 7 5 2 3
7 1 8 5 9 3 4 2 0 6
6 0 2 8 5 4 3 9 1 7
9 6 7 3 1 0 8 4 5 2
3 8 1 0 4 7 2 6 9 5
2 7 3 9 6 5 1 0 4 8
4 5 9 2 7 8 6 1 3 0
8 3 5 1 0 2 9 7 6 4
5 4 6 7 2 9 0 3 8 1
1 2 0 4 3 6 5 8 7 9

[1] Frank E. Bennett, Beiliang Du, Hantao Zhang, Existence of self-orthogonal diagonal Latin squares with a missing subsquare,
Discrete Mathematics 261 (2003) 69-86.

Best regards,
Mitsutoshi"

In the letter you see the decoding of the abbreviation SODLS(10).
SOLS(10) is the same, only for non-diagonal Latin squares of the 10th order.
In addition, the letter contains a link to the article from which the above example is taken. It seems to be from 2003.

From the letter, the definition of SODLS is also clear - it is such a DLS that is orthogonal to its transposed one.
The definition of SOSL is similar, only for LS.

Since the discussion in this paper is only about 10th order SOLS and SODLS (except for Chapter 3), I will write simply SOLS and SODLS, omitting the reference to order 10.
This was the first SODLS example that I found out about.
Illustration of the SODLS pair in the letter

0	7	6	9	3	2	4	8	5	1
9	1	0	6	8	7	5	3	4	2
4	8	2	7	1	3	9	5	6	0
6	5	8	3	0	9	2	1	7	4
8	9	5	1	4	6	7	0	2	3
1	3	4	0	7	5	8	2	9	6
7	4	3	8	2	1	6	9	0	5
5	2	9	4	6	0	1	7	3	8
2	0	1	5	9	4	3	6	8	7
3	6	7	2	5	8	0	4	1	9

0	9	4	6	8	1	7	5	2	3
7	1	8	5	9	3	4	2	0	6
6	0	2	8	5	4	3	9	1	7
9	6	7	3	1	0	8	4	5	2
3	8	1	0	4	7	2	6	9	5
2	7	3	9	6	5	1	0	4	8
4	5	9	2	7	8	6	1	3	0
8	3	5	1	0	2	9	7	6	4
5	4	6	7	2	9	0	3	8	1
1	2	0	4	3	6	5	8	7	9

Fig. 1

The second letter was the answer to the first, I quote:

"Dear Mitsutoshi,

Many thanks by that reference with an example of SODLS of order 10. I had an example obtained by Hedayat (1971) by using the sum composition technique, but it is not diagonal. The SODLS of order 10 seem to be very rare … in a reasonable time by using a backtracking program … none !"

Attached to the letter is an illustration of the SOLS pair in question. This seems to be from 1971! Showing illustration

Carry similar operations on the entries of Y and denote the resulting square by L_2. These squares are exhibited below:

$$L_1 = \begin{array}{ccccccccc|c}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 9 & 8 & 7 \\
3 & 6 & 5 & 4 & 0 & 8 & 7 & 1 & 9 & 2 \\
9 & 0 & 1 & 7 & 8 & 6 & 3 & 2 & 5 & 4 \\
7 & 9 & 6 & 5 & 2 & 3 & 1 & 0 & 4 & 8 \\
8 & 2 & 9 & 1 & 7 & 4 & 0 & 5 & 6 & 3 \\
1 & 3 & 4 & 9 & 5 & 2 & 8 & 6 & 7 & 0 \\
2 & 5 & 0 & 8 & 9 & 7 & 4 & 3 & 1 & 6 \\
5 & 4 & 7 & 6 & 3 & 9 & 2 & 8 & 0 & 1 \\
6 & 7 & 8 & 2 & 1 & 0 & 9 & 4 & 3 & 5 \\
\hline
4 & 8 & 3 & 0 & 6 & 1 & 5 & 7 & 2 & 9
\end{array}
\qquad
L_2 = \begin{array}{ccccccccc|c}
0 & 3 & 9 & 7 & 8 & 1 & 2 & 5 & 6 & 4 \\
1 & 6 & 0 & 9 & 2 & 3 & 5 & 4 & 7 & 8 \\
2 & 5 & 1 & 6 & 9 & 4 & 0 & 7 & 8 & 3 \\
3 & 4 & 7 & 5 & 1 & 9 & 8 & 6 & 2 & 0 \\
4 & 0 & 8 & 2 & 7 & 5 & 9 & 3 & 1 & 6 \\
5 & 8 & 6 & 3 & 4 & 2 & 7 & 9 & 0 & 1 \\
6 & 7 & 3 & 1 & 0 & 8 & 4 & 2 & 9 & 5 \\
9 & 1 & 2 & 0 & 5 & 6 & 3 & 8 & 4 & 7 \\
8 & 9 & 5 & 4 & 6 & 7 & 1 & 0 & 3 & 2 \\
\hline
7 & 2 & 4 & 8 & 3 & 0 & 6 & 1 & 5 & 9
\end{array}$$

The reader can verify for himself that L_1 and L_2 are both latin squares of order 10 with the desired properties viz, $L_1 \perp L_2$ and $L_2 = L_1^t$.

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Fig. 2

I have written in my working file a small study of the LS shown on the left in Fig. 2, with Belyshev's program.
Investigated LS

0 1 2 3 4 5 6 9 8 7
3 6 5 4 0 8 7 1 9 2
9 0 1 7 8 6 3 2 5 4
7 9 6 5 2 3 1 0 4 8
8 2 9 1 7 4 0 5 6 3
1 3 4 9 5 2 8 6 7 0
2 5 0 8 9 7 4 3 1 6
5 4 7 6 3 9 2 8 0 1
6 7 8 2 1 0 9 4 3 5
4 8 3 0 6 1 5 7 2 9

This LS has 10 orthogonal squares and gives a pseudo-triple with a maximal orthogonality characteristic of **82**.

«Name:a33.txt
1 - only the diagonal
Max=2000
1
76 70 74 90 60 80 76 66 70 66 :728
sq=10 64 67 70 73 82
cm=82 cmm=82
END»

So, we continue to watch the story. It is really very interesting.

On the boinc.ru forum (this forum was lost; on the restored boinc.ru forum, of course, this message is not present; this applies to all the following links to the boinc.ru forum) in the message
http://forum.boinc.ru/default.aspx?g=posts&m=87053#post87053
Belyshev gave a link to the SOLS list that he found on the Web:

"Here is http://www.vuuren.co.za/data/SOLS_Repository/SOLSNonRCParatopic10x10.txt
all 121642 representative classes of RC-paratopism SOLS10 are listed (caution, almost 12 MB). Not so much, but SODLS can be several orders of magnitude more (or less :)"

121642 SOLS were found, these are representatives of some "classes of RC-paratopism SOLS10".
I don't know what classes of RC-paratopism there are; possibly equivalence classes with respect to the isomorphism of LS. Well, the main thing is that these are SOLS, and, apparently, are significantly different, that is, not isomorphic.
At first Vatutin saw among these 121642 SOLS 71 DLS.
Well, it's easy to see.
I'll show you how the utility **GetType** by Harry White handles these SOLS

«Saturday 2018-12-01 09:28:29

Order? 10

Enter the name of the squares file: input
.. writing type information to file inputTypeDetail.txt

Counts

  121571 Latin
     71 diagonal Latin
  121642 natural \diagonal
  121642 self-orthogonal»

In the report we see: 71 diagonal Latin, 121642 self-orthogonal.
Thus, we immediately have 71 SODLS.
Next it was more interesting!

Belyshev answered here
http://forum.boinc.ru/default.aspx?g=posts&m=87070#post87070

"These are only those that lie on the surface. Need to dig deeper. :)

We have 121642 SOLS equivalence classes, each class includes up to 2 * (10!)^2 = 26336378880000 SOLS, of them up to 2 * 10! = 7257600 normalized (with the main diagonal of the type 0123456789) SOLS. As the representative of each class, the smallest lexicographically normalized SOLS was chosen.

It so happened that in 71 equivalence classes the smallest SOLS turned out to be DLS (and therefore SODLS). But every SODLS is SOLS, and therefore belongs to some SOLS equivalence class. And to find all the SODLS, you just need to find all the DLSs included in all these SOLS equivalence classes. To do this, you do not need to perform 7257600 transformations for each equivalence class, it will be enough to perform only 945."

And here
http://forum.boinc.ru/default.aspx?g=posts&m=87090#post87090
amazing finale!
Belyshev said:
“I wrote a program. 30534 SODLS were found (https://yadi.sk/d/l4Fus2cp3GTqe2), of which (https://yadi.sk/d/OiKwfw2b3Gan9x) were significantly different 30502. The operation time is 4 seconds.”

30502 CF SODLS were found in 4 seconds!
Well, plus more time for writing the program. I do not think this time is very long.
I note that by that time the DB CF ODLS of the manual project for the search for ODLS contained only about 10,000 CF ODLS (plus or minus), and the search was performed for more than a year.

Chapter 2. Properties of 30502 CF SODLS

Let's see how the 30502 CF SODLS found by Belyshev is handled by the utility **GetType** by Harry White

«Saturday 2018-12-01 10:23:31

Order? 10

Enter the name of the squares file: input
.. writing type information to file inputTypeDetail.txt

Counts

    30502 diagonal Latin
    30502 nfr
      161 self-orthogonal»

nfr - normalized on the first line (the first line is naturally ordered).
Why are only **161** self-orthogonal left? It's very simple: not all CFs are orthogonal to their transposed variant.
There are only 161 such CFs that are orthogonal to their transposed variant.
Harry's utility helped me quickly find an example of CF that is orthogonal to its transposed variant.

