Isothermal anelastic approximation

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Abstract

This paper constructs a variant of the anelastic approximation; its energetics and the conditions for its applicability, in addition, are elucidated. It is shown that the Boussinesq approximation is reproducible as a limiting case of this variant. On the other hand, it also turns out that the variant is inapplicable to an ideal gas.

1. Introduction

The anelastic approximation was devised by Ogura & Phillips (1962) to describe the motion of a thermally stratified ideal gas. It partly takes account of the compressibility of a gas, but excludes sound waves from the solutions of the governing equations.

Maruyama (2021) reconstructed this approximation in such a manner that it can be applied to any kind of fluid. He found, as a result, that the approximation has no direct relation to the classical Boussinesq approximation often used to study free convection.

The purpose of this paper is to present a variant of the anelastic approximation. The Boussinesq approximation is reproducible as a limiting case of this *isothermal* anelastic approximation, though this variant proves to be inapplicable to an ideal gas.

2. Isothermal anelastic approximation

We consider the motion of an inviscid fluid in a uniform gravitational field. The fluid is assumed to be contained in a fixed finite domain Ω. In the domain, the z-axis is taken vertically upwards: the unit vector in the positive z-direction is denoted by $\boldsymbol{k}$.

2.1. Equation of motion

When the velocity of the fluid is denoted by $\boldsymbol{u}$, the equation of motion is given by

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p/\rho - g\boldsymbol{k}, \tag{2.1}$$

in which D/Dt stands for the material derivative; p and ρ are, respectively, the pressure and the density of the fluid, and g is the acceleration due to gravity.

Now, let φ denote the specific Gibbs free energy of the fluid; φ satisfies the relation

$$d\varphi = -sdT + vdp, \tag{2.2}$$

where s and T are, respectively, the specific entropy and the temperature of the fluid; $v = 1/\rho$ denotes the specific volume of the fluid. In the following, all thermodynamic quantities are regarded as known functions of φ and T.

The relation (2.2) enables us to rewrite the equation of motion (2.1) in the form

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla\varphi - s\nabla T - g\boldsymbol{k}. \tag{2.3}$$

Let us now decompose φ and T as follows:

$$\varphi = \varphi_0 + \varphi', \quad T = T_0 + T'. \tag{2.4}$$

Here φ_0 and T_0 are defined by

$$\varphi_0 = -gz + c_1, \quad T_0 = c_2, \tag{2.5}$$

with c_1 and c_2 being constants. Then (2.3) can be written as

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla\varphi' - s\nabla T'. \tag{2.6}$$

However, regarded as a function of φ and T, s can also be decomposed as follows:

$$s = s_0 + s', \tag{2.7}$$

where s_0 is defined by

$$s_0 = s(\varphi_0, T_0). \tag{2.8}$$

We introduce here the following assumption:

$$|s'/s_0| \ll 1. \tag{2.9}$$

This assumption allows us to approximate (2.6) as follows:

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla\varphi' - s_0\nabla T'. \tag{2.10}$$

Finally, using the identity $s_0\nabla T' = \nabla(s_0 T') - T'\nabla s_0$, we get

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla(\varphi' + s_0 T') + T'\nabla s_0. \tag{2.11}$$

This is the equation of motion under the isothermal anelastic approximation: the last term of this equation represents the buoyancy force in the approximation.

2.2. Equation of continuity

In view of the decomposition (2.4), the density ρ of the fluid is decomposed as

$$\rho = \rho_0 + \rho', \tag{2.12}$$

where ρ_0 is defined by

$$\rho_0 = \rho(\varphi_0, T_0). \tag{2.13}$$

We now introduce the following assumption:

$$|\rho'/\rho_0| \ll 1. \tag{2.14}$$

Then ρ may be approximated as follows:

$$\rho = \rho_0. \tag{2.15}$$

Substituting (2.15) into the equation of continuity $\partial\rho/\partial t + \nabla\cdot(\rho\boldsymbol{u}) = 0$, we have

$$\nabla\cdot(\rho_0\boldsymbol{u}) = 0. \tag{2.16}$$

This is the equation of continuity under the isothermal anelastic approximation.

