

The Oscillator Propagator, Newton's Law and the Schrodinger Equation Part II

Francesco R. Ruggeri Hanwell, N.B. Mar. 29, 2021

In a series of notes (1), we argued that the time-independent Schrodinger equation may be considered as a statistical conservation of average energy equation. We further argued that $V(x) = \sum_k V_k \exp(ikx)$ delivers stochastic potential hits which create a momentum distribution $W(x) = \sum_p a(p) \exp(ipx)$ at each x . The form $\exp(ipx)$ describes a free particle and a statistical distribution is made of such free particles. The form $\exp(ipx)$ has a modulus of 1 indicating that $P(x) = \text{constant}$. $d/dx \exp(ipx) = ip$ suggesting the change in conditional probability with space is related to classical momentum. Given a momentum distribution, each p at x occurs at a different time. Thus, one does not follow a classical particle in time $x(t)$, nevertheless an average energy conservation equation applies.

Taking the time-independent Schrodinger equation $E W = -1/2m d/dx d/dx W + V(x) W$ and writing $EW = i d/dt \exp(-iEt)$, one obtains a time-dependent equation. We argue that if the time-independent case represents a classical conservation of average energy, the time-dependent equation should be linked to Newton's equation of motion. This immediately leads to a problem because we argue one does not want $x(t)$ for a statistical system. As a result, x in the time-dependent Schrodinger equation, which represented spatial position before, now represents a spatial variable that is independent of time. We consider the oscillator case and note that an amplitude is such a candidate where $x(t) = XA(t)$. Thus, the time-dependent Schrodinger equation ultimately becomes an equation in $A(t)$ with the amplitude cancelling, at least in the free particle and oscillator case. From such a consideration, one may work "backwards" and find a function for which the time-dependent Schrodinger equation (equating powers of X) becomes Newton's second law. From this, one may show that $\exp(i \text{Action})$ where $\text{Action} = \int_0^t L dt$ where L is the classical Lagrangian is this solution with d/dX applying to X which is an amplitude, not $x(t)$.

Statistical Approach to Bound State Quantum Mechanics

In classical statistical mechanics, a particle in a region with no potential has $P(x) = \text{constant}$ i.e. is equally likely to be found anywhere. This holds regardless of momentum. One may argue that p may represent an average of more complicated motion which still has $P(x) = \text{constant}$. For example, one may argue there is a conditional probability $P(p/x)$ linked to $P(x) = \text{constant}$, but differing for different p . For example, $d/dx P(p/x) = ip \exp(ipx)$ i.e. the rate of change of conditional probability may be proportional to momentum. The modulus of $\exp(ipx)$ is 1, so $P(x) = \text{constant}$ still hold.

If a potential $V(x)$ is introduced, one may retain a statistical free particle picture by considering stochastic hits $V(x) = \sum_k V_k \exp(ikx)$. These lead to a momentum distribution at each x of the form $P(p/x) = a(p) \exp(ipx) / W(x)$ where $W(x) = \sum_p a(p) \exp(ipx)$. By applying conservation of average energy, one obtains the time-independent Schrodinger equation:

$$E W = -\frac{1}{2} \frac{d}{dx} \frac{d}{dx} W + V(x) W \quad \text{or} \quad E = \left[\sum \text{over } p \quad \frac{p^2}{2m} a(p) \exp(ipx) \right] / W(x) + V(x) \quad ((1))$$

Thus, a statistical (stochastic) model may be linked to average classical conservation of kinetic energy. The question becomes: What does one do in the time-dependent case for which one has Newton's second law of motion? In the statistical picture, one does not follow a particle in time i.e. there is no $x(t)$. The momentum distribution at each x is made up of different p values occurring at x at different times.