This is CF SODLS

0 1 2 3 4 5 6 7 8 9
2 3 0 1 5 4 8 9 7 6
7 8 1 9 0 6 3 4 2 5
5 4 7 6 2 9 0 8 1 3
6 7 4 5 8 3 2 1 9 0
9 2 3 4 1 7 5 6 0 8
1 5 8 2 6 0 9 3 4 7
3 0 5 7 9 8 4 2 6 1
4 6 9 8 3 1 7 0 5 2
8 9 6 0 7 2 1 5 3 4

this is its transposed variant

0 2 7 5 6 9 1 3 4 8
1 3 8 4 7 2 5 0 6 9
2 0 1 7 4 3 8 5 9 6
3 1 9 6 5 4 2 7 8 0
4 5 0 2 8 1 6 9 3 7
5 4 6 9 3 7 0 8 1 2
6 8 3 0 2 5 9 4 7 1
7 9 4 8 1 6 3 2 0 5
8 7 2 1 9 0 4 6 5 3
9 6 5 3 0 8 7 1 2 4

I check with my program the orthogonality of these DLSs, the program gives

«ORTHOGONAL SQUARES!!!

0 12 27 35 46 59 61 73 84 98
21 33 8 14 57 42 85 90 76 69
72 80 11 97 4 63 38 45 29 56
53 41 79 66 25 94 2 87 18 30
64 75 40 52 88 31 26 19 93 7
95 24 36 49 13 77 50 68 1 82
16 58 83 20 62 5 99 34 47 71
37 9 54 78 91 86 43 22 60 15
48 67 92 81 39 10 74 6 55 23
89 96 65 3 70 28 17 51 32 44»

Checking 30502 CF SODLS for symmetry with Belyshev's program

«Поиск симметрий ЛК10 версия 3.0

Обработано ЛК: 30502
Время работы : 2.823 сек

Введите код симметрии: all

Квадратов с симметрией (27,27,27) найдено: 16 они записаны в файл symm_27_27_27.txt»

This means that these SODLS produced 16 standards with symmetry (27,27,27)!
Showing all these standards

0 1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9 0
2 0 1 5 8 3 9 4 7 6
3 9 4 7 2 8 5 0 6 1
4 6 9 1 0 7 3 5 2 8
5 7 0 8 6 4 2 9 1 3
6 8 7 9 1 2 4 3 0 5
7 5 6 0 3 9 8 1 4 2
8 4 5 2 9 0 1 6 3 7
9 3 8 6 7 1 0 2 5 4

0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	0
2	0	1	7	9	3	5	6	4	8
3	9	7	5	8	4	0	2	6	1
4	8	9	2	6	0	3	1	7	5
5	6	4	1	3	9	8	0	2	7
6	7	5	8	1	2	4	9	0	3
7	3	8	9	0	1	2	4	5	6
8	5	6	0	2	7	9	3	1	4
9	4	0	6	7	8	1	5	3	2

0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	0
2	0	1	9	6	8	5	4	7	3
3	4	5	8	7	9	0	6	2	1
4	8	7	2	9	3	1	5	0	6
5	6	9	0	8	4	2	3	1	7
6	9	4	7	2	1	8	0	3	5
7	5	8	1	0	2	3	9	6	4
8	7	6	5	3	0	9	1	4	2
9	3	0	6	1	7	4	2	5	8

0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	0
2	0	4	8	1	7	5	9	3	6
3	4	6	9	7	8	0	1	2	5
4	3	8	1	0	2	9	6	5	7
5	8	9	6	2	4	3	0	7	1
6	7	0	5	3	9	2	4	1	8
7	9	1	2	6	0	8	5	4	3
8	6	5	7	9	3	1	2	0	4
9	5	7	0	8	1	4	3	6	2

0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	0
2	0	4	9	6	8	1	5	3	7
3	4	8	1	7	0	9	6	2	5
4	6	5	2	0	1	8	9	7	3
5	8	9	6	3	7	4	0	1	2
6	3	7	5	8	9	2	4	0	1
7	9	0	8	1	3	5	2	4	6
8	5	1	7	9	2	0	3	6	4
9	7	6	0	2	4	3	1	5	8

0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	0
2	0	5	1	6	8	3	9	4	7
3	7	0	9	8	4	1	2	5	6
4	6	8	7	2	9	5	1	0	3
5	3	9	6	7	0	8	4	1	2
6	4	1	8	9	7	0	3	2	5
7	5	4	2	0	1	9	6	3	8
8	9	6	0	3	2	4	5	7	1
9	8	7	5	1	3	2	0	6	4

0 1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9 0
2 0 5 6 8 7 9 3 4 1
3 9 6 0 2 4 8 5 1 7
4 3 7 8 9 0 5 1 2 6
5 7 9 2 6 8 1 4 0 3
6 5 4 9 7 1 2 0 3 8
7 8 0 1 3 9 4 2 6 5
8 6 1 5 0 2 3 9 7 4
9 4 8 7 1 3 0 6 5 2

0 1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9 0
2 0 6 1 9 3 8 4 5 7
3 8 7 6 0 4 1 9 2 5
4 5 9 2 3 7 0 1 6 8
5 7 8 0 2 1 9 3 4 6
6 9 1 8 7 0 5 2 3 4
7 3 5 9 8 2 4 6 0 1
8 6 4 5 1 9 3 0 7 2
9 4 0 7 6 8 2 5 1 3

0 1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9 0
2 0 6 5 7 9 4 1 3 8
3 6 5 7 1 8 2 9 0 4
4 8 0 6 9 7 5 3 2 1
5 7 1 8 6 2 9 0 4 3
6 9 7 1 3 0 8 4 5 2
7 5 4 9 8 3 0 2 1 6
8 3 9 0 2 4 1 5 6 7
9 4 8 2 0 1 3 6 7 5

0 1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9 0
2 0 6 7 9 8 1 3 5 4
3 9 4 8 0 2 5 6 7 1
4 5 8 1 7 0 9 2 3 6
5 6 7 0 8 9 3 1 4 2
6 8 9 5 1 7 0 4 2 3
7 3 5 9 2 1 4 0 6 8
8 7 1 6 3 4 2 9 0 5
9 4 0 2 6 3 8 5 1 7

0 1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9 0
2 0 8 6 1 9 3 4 5 7
3 5 0 7 8 4 1 9 2 6
4 3 5 8 6 2 9 0 7 1
5 7 4 9 0 8 2 1 6 3
6 4 1 2 9 7 0 5 3 8
7 9 6 1 3 0 8 2 4 5
8 6 9 5 7 1 4 3 0 2
9 8 7 0 2 3 5 6 1 4

0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	0
2	0	9	1	6	7	8	4	5	3
3	4	8	5	9	2	0	1	7	6
4	3	5	8	7	0	9	2	6	1
5	9	0	2	1	8	4	6	3	7
6	8	7	0	2	9	5	3	1	4
7	6	4	9	0	1	3	5	2	8
8	7	1	6	3	4	2	9	0	5
9	5	6	7	8	3	1	0	4	2

0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	0
2	0	9	1	7	8	3	4	5	6
3	7	0	9	1	2	8	5	6	4
4	8	6	0	3	9	5	2	1	7
5	6	4	8	9	7	2	3	0	1
6	9	1	2	8	3	4	0	7	5
7	3	8	5	0	1	9	6	4	2
8	4	5	7	6	0	1	9	2	3
9	5	7	6	2	4	0	1	3	8

0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	0
2	0	9	6	8	7	4	1	3	5
3	7	6	8	0	9	1	5	4	2
4	5	8	7	6	2	9	3	0	1
5	8	0	1	7	3	2	9	6	4
6	3	7	5	9	1	0	4	2	8
7	9	4	0	2	8	5	6	1	3
8	4	5	9	1	0	3	2	7	6
9	6	1	2	3	4	8	0	5	7

0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	0
2	3	5	8	7	9	4	1	0	6
3	4	7	9	6	2	8	0	1	5
4	8	6	0	1	7	3	9	5	2
5	9	1	6	8	0	2	3	4	7
6	7	8	5	0	3	9	4	2	1
7	0	9	2	3	8	1	5	6	4
8	6	0	1	9	4	5	2	7	3
9	5	4	7	2	1	0	6	3	8

0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	0
2	4	6	9	7	8	1	5	0	3
3	6	1	5	9	7	0	2	4	8
4	0	5	7	8	2	9	3	1	6
5	9	0	8	1	4	3	6	7	2
6	7	8	0	2	1	4	9	3	5
7	5	9	2	3	0	8	4	6	1
8	3	7	1	6	9	2	0	5	4
9	8	4	6	0	3	5	1	2	7

In the topic "Symmetry (27,27,27)" I laid out 19 standards for symmetry (27,27,27)
https://boinc.progger.info/odlk/forum_thread.php?id=96&postid=2661#2661

Utility by V. Chirkov compares previously laid out 19 standards and found from SODLS 16 standards for symmetry (27,27,27):

«Имя входного файла ИСТОЧНИК (без расширения):symm_27_27_27
Имя входного файла ВЫЧИТАЕМОЕ (без расширения):input
symm_27_27_27.txt
Всего 16 квадратов (вход) в symm_27_27_27.txt
input.txt
Всего 19 квадратов (вход) в input.txt
Уникальных 0 квадратов (выход)»

All 16 standards from among the 19 standards found earlier!

