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Generalized sums of Fibonacci and Lucas Numbers

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Here we are proposing generalized sums for Fibonacci and Lucas numbers. In the case of the Fibonacci sequence, the generalized sum contains four Fibonacci numbers. For the Lucas sequence, numbers are three.

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In mathematics, a binary operation is a calculation that combines two elements to obtain another element. In particular, this operation has a peculiar meaning when it acts on a set in a manner that its two domains and its codomain are the same set. Of binary operations we have proposed several examples in some previous calculations for different sets of numbers (Mersenne, Fermat, q-numbers, repunits and others). These examples are generalizations of the sum, therefore they have been named as "generalized sums". The approach was inspired by the generalized sums used for entropy [1,2]. The analyses of sequences of integers and q-numbers have been collected in [3].

Let us repeat here just one of these generalized sums, that concerning the Mersenne numbers [4]. These numbers are given by: $M_n = 2^n - 1$. The generalized sum is:

$$M_m \oplus M_n = M_{m+n} = M_m + M_n + M_m M_n$$

In particular:

$$M_n \oplus M_1 = M_{n+1} = M_n + M_1 + M_n M_1$$

Being $M_1=1$, we have $M_{n+1}=2M_n+1$.

Let us consider here the Fibonacci numbers. The beginning of the sequence is 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... (sequence A000045 in the OEIS). In the sequence, $F_0=0$.

In a previous discussion [5], we have shown that the numbers of Fibonacci are forming a group. In fact, a Fibonacci number can be represented by a 2x2 symmetric matrix and the operation of the group is the product of matrices. This approach allowed to define the negaFibonacci numbers by means of the inverse of the Fibonacci matrices.

However, we would like to find a generalized sum, in the style of that obtained for the Mersenne numbers, also in the case of Fibonacci. We will show that we need the Lucas numbers. The same happens if we want the generalized sum of the Lucas numbers.

Let us consider $\varphi=(1+\sqrt{5})/2$ and $\psi=(1-\sqrt{5})/2$. A Fibonacci number is:

$$F_n=(\varphi^n-\psi^n)/\sqrt{5}$$

Let us introduce number $A_n=(\varphi^n+\psi^n)/\sqrt{5}$. We have that: $A_n=L_n/\sqrt{5}$, where

$$L_n=(\varphi^n+\psi^n)$$

is a Lucas number. The sequence of the first twelve Lucas numbers is: 2,1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ... (sequence A000032 in the OEIS). In the sequence $L_0=2$.

Let us calculate F_{n+m} .

$$F_{n+m}=\frac{\sqrt{5}}{2}\left(A_n F_m+A_m F_n\right)=\left(F_n L_m+F_m L_n\right)/2$$

Then, a binary operation requires the Lucas numbers too.

$$F_n \oplus F_m=F_{n+m}=\left(F_n L_m+F_m L_n\right)/2$$

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However, it is possible to see that $L_n = F_n + 2F_{n-1}$ (*)

Then: $$F_n \oplus F_m = \left(F_n F_m + 2F_n F_{m-1} + F_m F_n + 2F_m F_{n-1}\right)/2$$

The generalized sum for Fibonacci numbers is:

$$F_n \oplus F_m = F_n F_m + F_n F_{m-1} + F_m F_{n-1}$$

So that:

$$F_n \oplus F_1 = F_n F_1 + F_n F_0 + F_1 F_{n-1} = F_n + F_{n-1} = F_{n+1} \quad .$$

The generalized sum is not a binary operation, because we have involved four Fibonacci numbers. Besides F_n, F_m we have also F_{n-1}, F_{m-1} .

The same approach can be used for the Lucas Numbers:

$$L_n \oplus L_m = L_{n+m} = \left(L_n L_m + 5F_m F_n\right)/2$$

Since: $5F_n F_m = L_{n+m} - (-1)^m L_{n-m}$ (**),

$$L_n \oplus L_m = L_{n+m} = L_n L_m - (-1)^m L_{n-m}$$

So that:

$$L_n \oplus L_1 = L_{n+1} = L_n + L_{n-1} \quad .$$

Again, the generalized sum is not a binary operation, because we have involved three Lucas numbers. Besides F_n, F_m we have also F_{n-m} .

Of OEIS A000032, OEIS A00045 see please the detailed discussion and references given in the On-Line Encyclopedia of Integer Sequences. See please also https://en.wikipedia.org/wiki/Lucas_number for (*) and (**).

References

[1] Amelia Carolina Sparavigna, 2019. Composition Operations of Generalized Entropies Applied to the Study of Numbers, International Journal of Sciences, vol. 8(04), pages 87-92, DOI: 10.18483/ijSci.2044

[2] Amelia Carolina Sparavigna, 2019. Some Groupoids and their Representations by Means of Integer Sequences, International Journal of Sciences, vol. 8(10), pages 1-5, DOI: 10.18483/ijSci.2188

[3] Sparavigna, Amelia Carolina. (2020). Binary operations applied to numbers. Zenodo. http://doi.org/10.5281/zenodo.4155861

[4] Sparavigna, Amelia Carolina. (2018, May 20). On a generalized sum of the Mersenne Numbers. Zenodo. http://doi.org/10.5281/zenodo.1250048

[5] A.C. Sparavigna. On the group of the Fibonacci numbers, Zenodo, May 15, 2018, DOI: 10.5281/zenodo.1247352