2.3. Adiabatic equation

When the conduction of heat is neglected, the general equation of heat transfer (see Landau & Lifshitz 1987, § 49) takes the following form:

$$\rho_0 T\frac{Ds}{Dt} = 0. \tag{2.17}$$

where the approximation (2.15) has been used. In view of (2.4) and (2.7), this equation can be written, to the first order of primed variables, as follows:

$$\rho_0 T_0\frac{Ds_0}{Dt} + \rho_0 T'\frac{Ds_0}{Dt} + \rho_0 T_0\frac{Ds'}{Dt} = 0. \tag{2.18}$$

Alternatively, since $Ds_0/Dt = \boldsymbol{u}\cdot\nabla s_0$, we have

$$\rho_0 T_0\frac{Ds'}{Dt} = -\rho_0 T_0\boldsymbol{u}\cdot\nabla s_0 - \rho_0 T'\boldsymbol{u}\cdot\nabla s_0. \tag{2.19}$$

This gives the adiabatic equation under the isothermal anelastic approximation.

In order to close the system of equations, however, we need to express s' in (2.19) in terms of φ' and T'. To this end, we first note the following thermodynamic relations:

$$(\partial s/\partial\varphi)_T = -\beta, \quad (\partial s/\partial T)_\varphi = c_p/T - \beta s, \tag{2.20}$$

where $\beta = v^{-1}(\partial v/\partial T)_p$ is the thermal expansion coefficient, and c_p the specific heat at constant pressure. Hence, to the first order of φ' and T', s' can be written as

$$s' = (\partial s/\partial\varphi)_T|_{(\varphi_0,T_0)}\varphi' + (\partial s/\partial T)_\varphi|_{(\varphi_0,T_0)}T' = -\beta_0\varphi' + (c_{p0}/T_0 - \beta_0 s_0)T', \tag{2.21}$$

in which β_0 and c_{p0} are defined by

$$\beta_0 = \beta(\varphi_0, T_0), \quad c_{p0} = c_p(\varphi_0, T_0). \tag{2.22}$$

We have thus completed the formulation of the isothermal anelastic approximation.

2.4. Energetics of the isothermal anelastic approximation

Let us next investigate the energetics of the isothermal anelastic approximation. We first consider the specific internal energy e of the fluid: it satisfies the formula

$$e = \varphi - p/\rho + Ts. \tag{2.23}$$

Regarded as a function of φ and T, the pressure p in (2.23) can be decomposed as

$$p = p_0 + p', \tag{2.24}$$

with p_0 defined by

$$p_0 = p(\varphi_0, T_0). \tag{2.25}$$

Substituting (2.4), (2.7), and (2.24) into (2.23), and using the approximation (2.15), we get, to the first order of primed variables,

$$e = (\varphi_0 - p_0/\rho_0 + T_0 s_0) + (\varphi' - p'/\rho_0 + T_0 s' + s_0 T'). \tag{2.26}$$

However, from (2.2), the following thermodynamic relations are obtained:

$$(\partial p/\partial \varphi)_T = \rho, \quad (\partial p/\partial T)_\varphi = \rho s. \tag{2.27}$$

Accordingly, to the first order of φ' and T', p' is expressed as follows:

$$p' = (\partial p/\partial \varphi)_T|_{(\varphi_0,T_0)}\varphi' + (\partial p/\partial T)_\varphi|_{(\varphi_0,T_0)}T' = \rho_0\varphi' + \rho_0 s_0 T'. \tag{2.28}$$

This expression enables us to write

$$e = (\varphi_0 - p_0/\rho_0 + T_0 s_0) + T_0 s'. \tag{2.29}$$

Taking the material derivative of (2.29), and multiplying the result by ρ_0, we have

$$\rho_0 \frac{De}{Dt} = \rho_0 \boldsymbol{u} \cdot \nabla(\varphi_0 - p_0/\rho_0) + \rho_0 T_0 \boldsymbol{u} \cdot \nabla s_0 + \rho_0 T_0 \frac{Ds'}{Dt}. \tag{2.30}$$

Furthermore, the substitution of (2.19) into (2.30) leads to

$$\rho_0 \frac{De}{Dt} = \nabla \cdot \{\rho_0(\varphi_0 - p_0/\rho_0)\boldsymbol{u}\} - \rho_0 T' \boldsymbol{u} \cdot \nabla s_0, \tag{2.31}$$

where (2.16) has been used. We note, however, that $\rho_0 De/Dt$ can be written as

$$\rho_0 \frac{De}{Dt} = \rho_0 \frac{\partial e}{\partial t} + \rho_0 \boldsymbol{u} \cdot \nabla e = \frac{\partial}{\partial t}(\rho_0 e) + \nabla \cdot (\rho_0 e \boldsymbol{u}). \tag{2.32}$$