Chapter 3. Doubly SODLS

I'll start with the illustration (this is from the Harry White website
http://budshaw.ca/addenda/SODLSnotes.html)

For order 9, some SODLS are doubly self-orthogonal and some are not. Of the 224,832 nfr, natural orde
196,224 are singly SODLS. These numbers are confirmed by Francis Gaspalou, (June, 2016).
Example:

doubly SODLS

0	1	2	3	4	5	6	7	8
7	2	3	4	5	8	1	0	6
4	6	1	8	0	7	5	3	2
1	8	7	5	2	4	3	6	0
6	4	8	0	7	3	2	1	5
3	5	0	1	8	6	7	2	4
8	0	6	7	3	2	4	5	1
5	3	4	2	6	1	0	8	7
2	7	5	6	1	0	8	4	3

singly SODLS

0	1	2	3	4	5	6	7	8
3	2	7	5	8	0	4	1	6
6	5	1	8	2	7	0	3	4
8	0	6	4	7	2	3	5	1
2	7	0	1	3	4	8	6	5
1	8	3	7	5	6	2	4	0
7	3	4	6	1	8	5	0	2
4	6	5	2	0	1	7	8	3
5	4	8	0	6	3	1	2	7

Fig. 3

In the illustration you see doubly SODLS (left) and singly SODLS (right) 9th order.
This link also has a doubly SODLS definition: **Doubly Self-Orthogonal Diagonal Latin Square (orthogonal to its transpose and its antitranspose)**.

I will publish a letter from the Frenchman Francis Gaspalou dated **December 5, 2017**.

“Self Orthogonal Diagonal Latin Squares of order 9
Когда: 05 декабря 2017 в 16:10

Dear friends,

I have already announced the number of 470 essentially different SODLS of order 9 (cf email hereafter).
Today I inform you that the Russian Alex Belyshev found recently the same number.
This number of 470, found by two different programs, can then be considered as

established.

I remind that the 224,832 SODLS of order 9 are coming from a limited number of "essentially different SODLS" when applying the group of the 1,536 geometric transformations and the group of the 9! permutations.
For the low orders, we have

Order Nb of SODLS Nb of ess. diff. SODLS

4 2 1
5 4 1
6 0 0
7 64 2
8 1,152 8
9 224,832 470

You will find in attachment a list of these 470 SODLS.
I can give also a list of the 382 singly and a list of the 88 doubly to anyone who is interested (these lists were found also in June 2016)

Best regards
Francis

BTW: Alex confirmed also the number of 8 ess. diff. SODLS of order 8 I found in October 2010"

This letter was written to a community of researchers of magic and Latin squares.
Here we see that Francis Gaspalou found 470 substantially different SODLS 9th order. This result was confirmed by A. Belyshev.
And then Francis Gaspalou wrote

"You will find in attachment a list of these 470 SODLS.
I can give also a list of the 382 singly and a list of the 88 doubly to anyone who is interested (these lists were found also in June 2016)"

This is an interesting result. Thus, among 470 SODLS of order 9, there are 88 doubly SODLS and 382 singly SODLS.

Now let's see the sequence in OEIS https://oeis.org/A287761
A287761 Number of self-orthogonal diagonal Latin squares of order n with ordered first string.
1, 0, 0, 2, 4, 0, 64, 1152, 224832, 234255360

The sequence indicates the number of normalized SODLS for orders 1 to 10.
The letter from Francis Gaspalou just says "224,832 SODLS of order 9".
We see the same result on the Harry White website
http://budshaw.ca/SODLS.html

As I understand from the OEIS sequence, for order 10 there are 234,255,360 normalized SODLS.
I have not seen confirmation of this result in other sources.

In the OEIS sequence https://oeis.org/A329685

A329685 Number of main classes of self-orthogonal diagonal Latin squares of order n.
1, 0, 0, 1, 1, 0, 2, 8, 470, 30502

the numbers of the main SODLS classes of orders 1 to 10 are shown.
For order 9, we see the number of main classes 470. This is a confirmed result; Francis Gaspalou and Alexey Belyshev got the same result.
Belyshev received 30,502 main SODLS classes of order 10; he posted the representatives of these main classes (CF) (see the link at the end of the article).

Now about doubly SODLS of order 10.
As Francis Gaspalou told me, it's proven that doubly SODLS, as well as doubly SOLS, of the order of 10 do not exist.
I quote his letter

"Yes it is proven. See
Runming Lu, Sheng Liu and Jian Zhang
"Searching for Doubly Self-orthogonal Latin Squares", 2011
http://link.springer.com/chapter/10.1007%2F978-3-642-23786-7_41#page-1"

By the specified link we see

Fig. 4

Abstract

A Doubly Self Orthogonal Latin Square (DSOLS) is a Latin square which is orthogonal to its transpose to the diagonal and its transpose to the back diagonal. It is challenging to find a non-trivial DSOLS. For the orders n = 2 (mod 4), the existence of DSOLS(n) is unknown except for n = 2, 6. We propose an efficient approach and data structure based on a set system and exact cover, with which we obtained a new result, i.e., the non-existence of DSOLS(10).

As I understand it, this is from the next book
«Principles and Practice of Constraint Programming – CP 2011»
https://link.springer.com/book/10.1007/978-3-642-23786-7

Chapter 4. Quasi-SODLS

A complete closure from 121642 SOLS of the 10th order, performed by the Belyshev script zamyk.bat, yielded 33753 CF ODLS, including 79 groups of two pairs of ODLS.
Comparison with 30502 CF SODLS, previously laid out by Belyshev, using the utility by V. Chirkov

«Имя входного файла ИСТОЧНИК (без расширения):input
Имя входного файла ВЫЧИТАЕМОЕ (без расширения):kf_sodls
input.txt
Всего 33753 квадратов (вход) в input.txt
kf_sodls.txt
Всего 30502 квадратов (вход) в kf_sodls.txt
Уникальных 3251 квадратов (выход).»

Thus, an additional 3251 CF ODLS was obtained.
Then I found another 22 CF ODLS by post-processing (with my programs) all the solutions received.
The result was 33775 CF ODLS; among them are not only SODLS in the classical sense, also there are no SODLS at all.
Let's look at some examples.

Example 1

This example was shown above, I repeat it.

CF SODLS from received Belyshev

0 1 2 3 4 5 6 7 8 9
2 3 0 1 5 4 8 9 7 6
7 8 1 9 0 6 3 4 2 5
5 4 7 6 2 9 0 8 1 3
6 7 4 5 8 3 2 1 9 0
9 2 3 4 1 7 5 6 0 8
1 5 8 2 6 0 9 3 4 7
3 0 5 7 9 8 4 2 6 1
4 6 9 8 3 1 7 0 5 2
8 9 6 0 7 2 1 5 3 4

a transposed version of this DLS

0 2 7 5 6 9 1 3 4 8
1 3 8 4 7 2 5 0 6 9
2 0 1 7 4 3 8 5 9 6
3 1 9 6 5 4 2 7 8 0
4 5 0 2 8 1 6 9 3 7
5 4 6 9 3 7 0 8 1 2
6 8 3 0 2 5 9 4 7 1
7 9 4 8 1 6 3 2 0 5
8 7 2 1 9 0 4 6 5 3
9 6 5 3 0 8 7 1 2 4

Here, everything is accurate by definition: the original DLS is orthogonal to its transposed variant, that is, we have SODLS in the classical sense.

<u>Example 2</u>

Often the conversion of SODLS to canonical form (CF) violates the classical definition, that is, the obtained DLS is no longer orthogonal to its transpose variant, but it is orthogonal to its isomorph.
We look at an example

[DLK(1)]
0 2 9 4 3 6 7 8 5 1
3 1 5 7 6 9 8 4 0 2
9 8 2 1 7 4 0 3 6 5
1 9 8 3 0 2 4 5 7 6
5 6 7 2 4 8 9 1 3 0
7 4 3 6 9 5 1 0 2 8
4 3 1 5 8 0 6 2 9 7
8 0 6 9 5 3 2 7 1 4
6 7 4 0 2 1 5 9 8 3
2 5 0 8 1 7 3 6 4 9

[mate#1]
3 4 5 2 6 7 8 1 0 9
5 6 9 3 0 4 2 8 7 1
8 3 2 0 7 9 1 4 5 6
4 1 0 8 9 3 5 7 6 2
2 9 4 7 1 6 0 5 3 8
9 0 1 6 3 5 7 2 8 4
6 7 3 1 8 0 4 9 2 5
7 5 8 9 4 2 6 0 1 3
1 2 7 4 5 8 3 6 9 0
0 8 6 5 2 1 9 3 4 7

Original **DLK(1)** in the form of CF (format 2: the main diagonal is naturally ordered), it is not orthogonal to its transpose variant

0 3 9 1 5 7 4 8 6 2
2 1 8 9 6 4 3 0 7 5
9 5 2 8 7 3 1 6 4 0
4 7 1 3 2 6 5 9 0 8
3 6 7 0 4 9 8 5 2 1

6 9 4 2 8 5 0 3 1 7
7 8 0 4 9 1 6 2 5 3
8 4 3 5 1 0 2 7 9 6
5 0 6 7 3 2 9 1 8 4
1 2 5 6 0 8 7 4 3 9

However, the orthogonal DLS (**mate #1**) is isomorphic to the original **DLK(1)** with the following isomorphism (determined by the Belyshev program **avtoizor_lk**)

*T 9876543210 9876543210 7904518263

This isomorphism means the following combination of transformations:
a) transpose;
b) permutation of rows according to the scheme 9876543210;
c) column permutation according to the same scheme;
d) re-designation of elements according to the scheme 7904518263.

Pay attention to the specific scheme of permutation of rows and columns: the permutation is performed in the reverse order (in other words, reflection).
Applying all these transformations to the original DLS, you will get mate #1, orthogonal to it.