Time-Dependent Schrodinger Equation

The time-independent Schrodinger equation may be converted into a time-dependent equation easily, but a question remains as to its interpretation. In particular, $E W(x)$ may be written as: $i \frac{d}{dt} \exp(-iEt) W(x)$. This is simply a math procedure. If the time-independent Schrodinger equation represents average classical energy conservation, one might expect the time-dependent equation to be "linked" to Newton's equation of motion. This raises a problem, however, because in the time-independent picture one does not have $x(t)$. The system is stochastic. X in the the time-independent Schrodinger equation represents spatial position, but in classical physics spatial position changes with time. A way out of this dilemma, is to consider the idea of an amplitude as done in (2):

$$x(t) = X A(T_f - t) / A(T_f - T_i) + Y A(t - T_i) / A(T_f - T_i) \quad \text{Let } T_i = 0 \quad T_f = t \text{ in the Schrodinger equation } ((2))$$

In such a case X is like an amplitude. $V(x)$ in the time-independent Schrodinger equation utilizes X , the amplitude, so is no longer potential energy at a spatial position. The time-dependent equation is thus one in two variables X and t , but X is not a function of $t = T_f$, it is an amplitude. The solution of such an equation should depend on X and $t = T_f$. The time-independent and time-dependent equations are nevertheless still linked. If $\exp(-iE_n t) W_n(x)$ is a solution of the time-independent equation and $W_n(x)$ mathematically forms a complete orthonormal basis, then the solution of the time-dependent equation may be expanded as a series in $W_n(x)$. Interestingly, x which was a spatial position is now an amplitude, but from a mathematical point of view that is fine. Each $W_n(x)$ is associated with a different E_n (energy), so the solution of the time-dependent equation represents all energies. How can this be? If X , the amplitude cancels from the equation, then one has a Newton's equation for $A(t)$ such that one may use (2) to obtain $x(t)$.

Thus, one seeks a function such that the time-dependent Schrodinger equation, with X as an amplitude (see ((2))) yields Newton's law for $A(t)$. We consider the oscillator case:

$$i \frac{d}{dt} B(X, t) = -\frac{1}{2m} \frac{d}{dx} \frac{d}{dx} B(X, t) + .5k XX \quad ((3)) \quad X = \text{amplitude}$$

$$\text{A solution to } ((3)) \text{ for the } XX \text{ portion is: } B(X, t) = Q(t) \exp\{i \frac{dA}{dt} / A \frac{(m/2) XX}{A} \} \quad ((4))$$

$$\text{A full solution is: } B(X, t) = Q(t) \exp\{ [\frac{m/2}{A} \frac{dA}{dt} / A (XX + YY)] - \frac{2XY}{(\sqrt{k/m} \cdot 5m)} \} \quad ((5))$$

The XX portion allows, upon equating XX terms for XX to cancel out leaving Newton's law:

$$M \frac{d}{dt} \frac{dA}{dt} = -k A \quad ((6))$$

One may also see that that unusual $-1/2m \frac{d}{dX} \frac{d}{dX} B(t,X)$ term is closely associated with classical kinetic energy i.e. $m/2 \frac{dA}{dt} \frac{dA}{dt} \frac{XX}{AA} \quad ((7))$ (although there is the $A(T)A(T)$ in the denominator which is not part of kinetic energy).

The form $((5))$ may be shown to be equivalent to:

$\exp(i \text{ Action})$ where $\text{Action} = \int_0^T dt L$ where L is the classical Lagrangian.

Conclusion

In conclusion, one may start with a stochastic equation which represents average classical energy conservation i.e. the time-dependent Schrodinger equation and convert it into a time-dependent equation. In the time-dependent case, the spatial position x becomes a time-independent amplitude such that $x(t) = X A(t)$ (or more correctly $((2))$). The time dependent equation then represents Newton's second law for $A(t)$ (at least in the free particle and oscillator cases). Furthermore, this solution may be written as: $\exp(i \text{ Action})$.

References

1. Ruggeri, Francesco R. <https://www.zenodo.org/record/3945503#.YGI1qCjYrq8>
2. https://en.wikipedia.org/wiki/Path_integral_formulation#Free_particle