Hence, using (2.29), we can rewrite (2.31) as follows:

$$\frac{\partial}{\partial t}(\rho_0 e) + \nabla \cdot \big\{\rho_0 T_0 (s_0 + s')\boldsymbol{u}\big\} = -\rho_0 T' \boldsymbol{u} \cdot \nabla s_0. \tag{2.33}$$

Now, the integration of (2.33) over the domain Ω containing the fluid yields

$$\frac{d}{dt}\int_\Omega \rho_0 e\, dV = -\int_\Omega \rho_0 T' \boldsymbol{u}\cdot\nabla s_0\, dV. \tag{2.34}$$

Here it has been assumed that the normal component of $\boldsymbol{u}$ vanishes on the boundary of Ω. This is the equation for the rate of change of the internal energy of the fluid.

As for the potential energy of the fluid, it is evidently invariant:

$$\frac{d}{dt}\int_\Omega \rho_0 g z\, dV = 0. \tag{2.35}$$

This is a logical consequence of the approximation (2.15).

In order to find the equation for the rate of change of the kinetic energy of the fluid, we first rewrite, using (2.28), the equation of motion (2.11) in the following form:

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla(p'/\rho_0) + T'\nabla s_0. \tag{2.36}$$

Taking the inner product of (2.36) with $\rho_0\boldsymbol{u}$, we get, after some manipulation,

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho_0|\boldsymbol{u}|^2) + \nabla\cdot\left\{\rho_0(\tfrac{1}{2}|\boldsymbol{u}|^2 + p'/\rho_0)\boldsymbol{u}\right\} = \rho_0 T'\boldsymbol{u}\cdot\nabla s_0. \tag{2.37}$$

When the normal component of $\boldsymbol{u}$ vanishes on the boundary of Ω, (2.37) yields

$$\frac{d}{dt}\int_\Omega \tfrac{1}{2}\rho_0|\boldsymbol{u}|^2 dV = \int_\Omega \rho_0 T'\boldsymbol{u}\cdot\nabla s_0\, dV. \tag{2.38}$$

We have thus obtained the equation for the rate of change of the kinetic energy of the fluid: the term on the right-hand side represents the work done by the buoyancy force.

Adding all the energy equations (2.34), (2.35), and (2.38), we finally get

$$\frac{d}{dt}\int_\Omega \rho_0\left(\tfrac{1}{2}|\boldsymbol{u}|^2 + gz + e\right) dV = 0. \tag{2.39}$$

Thus the total energy of the fluid is conserved; this result confirms that the isothermal anelastic approximation is consistent with the conservation law of energy.

Furthermore, by comparing (2.34) and (2.38), we recognize that the work done by the buoyancy force in the isothermal anelastic approximation corresponds to the conversion between kinetic and internal energy. This conclusion coincides with that arrived at by Maruyama (2014) concerning the buoyancy force in the Boussinesq approximation.

2.5. Applicability of the isothermal anelastic approximation

The isothermal anelastic approximation was derived on the basis of the assumptions (2.9) and (2.14). Our objective in the following is to clarify under what conditions these two assumptions are justifiable; we begin by considering (2.14).

We first direct our attention to the following thermodynamic relations:

$$(\partial\rho/\partial\varphi)_T = \rho(\gamma/a^2), \quad (\partial\rho/\partial T)_\varphi = \rho\left\{s(\gamma/a^2) - \beta\right\}, \tag{2.40}$$

in which γ is the ratio of specific heats, and a the speed of sound. On the basis of these relations, ρ' in (2.14) can be expressed, to the first order of φ' and T', as follows:

$$\begin{aligned}\rho' &= (\partial\rho/\partial\varphi)_T|_{(\varphi_0,T_0)}\varphi' + (\partial\rho/\partial T)_\varphi|_{(\varphi_0,T_0)}T' \\ &= \rho_0(\gamma_0/a_0^2)\varphi' + \rho_0\left\{s_0(\gamma_0/a_0^2) - \beta_0\right\}T',\end{aligned} \tag{2.41}$$

where we have introduced the notation

$$\gamma_0 = \gamma(\varphi_0, T_0), \quad a_0 = a(\varphi_0, T_0). \tag{2.42}$$

This expression for ρ' enables us to estimate the left-hand side of (2.14) as

$$\begin{aligned}|\rho'/\rho_0| &= O\{\gamma_0(gH/a_0^2)(\Delta\varphi'/gH)\} \\ &\quad + O\{\gamma_0(gH/a_0^2)(s_0\Delta T'/gH)\} + O(\beta_0\Delta T').\end{aligned} \tag{2.43}$$