I did the following check for this orthogonal pair.
Using the Belyshev program **izomorfDLK10A** I found all 15360 DLSs of this class of isomorphism (to which the shown ODLSs belong).
Program protocol:

«Программа поиска нормализованых изоморфов данного ДЛК10:

0 2 9 4 3 6 7 8 5 1
3 1 5 7 6 9 8 4 0 2
9 8 2 1 7 4 0 3 6 5
1 9 8 3 0 2 4 5 7 6
5 6 7 2 4 8 9 1 3 0
7 4 3 6 9 5 1 0 2 8
4 3 1 5 8 0 6 2 9 7
8 0 6 9 5 3 2 7 1 4
6 7 4 0 2 1 5 9 8 3
2 5 0 8 1 7 3 6 4 9

Уникальных изоморфов: 15360
Они записаны в файл out_EXKFMD.txt»

Then I checked the received 15360 DLS with the Harry White **GetType** utility, here is the result of the check

«Thursday 2020-04-02 08:44:44

Order? 10

Enter the name of the squares file: input
.. writing type information to file inputTypeDetail_7.txt

Counts

15360 diagonal Latin
15360 nfr
7680 self-orthogonal»

There is no doubt that this is the SODLS equivalence class (as defined by SODLS).

Example 3

[DLK(1)]
0 2 3 4 5 6 7 8 9 1
2 1 7 6 8 3 5 9 0 4
5 4 2 9 1 0 8 3 7 6
8 6 0 3 2 9 4 5 1 7
7 5 9 0 4 2 3 1 6 8
1 8 6 2 7 5 9 0 4 3
4 7 1 5 9 8 6 2 3 0
9 3 8 1 6 4 0 7 5 2
3 9 4 7 0 1 2 6 8 5
6 0 5 8 3 7 1 4 2 9

[mate#1]
1 4 2 5 6 0 3 7 8 9
2 3 8 6 5 1 7 9 4 0
0 7 6 1 4 3 8 5 9 2
3 1 7 0 9 4 2 8 6 5
6 2 3 9 8 7 4 0 5 1
7 9 4 8 1 5 0 2 3 6
4 0 5 3 2 6 9 1 7 8
5 8 0 2 7 9 6 4 1 3
9 6 1 7 0 8 5 3 2 4
8 5 9 4 3 2 1 6 0 7

In this example, the original DLS is orthogonal to its isomorph, with the isomorphism as follows:

*T 0423186759 0423186759 1860329457

Here, rows and columns are rearranged in the same pattern, but (!) this scheme is not the same as in the previous example.
The row/column permutation scheme 0423186759 is not a reflection.

And now we will check for the main DLS of this orthogonal pair (**DLK(1)**), as in the previous example.
IzomorfDLK10A program gives

«Программа поиска нормализованых изоморфов данного ДЛК10:

0 2 3 4 5 6 7 8 9 1
2 1 7 6 8 3 5 9 0 4
5 4 2 9 1 0 8 3 7 6
8 6 0 3 2 9 4 5 1 7
7 5 9 0 4 2 3 1 6 8
1 8 6 2 7 5 9 0 4 3
4 7 1 5 9 8 6 2 3 0
9 3 8 1 6 4 0 7 5 2

3 9 4 7 0 1 2 6 8 5
6 0 5 8 3 7 1 4 2 9

Уникальных изоморфов: 15360
Они записаны в файл out_ODUJMD.txt»

Checking with the utility by Harry White gives

«Friday 2020-04-03 05:50:17

Order? 10

Enter the name of the squares file: input
.. writing type information to file inputTypeDetail_7.txt

Counts

15360 diagonal Latin
15360 nfr»

As you can see, there is no self-orthogonal in the classical sense.
However, the original DLS is orthogonal to its isomorph.

I will give ***working definitions*** (for use in this study of 10th-order SODLS).
The isomorphisms of type
*T 9876543210 9876543210 7904518263
where the rows/columns are rearranged according to the same scheme and this reflection (that is, the rearrangement of rows/columns in the reverse order) is called **isomorphism of type A**.
All type A isomorphisms are inherent in SODLS in the classical sense.
This conclusion is drawn from an analysis of empirical data.

The isomorphisms of type
*T 0423186759 0423186759 1860329457
where the rows and columns are rearranged in the same way, but this is not a reflection, we will call **isomorphisms of type B**.

<u>Definition of quasi-SODLS</u>: DLSs that are isomorphic to their orthogonal square with type B isomorphism (or with several isomorphisms including type B isomorphism but not including type A isomorphisms or just transpose), we call **quasi-SODLS**.

An example for parentheses is shown below.
For SOLS, this definition can be extended. Chapter 6 shows examples of such an extension.
I note that there are also isomorphisms: transpose with a re-designation of elements that are equivalent only to transpose.
An example of such an isomorphism:
*T 0123456789 0123456789 0412963578.
Such isomorphisms are also excluded in the definition of quasi-SODLS, since they are inherent in classical SODLS.

I analyzed all the orthogonal pairs obtained from the classic SODLS with Belyshev's **zamyk.bat** script.
For example, isomorphisms similar to those shown in Example 3 (**type B**):

*T 0423186759 0423186759 1860329457
*T 0423186759 0423186759 8210364975
*T 0423186759 0423186759 8254617903
*T 0423186759 0423186759 4739815026
*T 0423186759 0423186759 4763928150
*T 0423186759 0423186759 2374810659
*T 0423186759 0423186759 1628590473
*T 0423186759 0423186759 4368072915
*T 0423186759 0423186759 3627510498
*T 0423186759 0423186759 4170562389
*T 0423186759 0423186759 4128035796
*T 0423186759 0423186759 5413068927
*T 0423186759 0423186759 1857369024
*T 0423186759 0423186759 3205749618
*T 0423186759 0423186759 3754128906
*T 0423186759 0423186759 5136407928
*T 0423186759 0423186759 3748610295
*T 0423186759 0423186759 4705618392
*T 0423186759 0423186759 3147809256
*T 0423186759 0423186759 2130758694
*T 0423186759 0423186759 5690437821
*T 0423186759 0423186759 4520196738
*T 0423186759 0423186759 5240687391
*T 0423186759 0423186759 4290615378

Each isomorphism corresponds to a unique orthogonal pair of quasi-SODLS.
Isomorphisms differ only in the re-designation of elements.

A very interesting question is: are there other orthogonal quasi-SODLS pairs besides those obtained from classical SODLS closures?
My hypothesis: such quasi-SODLS do not exist.
The hypothesis needs to be proved or refuted.
To refute it is enough to find one counterexample.
That is: we need to find a DLS that is orthogonal to its isomorph (with any isomorphism!), And the canonical form (CF) of this DLS is not contained in the closure of all known SODLS.
All known CFs of SODLS and CFs of quasi-SODLS are contained in the set 33775 CFs of the ODLS, which is attached to the article.

Exploring quasi-SODLS, I discovered another case where there are three isomorphisms connecting isomorphic orthogonal squares.

<u>Example 4</u>

[DLK(1)]
0 3 6 7 5 9 8 2 4 1
7 1 8 9 6 4 2 5 0 3
6 8 2 0 9 7 5 3 1 4
1 9 5 3 8 2 4 0 7 6
3 7 1 2 4 6 9 8 5 0
8 0 3 4 7 5 1 6 9 2
9 5 4 8 3 0 6 1 2 7
2 4 9 5 0 1 3 7 6 8
4 2 7 6 1 3 0 9 8 5
5 6 0 1 2 8 7 4 3 9

[mate#1]
4 3 5 1 6 2 7 8 9 0
0 9 2 7 8 1 3 4 5 6
2 0 6 9 5 8 1 7 3 4
8 4 0 5 3 9 6 2 7 1
1 6 4 0 2 3 9 5 8 7
6 1 8 3 4 7 2 9 0 5
3 5 7 8 9 6 0 1 4 2
7 8 1 2 0 5 4 3 6 9
5 2 9 4 7 0 8 6 1 3
9 7 3 6 1 4 5 0 2 8

Here, the main DLS is also orthogonal to its isomorph.
The program for determining isomorphism yields three isomorphisms at once:

*T 1306897425 2081793645 0153682947
*T 4697083521 4986071352 2531467980
*T 5783906124 5783906124 7315840962

The third isomorphism is a type B isomorphism already known to us, when rows and columns are rearranged according to the same scheme, but this is not a reflection.
And in the first two isomorphisms, rows and columns are rearranged according to different schemes.
It is interesting to note that both ODLS in this example have symmetry (27,27,27).

I will show two more similar quasi-SODLS pairs

Example 5

[DLK(1)]
0 2 3 4 6 7 8 5 9 1
7 1 9 0 5 4 3 6 2 8
6 4 2 8 7 0 9 3 1 5
1 5 0 3 2 9 7 8 4 6
9 7 8 1 4 6 0 2 5 3
3 8 6 7 9 5 2 1 0 4
4 9 7 5 1 8 6 0 3 2
2 3 4 9 8 1 5 7 6 0
5 0 1 6 3 2 4 9 8 7
8 6 5 2 0 3 1 4 7 9

[mate#1]
4 1 2 3 5 6 0 7 8 9
8 6 3 7 0 9 1 4 2 5
2 7 8 4 1 3 9 5 0 6
1 2 5 9 3 0 7 6 4 8
6 4 7 5 2 1 8 9 3 0
7 8 0 2 4 5 6 3 9 1
0 5 9 8 7 2 3 1 6 4
5 3 6 1 9 8 4 0 7 2
9 0 4 6 8 7 5 2 1 3
3 9 1 0 6 4 2 8 5 7

Here the orthogonal square **mate #1** can be obtained from the main DLS by the following isomorphisms

*T 2587619430 9764381502 5698147230
*T 8694372150 9508714326 1869705432
*T 9321548760 9321548760 7986521034

Example 6

[DLK(1)]
0 6 7 4 3 9 2 8 5 1
3 1 0 5 9 8 4 6 2 7
1 9 2 0 5 7 8 3 6 4
6 7 8 3 1 2 0 9 4 5
5 0 3 2 4 6 9 1 7 8
8 2 4 9 7 5 3 0 1 6
7 3 1 8 0 4 6 5 9 2
2 4 9 6 8 1 5 7 3 0
9 5 6 1 2 0 7 4 8 3
4 8 5 7 6 3 1 2 0 9

[mate#1]
3 6 5 1 7 0 2 4 8 9
4 5 0 9 8 1 6 2 7 3
6 3 4 7 0 2 9 1 5 8
9 0 6 8 1 3 5 7 4 2
5 9 3 6 2 8 4 0 1 7
2 1 9 5 6 7 0 8 3 4
7 2 8 3 4 5 1 6 9 0
8 7 2 0 5 4 3 9 6 1
1 4 7 2 9 6 8 3 0 5
0 8 1 4 3 9 7 5 2 6

Here the orthogonal square **mate #1** can be obtained from the main DLS by the following isomorphisms

*T 0853726419 0853726419 3078941256
*T 5089423176 2459813706 7421956308
*T 8506129743 1756409823 2536908741

All ODLS in examples 5 - 6 also have symmetry (27,27,27).
There are a couple more of these quasi-SODLS pairs.

