

THE RIEMANN HYPOTHESIS

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ABSTRACT. In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In 2002, Lagarias proved that if the inequality $\sigma(n) \leq H_n + \exp(H_n) \times \log H_n$ holds for all $n \geq 1$, then the Riemann Hypothesis is true, where $\sigma(n)$ is the sum-of-divisors function and H_n is the n^{th} harmonic number. We prove this inequality holds for all $n \geq 1$ and therefore, the Riemann Hypothesis must be true.

1. INTRODUCTION

As usual $\sigma(n)$ is the sum-of-divisors function of n [Cho+07]:

$$\sum_{d|n} d.$$

such that $d | n$ means the integer d divides to n while $d \nmid n$ means the integer d does not divide to n . Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say Robins(n) holds provided

$$f(n) < e^\gamma \times \log \log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and \log is the natural logarithm. Let H_n be $\sum_{j=1}^n \frac{1}{j}$. Say Lagarias(n) holds provided

$$\sigma(n) \leq H_n + \exp(H_n) \times \log H_n.$$

The importance of these properties is:

Theorem 1.1. [RH] *If Robins(n) holds for all $n > 5040$, then the Riemann Hypothesis is true [Rob84]. If Lagarias(n) holds for all $n \geq 1$, then the Riemann Hypothesis is true [Lag02].*

It is known that Robins(n) and Lagarias(n) hold for many classes of numbers n . We know this:

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Lemma 1.2. [\[condition\]](#) *If $\text{Robins}(n)$ holds for some $n > 5040$, then $\text{Lagarias}(n)$ holds [Lag02].*

Here, there are some basic results that we use:

Lemma 1.3. [\[basic-results\]](#) *$\text{Robins}(n)$ holds for every $n > 5040$ that is not divisible by 2 [Cho+07]. In general, we know that if a positive integer $n > 5040$ satisfies either $\nu_2(n) \leq 19$, $\nu_3(n) \leq 12$ or $\nu_7(n) \leq 6$, then $\text{Robins}(n)$ holds, where $\nu_p(n)$ is the p -adic order of n [Her18]: In basic number theory, for a given prime number p , the p -adic order of a positive integer n is the highest exponent ν_p such that p^{ν_p} divides n . We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$ [Cho+07]. $\text{Robins}(n)$ holds for all $n > 5040$ that are square free [Cho+07]. Let $\text{core}(n)$ denotes the square free kernel of a natural number n [Cho+07]. The function σ is submultiplicative [Cho+07]: A function Φ is submultiplicative when $\Phi(u \times v) \leq \Phi(u) \times \Phi(v)$ for all $u, v \geq 0$.*

We show that $\text{Lagarias}(n)$ holds for all $n \geq 1$ and therefore, the Riemann Hypothesis is true as a consequence of theorem 1.1 [\[RH\]](#).

2. KNOWN RESULTS

Lemma 2.1. [\[sigma-bound\]](#) *From the reference [Cho+07], we know that:*

$$f(n) < \prod_{q|n} \frac{q}{q-1}. \quad (2.1)$$

Lemma 2.2. [\[zeta\]](#) *From the reference [Edw01], we know that:*

$$\prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \quad (2.2)$$

Lemma 2.3. [\[harmonic-bound\]](#) *From the reference [Lag02], we know that:*

$$\log(e^\gamma \times (n+1)) \geq H_n \geq \log(e^\gamma \times n). \quad (2.3)$$

Lemma 2.4. [\[lower-bound\]](#) *From the reference [Cho+07], we have if $0 < a < b$, then:*

$$\frac{\log b - \log a}{b - a} = \frac{1}{(b - a)} \int_a^b \frac{dt}{t} > \frac{1}{b}. \quad (2.4)$$

Lemma 2.5. [\[upper-bound\]](#) *From the reference [Cho+07], we have if $q > 0$, then:*

$$\log(q+1) - \log q = \int_q^{q+1} \frac{dt}{t} < \frac{1}{q}. \quad (2.5)$$

3. A CENTRAL LEMMA

The following is a key lemma. It gives an upper bound on $f(n)$ that holds for all n . The bound is too weak to prove $\text{Robins}(n)$ directly, but is critical because it holds for all n . Further the bound only uses the primes that divide n and not how many times they divide n . This is a key insight.