Here H denotes the vertical extent of the domain Ω containing the fluid; $\Delta\varphi'$ and $\Delta T'$ are the characteristic scales of φ' and T', respectively. Now, after Maruyama (2021), we assume that the following condition applies:

$$(gH)^{1/2}/a_0 \leq O(1). \tag{2.44}$$

Then, since $\gamma_0 = O(1)$, (2.14) is satisfied under the following conditions:

$$\Delta\varphi'/gH \ll 1, \tag{2.45}$$

$$s_0\Delta T'/gH \ll 1, \tag{2.46}$$

$$\beta_0\Delta T' \ll 1. \tag{2.47}$$

On the other hand, using (2.21), we can estimate the left-hand side of (2.9) as

$$\begin{aligned}|s'/s_0| &= O\{(c_{p0}/s_0)(\Gamma_0 H/\Delta T')(\Delta T'/T_0)(\Delta\varphi'/gH)\} \\ &\quad + O\{(c_{p0}/s_0)(\Delta T'/T_0)\} + O(\beta_0\Delta T'),\end{aligned} \tag{2.48}$$

where $\Gamma_0 = \beta_0 T_0 g/c_{p0}$ stands for the adiabatic lapse rate. Since the motion of the fluid is adiabatic, we expect that the following condition applies:

$$\Gamma_0 H/\Delta T' \leq O(1). \tag{2.49}$$

Furthermore, since $c_{p0} = T_0(\partial s/\partial T)_p|_{(\varphi_0,T_0)}$, it is reasonable to assume that

$$c_{p0}/s_0 \leq O(1). \tag{2.50}$$

Then we see from (2.48) that (2.9) is satisfied when the condition

$$\Delta T'/T_0 \ll 1 \tag{2.51}$$

holds together with (2.45) and (2.47).

We should note, however, that the following inequality can be obtained from (2.6):

$$|\nabla\varphi'| \le |\partial \boldsymbol{u}/\partial t| + |(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}| + |s\nabla T'|. \tag{2.52}$$

Denoting by L the length scale characteristic of the motion of the fluid, we can write

$$|\nabla\varphi'| = O(\Delta\varphi'/L), \quad |\nabla T'| = O(\Delta T'/L). \tag{2.53}$$

Also, if U denotes the velocity scale characteristic of the motion, we obtain

$$|\partial \boldsymbol{u}/\partial t| = O(U/\tau), \quad |(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}| = O(U^2/L), \tag{2.54}$$

where τ stands for the time scale characteristic of the motion. We now assume that c_1 and c_2 in (2.5) can be chosen so that the following condition is fulfilled:

$$s/s_0 \le O(1). \tag{2.55}$$

Then it follows from (2.52), (2.53), and (2.54) that, when the conditions

$$U/(gH)^{1/2} \ll 1, \quad (L/\tau)/(gH)^{1/2} \ll 1 \tag{2.56}$$

hold together with (2.46), the condition (2.45) is satisfied.

We should also note that (2.46) can be expressed in the form

$$\beta_0 \Delta T' \ll (c_{p0}/s_0)(\Gamma_0 H/\Delta T')(\Delta T'/T_0). \tag{2.57}$$

Hence (2.47) is satisfied under the conditions (2.46), (2.49), (2.50), and (2.51).

In summary, the isothermal anelastic approximation is applicable when the following conditions hold: (2.44), (2.46), (2.49), (2.50), (2.51), (2.55), and (2.56).

3. Application to a shallow layer of fluid

In this section, we apply the isothermal anelastic approximation to a "shallow" layer of fluid; our aim is to reproduce the Boussinesq approximation as a limiting case.