The isomorphisms of type
*T 1306897425 2081793645 0153682947
when rows and columns are rearranged according to different schemes, we will call **isomorphisms of type C**.

Finally, the last example, also three isomorphisms connect ODLS in an orthogonal pair

*T 0321548769 0321548769 2603795814
*T 0574863912 0574863912 2541637098
*T 0896315247 0896315247 2789516430

but in each of these isomorphisms, the rows and columns are rearranged in the same way (type B isomorphisms).
Unfortunately, I did not write down the ODLS of this pair when I was looking for isomorphisms.

Chapter 5. Groups of two ODLS pairs containing SODLS and quasi-SODLS

In the previous chapters, only orthogonal ODLS pairs were considered, which are SODLS and quasi-SODLS, the so-called "odnushki".
There are also groups of two pairs of ODLS containing SODLS and quasi-SODLS, the so-called "dvushki".
Moreover, among the two orthogonal squares only one is isomorphic to the main DLS.
I will show examples.

1. A group of two pairs of ODLS with SODLS

[DLK(2)]
0 3 4 6 7 2 9 8 5 1
2 1 7 9 5 3 8 6 0 4
8 6 2 5 0 9 1 3 4 7
1 5 6 3 9 7 4 0 2 8
9 8 3 2 4 6 5 1 7 0
6 7 1 0 8 5 2 4 9 3
4 9 0 7 3 8 6 5 1 2
3 4 9 8 2 1 0 7 6 5
7 2 5 4 1 0 3 9 8 6
5 0 8 1 6 4 7 2 3 9

[mate#1]
3 4 5 2 0 6 7 8 1 9
8 0 6 9 5 1 2 7 4 3
7 1 4 6 8 0 3 2 9 5
6 2 9 8 4 3 1 5 7 0
1 9 3 0 7 5 8 4 2 6
4 8 2 1 6 9 5 0 3 7
2 5 0 7 9 4 6 3 8 1
5 6 8 3 2 7 9 1 0 4
9 3 7 4 1 2 0 6 5 8
0 7 1 5 3 8 4 9 6 2

[mate#2]
4 5 6 7 0 8 3 9 2 1
9 7 8 1 6 2 0 3 5 4
3 2 5 8 9 7 4 0 1 6
8 0 1 9 5 4 2 6 3 7
2 1 4 0 3 6 9 5 7 8
5 9 0 2 8 1 6 7 4 3
0 6 7 3 1 5 8 4 9 2
6 8 9 4 7 3 1 2 0 5
1 4 3 5 2 0 7 8 6 9
7 3 2 6 4 9 5 1 8 0

The main **DLK(2)** - of the “dvushka” and the orthogonal square **mate #2** are isomorphic with the following isomorphism

*T 0123456789 0123456789 4759318260

I converted the orthogonal squares so that self-orthogonal became obvious (the designations of the squares are preserved, as in Belyshev)

DLK(2)
0 3 4 6 7 2 9 8 5 1
2 1 7 9 5 3 8 6 0 4
8 6 2 5 0 9 1 3 4 7
1 5 6 3 9 7 4 0 2 8
9 8 3 2 4 6 5 1 7 0
6 7 1 0 8 5 2 4 9 3
4 9 0 7 3 8 6 5 1 2
3 4 9 8 2 1 0 7 6 5
7 2 5 4 1 0 3 9 8 6
5 0 8 1 6 4 7 2 3 9

mate#1
4 0 2 7 9 8 1 6 5 3
6 9 8 3 2 5 7 1 0 4
1 5 0 8 6 9 4 7 3 2
8 7 3 6 0 4 5 2 1 9
5 3 4 9 1 2 6 0 7 8
0 6 7 5 8 3 2 9 4 1
7 2 9 1 3 0 8 4 6 5
2 8 6 4 7 1 3 5 9 0
3 4 1 0 5 7 9 8 2 6
9 1 5 2 4 6 0 3 8 7

mate#2
0 2 8 1 9 6 4 3 7 5
3 1 6 5 8 7 9 4 2 0
4 7 2 6 3 1 0 9 5 8
6 9 5 3 2 0 7 8 4 1
7 5 0 9 4 8 3 2 1 6
2 3 9 7 6 5 8 1 0 4
9 8 1 4 5 2 6 0 3 7
8 6 3 0 1 4 5 7 9 2
5 0 4 2 7 9 1 6 8 3
1 4 7 8 0 3 2 5 6 9

We look at an illustration in this format

Fig. 5

By the way, the isomorphism connecting the main DLS and orthogonal square **mate #2** is as follows

*T 0123456789 0123456789 0123456789

This is understandable: only transpose - as defined by SODLS.

And one more interesting point: this “dvushka” produces a second “dvushka” , which also belongs to SODLS, when the Belyshev script closes

[DLK(2)]
0 3 4 6 7 2 9 8 5 1
2 1 7 9 5 3 8 6 0 4
8 6 2 5 0 9 1 3 4 7
9 5 6 3 1 7 4 0 2 8
1 8 3 2 4 6 5 9 7 0
6 7 1 0 8 5 2 4 9 3
4 9 0 7 3 8 6 5 1 2
3 4 9 8 2 1 0 7 6 5
7 2 5 4 9 0 3 1 8 6
5 0 8 1 6 4 7 2 3 9

[mate#1]
5 4 6 8 9 1 0 3 2 7
3 8 1 7 6 2 9 0 4 5

0 2 4 1 3 8 5 9 7 6
1 9 7 3 4 5 2 6 0 8
2 7 5 9 0 6 3 4 8 1
4 3 9 2 1 7 6 8 5 0
9 6 8 0 7 4 1 5 3 2
6 1 3 5 8 0 7 2 9 4
7 5 0 4 2 9 8 1 6 3
8 0 2 6 5 3 4 7 1 9

[mate#2]
5 4 6 8 9 1 0 3 2 7
3 9 1 7 6 2 8 0 4 5
0 2 4 1 3 9 5 8 7 6
1 8 7 3 4 5 2 6 0 9
2 7 5 9 0 6 3 4 8 1
4 3 8 2 1 7 6 9 5 0
8 6 9 0 7 4 1 5 3 2
6 1 3 5 8 0 7 2 9 4
7 5 0 4 2 8 9 1 6 3
9 0 2 6 5 3 4 7 1 8

as well as "odnushka", also SODLS (to be shown later).
In this "dvushka", the main DLS is isomorphic to the orthogonal square **mate #2** with such an isomorphism

*T 0123456789 0123456789 5943071268

In this case, a complete closure gives only 3 CF ODLS also belongs to SODLS

«Найдено марьяжных КФ:
count[1] = 1
count[2] = 2
Всего: 3
Найдено соквадратов: 5
КФ соквадратов: 3»

The most interesting: these two "dvushki" are relatives :)
Each "dvushka" separately gives 2 CF ODLS. It would seem that together they should give 4 CF.
But no! Together they give only 2 CF.
It turns out that **mate #1** of the first "dvushka" is isomorphic to the main DLS of the second "dvushka", and **mate #1** of the second "dvushka" is isomorphic to the main DLS of the first "dvushka".
And unique only main DLSs of "dvushka".
We look at the illustration of both "dvushka" in a format with obvious self-orthogonality

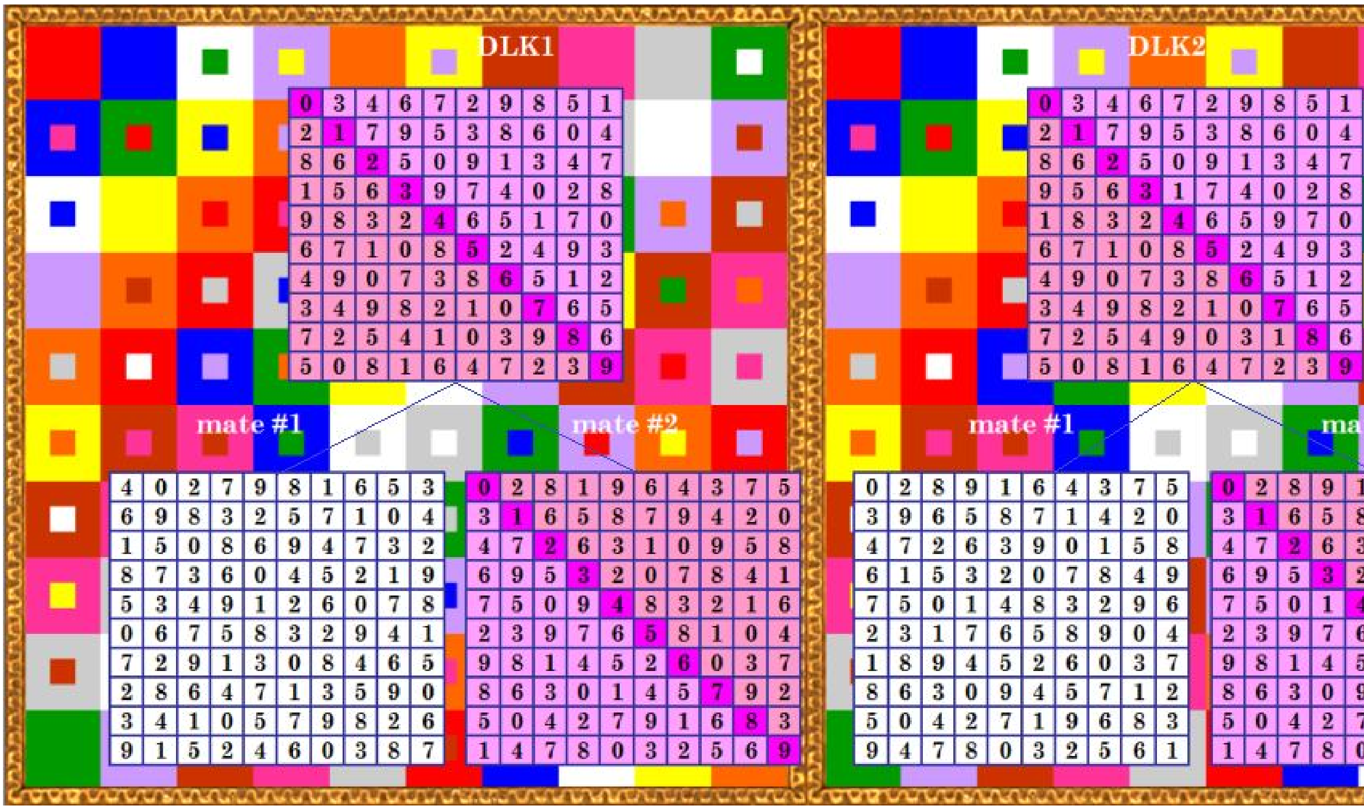

Fig. 6

Well, not without reason these two "dvushki" are relatives, because the second "dvushka" is generated by the first in the closure.
To these two "dvushki", relatives should also add a "odnushka", which is also generated by the first "dvushka" in the closure.
"Odnushka" also belongs to SODLS and adds one CF ODLS to this family of relatives.
Illustration with obvious self-orthogonality

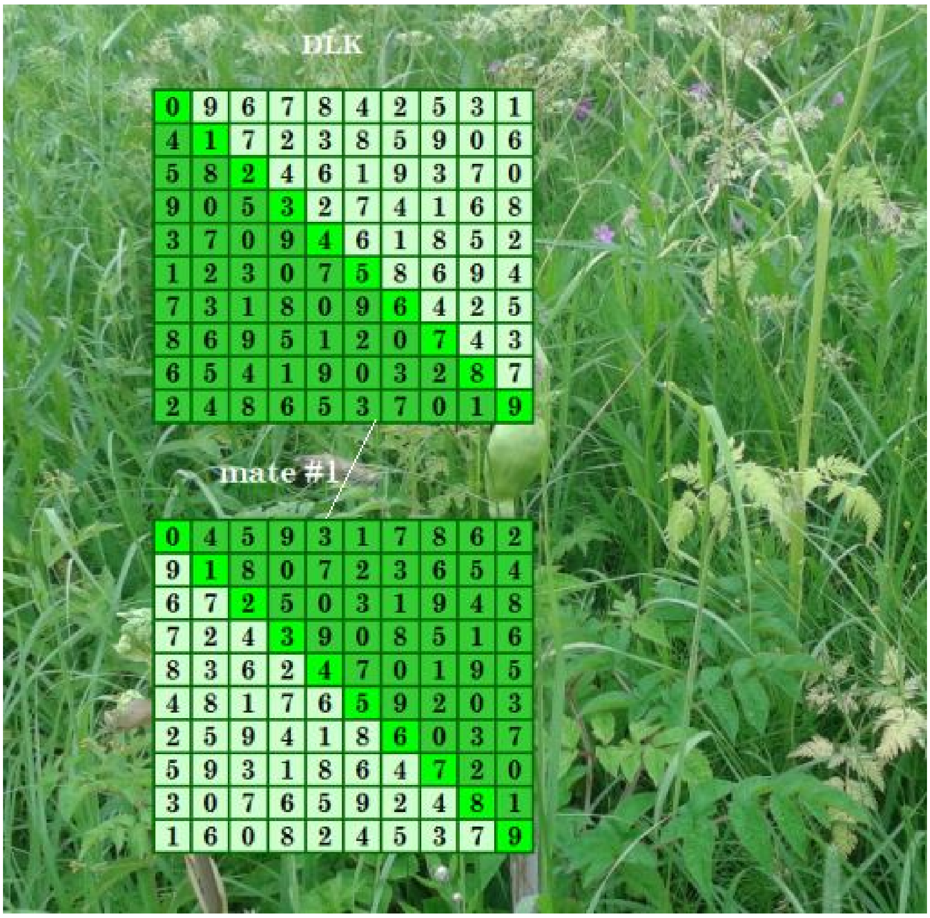

Fig. 7

Thus, we can talk about a family of ODLS configurations obtained in the closure from the first "dvushka". All three configurations contain a pair of SODLS.
Together, all three configurations give 3 CF ODLS.
If there were not relatives and not SODLS, but full-fledged "dvushki" and "odnushka", then there would be 8 CF ODLS.

2. Groups of two pairs of ODLS containing quasi-SODLS

Only three groups of ODLS pairs (quads) containing quasi-SODLS were found.
I will show them all in a format from the Belyshev program; isomorphism is indicated, which connects the main DLS of the "dvushka" and the orthogonal square isomorphic to it.

1.

[DLK(2)]
0 2 3 4 5 7 8 6 9 1
9 1 4 6 8 3 7 2 0 5
1 8 2 0 9 6 5 3 4 7
7 6 9 3 2 0 4 1 5 8
3 5 1 8 4 2 9 0 7 6
8 3 6 2 7 5 0 9 1 4
4 7 0 5 1 9 6 8 3 2
5 4 8 9 0 1 2 7 6 3
2 9 7 1 6 4 3 5 8 0

6 0 5 7 3 8 1 4 2 9

[mate#1]
8 0 2 9 5 1 3 6 7 4
6 3 5 1 7 4 2 8 9 0
0 2 1 5 3 9 8 7 4 6
3 7 8 0 4 2 6 5 1 9
1 6 7 4 2 5 9 3 0 8
5 9 3 6 8 7 0 4 2 1
7 5 6 3 9 0 4 1 8 2
4 8 0 2 1 6 7 9 5 3
9 1 4 8 0 3 5 2 6 7
2 4 9 7 6 8 1 0 3 5

[mate#2]
8 0 2 9 5 1 3 6 7 4
6 3 5 1 7 4 2 8 9 0
0 2 1 5 3 8 9 7 4 6
3 7 9 0 4 2 6 5 1 8
1 6 7 4 2 5 8 3 0 9
5 9 3 6 8 7 0 4 2 1
7 5 6 3 9 0 4 1 8 2
4 8 0 2 1 6 7 9 5 3
9 1 4 8 0 3 5 2 6 7
2 4 8 7 6 9 1 0 3 5

*T 0423186759 0423186759 8210364975

2.

[DLK(2)]
0 2 3 4 5 7 8 6 9 1
9 1 4 6 8 3 7 2 0 5
1 8 2 7 9 6 5 3 4 0
7 6 9 3 2 0 4 1 5 8
3 5 1 8 4 2 9 0 7 6
8 3 6 2 7 5 0 9 1 4
4 0 7 5 1 9 6 8 3 2
5 4 8 9 0 1 2 7 6 3
2 9 0 1 6 4 3 5 8 7
6 7 5 0 3 8 1 4 2 9

[mate#1]
8 4 2 9 3 5 6 1 0 7
1 6 3 5 0 7 2 8 9 4
4 2 5 3 6 9 8 0 7 1
6 0 8 4 7 2 1 3 5 9
5 1 0 7 2 3 9 6 4 8
3 9 6 1 8 0 4 7 2 5
0 3 1 6 9 4 7 5 8 2
7 8 4 2 5 1 0 9 3 6
9 5 7 8 4 6 3 2 1 0
2 7 9 0 1 8 5 4 6 3

[mate#2]
8 4 2 9 3 5 6 1 0 7
1 6 3 5 0 7 2 8 9 4
4 2 5 3 6 8 9 0 7 1
6 0 9 4 7 2 1 3 5 8
5 1 0 7 2 3 8 6 4 9
3 9 6 1 8 0 4 7 2 5
0 3 1 6 9 4 7 5 8 2
7 8 4 2 5 1 0 9 3 6
9 5 7 8 4 6 3 2 1 0
2 7 8 0 1 9 5 4 6 3

*T 0423186759 0423186759 8254617903

3.