The density $\rho = \rho_0$ of the fluid considered in the previous section varies with height:

$$\nabla\rho_0 = (\partial\rho/\partial\varphi)_T|_{(\varphi_0,T_0)}\nabla\varphi_0 + (\partial\rho/\partial T)_\varphi|_{(\varphi_0,T_0)}\nabla T_0 = -(\rho_0\gamma_0 g/a_0^2)\boldsymbol{k}. \tag{3.1}$$

However, ρ_0 may be regarded as constant when the following condition is fulfilled:

$$\Delta\rho_0/\rho_0 \ll 1. \tag{3.2}$$

Here $\Delta\rho_0$ denotes the variation scale of ρ_0 over the vertical extent H of the domain Ω containing the fluid: it is reasonable to put

$$\Delta\rho_0 = |\nabla\rho_0|H = \rho_0\gamma_0 gH/a_0^2. \tag{3.3}$$

Since $\gamma_0 = O(1)$, we realize that ρ_0 may be regarded as constant under the condition

$$(gH)^{1/2}/a_0 \ll 1. \tag{3.4}$$

In this case, the equation of continuity $\partial\rho/\partial t + \nabla\cdot(\rho\boldsymbol{u}) = 0$ reduces to

$$\nabla\cdot\boldsymbol{u} = 0. \tag{3.5}$$

The equation of motion, written in the form (2.36), also reduces to

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p'/\rho_0 + \beta_0 T' g\boldsymbol{k}, \tag{3.6}$$

where we have used the following expression obtained from (2.5) and (2.20):

$$\nabla s_0 = (\partial s/\partial\varphi)_T|_{(\varphi_0,T_0)}\nabla\varphi_0 + (\partial s/\partial T)_\varphi|_{(\varphi_0,T_0)}\nabla T_0 = \beta_0 g\boldsymbol{k}. \tag{3.7}$$

These equations are the same as those applying under the Boussinesq approximation.

We next turn our attention to the general equation of heat transfer (2.17). With the aid of (2.20), it can be expressed, in terms of φ and T, as follows:

$$-\rho_0\beta T\frac{D\varphi}{Dt} + (\rho_0 c_p - \rho_0\beta T s)\frac{DT}{Dt} = 0. \tag{3.8}$$

Similarly to (2.4) and (2.7), c_p and β in this equation can be decomposed as

$$c_p = c_{p0} + c_p', \quad \beta = \beta_0 + \beta'. \tag{3.9}$$

Hence, to the first order of primed variables, we have

$$\rho_0 c_{p_0}\frac{DT'}{Dt} = \rho_0\beta_0 T_0\left\{\frac{D\varphi_0}{Dt} + \frac{T'}{T_0}\frac{D\varphi_0}{Dt} + \frac{\beta'}{\beta_0}\frac{D\varphi_0}{Dt} + \frac{D\varphi'}{Dt} + s_0\frac{DT'}{Dt}\right\}. \tag{3.10}$$

Now, let $\Delta\varphi_0$ denote the variation scale of φ_0: it is evident from (2.5) that

$$\Delta\varphi_0 = gH. \tag{3.11}$$

On the other hand, to the first order of φ' and T', β' is expressed as

$$\beta' = (\partial\beta/\partial\varphi)_T|_{(\varphi_0,T_0)}\varphi' + (\partial\beta/\partial T)_\varphi|_{(\varphi_0,T_0)}T'. \tag{3.12}$$

Accordingly, we can estimate β'/β_0 as follows:

$$\begin{aligned}|\beta'/\beta_0| = {} & O\left\{\left|gH(\partial\beta/\partial\varphi)_T|_{(\varphi_0,T_0)}/\beta_0\right|(\Delta\varphi'/gH)\right\} \\ & + O\left\{\left|T_0(\partial\beta/\partial T)_\varphi|_{(\varphi_0,T_0)}/\beta_0\right|(\Delta T'/T_0)\right\}.\end{aligned} \tag{3.13}$$

We assume here that the following conditions apply:

$$\left|gH(\partial\beta/\partial\varphi)_T|_{(\varphi_0,T_0)}/\beta_0\right| \le O(1), \quad \left|T_0(\partial\beta/\partial T)_\varphi|_{(\varphi_0,T_0)}/\beta_0\right| \le O(1). \tag{3.14}$$

Then, among the terms in the braces of (3.10), those containing primed variables can be ignored in comparison with the first, in view of the conditions (2.45), (2.46), and (2.51) that hold under the isothermal anelastic approximation. Thus we have

$$\rho_0 c_{p0} \frac{DT'}{Dt} = \rho_0 \beta_0 T_0 \frac{D\varphi_0}{Dt}. \tag{3.15}$$

The term on the right-hand side, which can be written as $-\rho_0\beta_0 T_0 g\boldsymbol{u}\cdot\boldsymbol{k}$, represents the adiabatic heating and cooling due to the vertical motion of the fluid.