[DLK(2)]
0 7 8 2 3 4 9 5 6 1
4 1 3 8 9 7 2 6 0 5
9 8 2 6 5 0 7 3 1 4
8 5 6 3 1 2 4 9 7 0
2 3 9 0 4 6 8 1 5 7
6 9 1 7 8 5 3 0 4 2
5 4 0 9 7 1 6 2 3 8
1 6 5 4 2 8 0 7 9 3
3 2 7 1 0 9 5 4 8 6
7 0 4 5 6 3 1 8 2 9

[mate#1]
3 2 4 5 6 1 8 9 7 0
9 6 3 2 5 7 1 0 8 4
2 1 0 8 3 4 6 5 9 7
6 5 9 7 8 2 3 4 0 1
8 4 7 9 2 6 0 3 1 5
5 0 1 3 7 8 9 2 4 6
0 8 6 1 9 5 4 7 2 3
7 3 2 0 4 9 5 1 6 8
1 9 8 4 0 3 7 6 5 2
4 7 5 6 1 0 2 8 3 9

[mate#2]
3 2 4 5 6 1 8 9 7 0
8 6 3 2 5 7 1 0 9 4
2 1 0 9 3 4 6 5 8 7
6 5 8 7 9 2 3 4 0 1
9 4 7 8 2 6 0 3 1 5
5 0 1 3 7 8 9 2 4 6
0 9 6 1 8 5 4 7 2 3
7 3 2 0 4 9 5 1 6 8
1 8 9 4 0 3 7 6 5 2
4 7 5 6 1 0 2 8 3 9

*T 0678541239 0678541239 3415826079

In all three examples, an isomorphism of type B.

Chapter 6. Quasi-SOLS

First, consider the LS shown in Fig. 2 left

0 1 2 3 4 5 6 9 8 7
3 6 5 4 0 8 7 1 9 2
9 0 1 7 8 6 3 2 5 4
7 9 6 5 2 3 1 0 4 8
8 2 9 1 7 4 0 5 6 3
1 3 4 9 5 2 8 6 7 0
2 5 0 8 9 7 4 3 1 6
5 4 7 6 3 9 2 8 0 1
6 7 8 2 1 0 9 4 3 5
4 8 3 0 6 1 5 7 2 9

This LS has 10 orthogonal squares, of which only 5 are unique (substantially different).
One of the orthogonal square we see in Fig. 2, this is a transposed variant of the original LS

0 3 9 7 8 1 2 5 6 4
1 6 0 9 2 3 5 4 7 8
2 5 1 6 9 4 0 7 8 3
3 4 7 5 1 9 8 6 2 0
4 0 8 2 7 5 9 3 1 6
5 8 6 3 4 2 7 9 0 1
6 7 3 1 0 8 4 2 9 5
9 1 2 0 5 6 3 8 4 7
8 9 5 4 6 7 1 0 3 2
7 2 4 8 3 0 6 1 5 9

The original LS and its transposed variant have the following symmetries (Belyshev's program **find_symm_3.0**)

«Поиск симметрий ЛК10 версия 3.0

Обработано ЛК: 2
Время работы : 0 сек

Введите код симметрии: all

Квадратов с симметрией (27,27,27) найдено: 1 они записаны в файл symm_27_27_27.txt
Квадратов с симметрией (30,30,30) найдено: 1 они записаны в файл symm_30_30_30.txt»

Let's look at the isomorphisms connecting the original LS and its transposed variant, there 9 of them:

*T 0123456789 0123456789 0123456789
*T 1234567809 1234567809 3678210549
*T 2345678019 2345678019 8054763129
*T 3456780129 3456780129 4312508679
*T 4567801239 4567801239 2867134059

*T 5678012349 5678012349 7405682319
*T 6780123459 6780123459 5231047869
*T 7801234569 7801234569 1786325409
*T 8012345679 8012345679 6540871239

The first isomorphism in the list is only transposition, all other isomorphisms are of type B.
Thus, the first orthogonal pair is SOLS in the classical sense.
And now let's look at the remaining orthogonal squares of the original LS, here are shown only significantly different (not isomorphic):

0 3 9 7 8 1 2 5 6 4
8 4 0 2 9 3 5 1 7 6
3 6 7 9 4 5 0 2 8 1
1 2 3 6 0 9 8 4 5 7
2 5 8 4 7 6 1 9 0 3
6 1 4 0 3 8 9 7 2 5
4 7 2 5 1 0 3 6 9 8
5 9 6 1 2 4 7 8 3 0
9 8 1 3 5 7 6 0 4 2
7 0 5 8 6 2 4 3 1 9

0 3 9 7 8 1 2 5 6 4
1 4 0 5 7 3 8 2 9 6
3 6 8 1 2 9 0 4 5 7
7 1 3 6 0 2 9 8 4 5
8 7 2 4 5 6 1 9 0 3
6 5 1 0 3 8 4 7 2 9
2 8 5 9 4 0 3 6 7 1
4 2 6 8 9 7 5 1 3 0
5 9 7 3 1 4 6 0 8 2
9 0 4 2 6 5 7 3 1 8

0 3 9 7 8 1 2 5 6 4
5 9 6 1 4 0 3 2 7 8
4 7 1 5 3 8 9 6 2 0
2 8 4 3 7 6 0 1 5 9
7 2 0 8 6 4 5 9 3 1
6 4 2 9 0 5 8 7 1 3
3 5 8 2 1 9 7 0 4 6
8 6 7 0 2 3 1 4 9 5
1 0 5 4 9 2 6 3 8 7
9 1 3 6 5 7 4 8 0 2

0 3 9 7 8 1 2 5 6 4
3 7 5 6 1 4 9 2 8 0
4 5 1 8 3 9 0 6 2 7
2 0 8 9 4 5 6 7 3 1
5 2 7 4 6 0 8 3 1 9
9 8 2 1 0 3 7 4 5 6
8 6 4 2 9 7 5 1 0 3
7 4 3 0 2 6 1 8 9 5
6 1 0 5 7 2 3 9 4 8
1 9 6 3 5 8 4 0 7 2

All these orthogonal squares are not isomorphic to the original LS.
Among the remaining 5 orthogonal co-squares (isomorphic to those shown), there is also one orthogonal square that is isomorphic to the original LS

0 3 9 7 8 1 2 5 6 4
4 1 5 3 9 8 6 7 0 2
1 2 6 0 5 3 8 4 9 7
3 6 7 4 1 0 5 8 2 9
7 5 4 9 2 6 1 0 8 3
8 9 0 2 3 7 4 6 1 5
6 8 3 1 7 5 9 2 4 0
2 4 8 5 6 9 0 3 7 1
9 7 2 8 0 4 3 1 5 6
5 0 1 6 4 2 7 9 3 8

with the following isomorphisms

*T 0123456789 7801234569 9567403128
*T 1234567809 8012345679 7312659048
*T 2345678019 0123456789 2904137568
*T 3456780129 1234567809 4756092318
*T 4567801239 2345678019 6231574908
*T 5678012349 3456780129 1490326758
*T 6780123459 4567801239 0675941238
*T 7801234569 5678012349 5123760498
*T 8012345679 6780123459 3049215678

All these isomorphisms are of type C.
Thus, we have a group of 10 pairs of OLS; the main LS is isomorphic to two of the orthogonal squares; one of the orthogonal pairs is SOLS, the second is quasi-SOLS, in an expanded sense than the definition of quasi-SODLS given above.

Now let's look at Lyamzin's LS, I found it on the Internet a long time ago, when I was just starting to work on 10th-order LS

0	1	2	3	4	5	6	7	8	9
1	0	5	8	3	2	7	9	6	4
2	5	0	6	9	4	3	8	1	7
3	8	6	0	7	1	5	4	9	2
4	3	9	7	0	8	2	6	5	1
5	2	4	1	8	0	9	3	7	6
6	7	3	5	2	9	0	1	4	8
7	9	8	4	6	3	1	0	2	5
8	6	1	9	5	7	4	2	0	3
9	4	7	2	1	6	8	5	3	0

Fig. 8

This is a search for pseudo-triple from LS Lyamzin by Belyshev's program

(1-0,2-142 tr=872) cm=82 sq=2

```
 0 11 22 33 44 55 66 77 88 99
12  5 50 86 39 24 73 98 61 47
23 58  6 60 97 41 35 84 19 72
34 83 69  7 70 18 52 46 95 21
45 32 94 71  8 80 29 63 57 16
56 27 43 15 82  9 90 31 74 68
67 79 38 54 26 93  1 10 42 85
78 96 81 49 65 37 14  2 20 53
89 64 17 92 51 76 48 25  3 30
91 40 75 28 13 62 87 59 36  4

 0 11 22 33 44 55 66 77 88 99
19  4 57 82 31 26 78 95 63 40
21 50  5 68 93 42 37 89 16 74
32 85 60  6 79 14 53 48 91 27
43 38 96 70  7 81 25 64 59 12
54 23 49 17 80  8 92 36 75 61
65 72 34 51 28 90  9 13 47 86
76 97 83 45 62 39 10  1 24 58
87 69 18 94 56 73 41 20  2 35
98 46 71 29 15 67 84 52 30  3
```

Here, we see that Lyamzin's LS has two orthogonal squares (sq=2).
I will show this OLS group in another format

```
LS
0 1 2 3 4 5 6 7 8 9
1 0 5 8 3 2 7 9 6 4
2 5 0 6 9 4 3 8 1 7
3 8 6 0 7 1 5 4 9 2
4 3 9 7 0 8 2 6 5 1
5 2 4 1 8 0 9 3 7 6
6 7 3 5 2 9 0 1 4 8
7 9 8 4 6 3 1 0 2 5
8 6 1 9 5 7 4 2 0 3
9 4 7 2 1 6 8 5 3 0

sq1
 0 1 2 3 4 5 6 7 8 9
 2 5 0 6 9 4 3 8 1 7
 3 8 6 0 7 1 5 4 9 2
 4 3 9 7 0 8 2 6 5 1
 5 2 4 1 8 0 9 3 7 6
 6 7 3 5 2 9 0 1 4 8
 7 9 8 4 6 3 1 0 2 5
 8 6 1 9 5 7 4 2 0 3
 9 4 7 2 1 6 8 5 3 0
 1 0 5 8 3 2 7 9 6 4

sq2
 0 1 2 3 4 5 6 7 8 9
 9 4 7 2 1 6 8 5 3 0
 1 0 5 8 3 2 7 9 6 4
 2 5 0 6 9 4 3 8 1 7
```