However, the term on the right-hand side of (3.15) can further be ignored when

$$\Gamma_0 H/\Delta T' \ll 1. \tag{3.16}$$

In this case, (3.15) reduces to

$$\rho_0 c_{p0} \frac{DT'}{Dt} = 0. \tag{3.17}$$

This coincides with the adiabatic equation under the Boussinesq approximation.

4. Summary and discussion

The isothermal anelastic approximation, a variant of the anelastic approximation, has been formulated; its energetics and the conditions for its applicability have in addition been elucidated. It has also been demonstrated that the Boussinesq approximation can be reproduced as a limiting case of this approximation.

4.1. Another form of the equation of motion

The equation of motion under the isothermal anelastic approximation can be written, in terms of p' and ρ' defined respectively by (2.28) and (2.41), in the following form:

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p'/\rho_0 - (\rho' g/\rho_0)\boldsymbol{k}. \tag{4.1}$$

As explained below, this form of the equation of motion can be derived from (2.36).

We first rewrite the first term on the right-hand side of (2.36) as follows:

$$-\nabla(p'/\rho_0) = -\nabla p'/\rho_0 + (p'/\rho_0)(\nabla\rho_0/\rho_0). \tag{4.2}$$

However, since $p'/\rho_0 = \varphi' + s_0 T'$ and $\nabla\rho_0/\rho_0 = -(\partial\rho/\partial\varphi)_T|_{(\varphi_0,T_0)}(g/\rho_0)\boldsymbol{k}$, we get

$$-\nabla(p'/\rho_0) = -\nabla p'/\rho_0 - \left\{(\partial\rho/\partial\varphi)_T|_{(\varphi_0,T_0)}\varphi' + s_0(\partial\rho/\partial\varphi)_T|_{(\varphi_0,T_0)}T'\right\}(g/\rho_0)\boldsymbol{k}. \tag{4.3}$$

The second term on the right-hand side of (2.36) can also be rewritten as

$$T'\nabla s_0 = -\rho_0(\partial s/\partial\varphi)_T|_{(\varphi_0,T_0)}T'(g/\rho_0)\boldsymbol{k}. \tag{4.4}$$

Substituting (4.3) and (4.4) into (2.36), we obtain

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p'/\rho_0 - \left[(\partial\rho/\partial\varphi)_T|_{(\varphi_0,T_0)}\varphi' + \{\partial(\rho s)/\partial\varphi\}_T\,|_{(\varphi_0,T_0)}T'\right](g/\rho_0)\boldsymbol{k}. \tag{4.5}$$

It follows from (2.27), however, that

$$\{\partial(\rho s)/\partial\varphi\}_T = (\partial\rho/\partial T)_\varphi. \tag{4.6}$$

The substitution of this thermodynamic relation into (4.5) yields

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p'/\rho_0 - \left\{(\partial\rho/\partial\varphi)_T|_{(\varphi_0,T_0)}\varphi' + (\partial\rho/\partial T)_\varphi|_{(\varphi_0,T_0)}T'\right\}(g/\rho_0)\boldsymbol{k}. \tag{4.7}$$

In light of (2.41), we observe that (4.7) is the same equation as (4.1).

It should be stressed finally that, despite this result, the density of a fluid under the isothermal anelastic approximation is given by ρ_0, not by $\rho_0 + \rho'$.

4.2. On the condition (2.46)

Of the conditions for the applicability of the isothermal anelastic approximation, we focus attention on the condition (2.46): it can also be expressed in the form (2.57).

So long as (2.49) and (2.50) applies, (2.57) requires that

$$\beta_0 \Delta T' \ll \Delta T'/T_0. \tag{4.8}$$

This is an excessively restrictive condition. Indeed, this condition can never be satisfied by an ideal gas since $\beta_0 = 1/T_0$ for an ideal gas; it therefore follows that the isothermal anelastic approximation is inapplicable to an ideal gas.

The condition (2.46), however, is unnecessary for the applicability of the Boussinesq approximation (see Maruyama 2019); it is replaced by the less severe condition (2.47). The Boussinesq approximation is therefore applicable also to an ideal gas.

Hence we may conclude as follows: the Boussinesq approximation is only fractionally included in the isothermal anelastic approximation, although it can be reproduced as a limiting case of this variant of the anelastic approximation.

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