3 8 6 0 7 1 5 4 9 2
4 3 9 7 0 8 2 6 5 1
5 2 4 1 8 0 9 3 7 6
6 7 3 5 2 9 0 1 4 8
7 9 8 4 6 3 1 0 2 5
8 6 1 9 5 7 4 2 0 3

Here, both orthogonal squares are isomorphic to the main LS and are isomorphic to each other; it is obvious that they are obtained from the main LS (as well as from each other) by rearranging the rows. But not just rows swapping!
The isomorphisms connecting the main LS with the orthogonal squadron **sq1** are as follows:

** 0123456789 0912345678 0234567891
** 0234567891 0123456789 0123456789
** 0345678912 0234567891 0912345678
** 0456789123 0345678912 0891234567
** 0567891234 0456789123 0789123456
** 0678912345 0567891234 0678912345
** 0789123456 0678912345 0567891234
** 0891234567 0789123456 0456789123
** 0912345678 0891234567 0345678912
*T 0123456789 0912345678 0234567891
*T 0234567891 0123456789 0123456789
*T 0345678912 0234567891 0912345678
*T 0456789123 0345678912 0891234567
*T 0567891234 0456789123 0789123456
*T 0678912345 0567891234 0678912345
*T 0789123456 0678912345 0567891234
*T 0891234567 0789123456 0456789123
*T 0912345678 0891234567 0345678912

The isomorphisms connecting the main LS with the orthogonal squadron **sq2** are as follows:

** 0123456789 0234567891 0912345678
** 0234567891 0345678912 0891234567
** 0345678912 0456789123 0789123456
** 0456789123 0567891234 0678912345
** 0567891234 0678912345 0567891234
** 0678912345 0789123456 0456789123
** 0789123456 0891234567 0345678912
** 0891234567 0912345678 0234567891
** 0912345678 0123456789 0123456789
*T 0123456789 0234567891 0912345678
*T 0234567891 0345678912 0891234567
*T 0345678912 0456789123 0789123456
*T 0456789123 0567891234 0678912345
*T 0567891234 0678912345 0567891234
*T 0678912345 0789123456 0456789123
*T 0789123456 0891234567 0345678912
*T 0891234567 0912345678 0234567891
*T 0912345678 0123456789 0123456789

The isomorphisms connecting the orthogonal squares **sq1** and **sq2** are as follows:

** 0123456789 0345678912 0891234567

** 0234567891 0456789123 0789123456
** 0345678912 0567891234 0678912345
** 0456789123 0678912345 0567891234
** 0567891234 0789123456 0456789123
** 0678912345 0891234567 0345678912
** 0789123456 0912345678 0234567891
** 0891234567 0123456789 0123456789
** 0912345678 0234567891 0912345678
*T 0123456789 0123456789 0912345678
*T 0234567891 0234567891 0891234567
*T 0345678912 0345678912 0789123456
*T 0456789123 0456789123 0678912345
*T 0567891234 0567891234 0567891234
*T 0678912345 0678912345 0456789123
*T 0789123456 0789123456 0345678912
*T 0891234567 0891234567 0234567891
*T 0912345678 0912345678 0123456789

This LS expands the concept of quasi-SOLS: here we have not only isomorphisms with transpose, permutation of rows/columns and re-designation of elements, but also isomorphisms without transpose - only permutation of rows/columns and re-designation of elements.
There is even a case of only line swapping:
between the main LS and sq1
** 0234567891 0123456789 0123456789
between the main LS and sq2
** 0912345678 0123456789 0123456789

Unfortunately, **sq1** and **sq2** are not orthogonal (orthogonality is only in **82** cells out of 100). The Lyamzin's square until did not reach a little to the MOLS triple.
But this is a good approximation to the three MOLS. The next approximation to the MOLS triple is my pseudo-triple with orthogonality characteristic **88** (shown above).
Finally, the last known approximation is the pseudo-triple with orthogonality characteristic **91** (the author of this pseudo-triple J. Wanless).

We look at the symmetries that the Lyamzin square has

«Поиск симметрий ЛК10 версия 3.0

Обработано ЛК: 1
Время работы: 0.015 сек

Введите код симметрии: all

Квадратов с симметрией (27,27,27) найдено: 1 они записаны в файл symm_27_27_27.txt
Квадратов с симметрией (30,30,30) найдено: 1 они записаны в файл symm_30_30_30.txt
Квадратов с симметрией (1,1,1)+ найдено: 1 они записаны в файл symm_1_1_1p.txt
Квадратов с симметрией (1,27,27)+ найдено: 1 они записаны в файл symm_1_27_27p.txt
Квадратов с симметрией (1,30,30)+ найдено: 1 они записаны в файл symm_1_30_30p.txt»

And these are the symmetries of the orthogonal squares **sq1** and **sq2**

«Поиск симметрий ЛК10 версия 3.0

Обработано ЛК: 1
Время работы: 0.015 сек

Введите код симметрии: all

Квадратов с симметрией (27,27,27) найдено: 1 они записаны в файл symm_27_27_27.txt
Квадратов с симметрией (30,30,30) найдено: 1 они записаны в файл symm_30_30_30.txt
Квадратов с симметрией (1,1,1)+ найдено: 1 они записаны в файл symm_1_1_1p.txt
Квадратов с симметрией (1,27,27)+ найдено: 1 они записаны в файл symm_1_27_27p.txt
Квадратов с симметрией (1,30,30)+ найдено: 1 они записаны в файл symm_1_30_30p.txt»

The same list of symmetries is as for the main LS, which is not surprising, since sq1 and sq2 are isomorphic to the main LS.

Finally, check LS Lyamzin by Harry White's utility **GetType**

«Tuesday 2020-04-07 13:04:15

Order? 10

Enter the name of the squares file: input
.. writing type information to file inputTypeDetail_8.txt

Counts
------
    1 Latin
    1 nfr
    1 nfc
    1 nfr nfc
    1 self-transpose»

nfr means normalized LS (the first row is naturally ordered); **nfc** means the first column is naturally ordered;
self-transpose means the coincidence of the LS with its transposed variant (i.e., the LS is diagonally symmetric).
I assume that **nfr nfc** means reduced LS, that is, with normalized (naturally ordered) first row and first column.

Conclusion

This article attempts to classify SOLS and SODLS 10th order.
Classification is done using empirical data.
Perhaps, using a theoretical approach, a more perfect classification can be made. I think that such a classification will appear over time.
The main essence of the article is the hypothesis that there are no other quasi-SODLS of the 10th order other than those obtained by closing classical SODLS.
It is clear that the hypothesis must be proved or disproved. To refute the hypothesis, it is enough to find one counterexample.
The article uses some elements of Belyshev's theory of LS without a detailed explanation, which may complicate the reading of the article.

Unfortunately, I do not know Belyshev's articles describing these elements of the theory so that I can send the reader these articles.
All this was described on the boinc.ru forum, but it also disappeared when the forum disappeared.
The newly created boinc.ru forum has not these descriptions.
Belyshev's theory of LS is large, a lot has been created for symmetry.
Not without reason did I advise him not to confine himself to a forum description (where torn pieces were obtained, difficult to understand), but to write complete articles on each topic.
I have 224832 normalized SODLS of order 9, as well as 470 main classes of this order; they were sent to me by Francis Gaspalou when we discussed with him the search for the main classes.
If you are interested in these SODLS, write to me.

References

Latin Square Articles
https://boinc.progger.info/odlk/forum_thread.php?id=54
http://web.archive.org/web/20191225122419/https://boinc.progger.info/odlk/forum_thread.php?id=54
About pseudo-triples
https://boinc.progger.info/odlk/forum_thread.php?id=101
http://web.archive.org/web/20200213111643/https://boinc.progger.info/odlk/forum_thread.php?id=101
The symmetry (27,27,27)
https://boinc.progger.info/odlk/forum_thread.php?id=96
http://web.archive.org/web/20200204080544/https://boinc.progger.info/odlk/forum_thread.php?id=96
SOLS(10) and SODLS(10)
https://boinc.progger.info/odlk/forum_thread.php?id=97
http://web.archive.org/web/20200212035451/https://boinc.progger.info/odlk/forum_thread.php?id=97
Harry White. Doubly Self-Orthogonal
http://budshaw.ca/addenda/SODLSnotes.html
Harry White. Self-orthogonal Diagonal Latin Squares
http://budshaw.ca/SODLS.html
121642 representatives of RC-paratopism class SOLS of order 10
http://www.vuuren.co.za/data/SOLS_Repository/SOLSNonRCParatopic10x10.txt
30534 SODLS of the 10th order, found by A. Belyshev from 121642 SOLS of the 10th order
https://yadi.sk/d/l4Fus2cp3GTqe2
30502 CF SODLS 10th order found by A. Belyshev from 121642 SOLS 10th order
https://yadi.sk/d/OiKwfw2b3Gan9x
Number of self-orthogonal diagonal Latin squares of order n with ordered first string
https://oeis.org/A287761
Number of main classes of self-orthogonal diagonal Latin squares of order n
https://oeis.org/A329685
Number of main classes of doubly self-orthogonal diagonal Latin squares of order n
https://oeis.org/A333366

Note: links to the forum of the ODLK BOINC project are given in two variant
a) to the forum;
b) to the web archive.

Translated by Natalia Makarova, edited by Franz-Xaver Harvanek.

May 16, 2020

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