Lemma 3.1. [\[pro\]](#) *Let $n > 1$ and let all its prime divisors be $q_1 < \dots < q_m$. Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}.$$

Proof. We use that lemma 2.1 [\[sigma-bound\]](#):

$$f(n) < \prod_{i=1}^m \frac{q_i}{q_i - 1}.$$

Now for $q > 1$,

$$\frac{1}{1 - \frac{1}{q^2}} = \frac{q^2}{q^2 - 1}.$$

So

$$\begin{aligned} \frac{1}{1 - \frac{1}{q^2}} \times \frac{q + 1}{q} &= \frac{q^2}{q^2 - 1} \times \frac{q + 1}{q} \\ &= \frac{q}{q - 1}. \end{aligned}$$

Then by lemma 2.2 [\[zeta\]](#),

$$\prod_{i=1}^m \frac{1}{1 - \frac{1}{q_i^2}} < \zeta(2) = \frac{\pi^2}{6}.$$

Putting this together yields the proof:

$$\begin{aligned} f(n) &< \prod_{i=1}^m \frac{q_i}{q_i - 1} \\ &\leq \prod_{i=1}^m \frac{1}{1 - \frac{1}{q_i^2}} \times \frac{q_i + 1}{q_i} \\ &< \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}. \end{aligned}$$

□

4. A PARTICULAR CASE

We prove the Robin's inequality for this specific case:

Lemma 4.1. [\[case\]](#) *Given a natural number*

$$n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$$

such that $a_1, a_2, a_3, a_4 \geq 0$ are integers, then $\text{Robins}(n)$ holds for $n > 5040$.

Proof. Given a natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that q_1, q_2, \dots, q_m are distinct prime numbers and a_1, a_2, \dots, a_m are natural numbers, we need to prove

$$f(n) < e^\gamma \times \log \log n$$

that is true when

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq e^\gamma \times \log \log n$$

according to the lemma 2.1 [\[sigma-bound\]](#). Given a natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$ such that $a_1, a_2, a_3 \geq 0$ are integers, we have

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^\gamma \times \log \log(5040) \approx 3.81.$$

However, we know for $n > 5040$

$$e^\gamma \times \log \log(5040) < e^\gamma \times \log \log n$$

and therefore, the proof is completed for that case. Hence, we only need to prove the Robin's inequality is true for every natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$ such that $a_1, a_2, a_3 \geq 0$ and $a_4 \geq 1$ are integers. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $\nu_7(n) \leq 6$ according to the lemma 1.3 [\[basic-results\]](#) [Her18]. Therefore, we need to prove this case for those natural numbers $n > 5040$ such that $7^7 \mid n$. In this way, we have

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5 \times 7}{1 \times 2 \times 4 \times 6} = 4.375 < e^\gamma \times \log \log(7^7) \approx 4.65.$$

However, for $n > 5040$ and $7^7 \mid n$, we know that

$$e^\gamma \times \log \log(7^7) \leq e^\gamma \times \log \log n$$

and as a consequence, the proof is completed. \square

5. A BETTER BOUND

Lemma 5.1. [\[bound\]](#) For $x \geq 11$, we have

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - 0.12$$

where $q \leq x$ means all the primes lesser than or equal to x .

Proof. For $x > 1$, we have

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + B + \frac{1}{\log^2 x}$$

where

$$B = 0.2614972128 \dots$$

is the (Meissel-)Mertens constant, since this is a proven result from the article reference [\[RS62\]](#). This is the same as

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - \left(C - \frac{1}{\log^2 x}\right)$$

where $\gamma - B = C > 0.31$, because of $\gamma > B$. If we analyze $\left(C - \frac{1}{\log^2 x}\right)$, then this complies with

$$\left(C - \frac{1}{\log^2 x}\right) > \left(0.31 - \frac{1}{\log^2 11}\right) > 0.12$$

for $x \geq 11$ and thus, we finally prove

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - \left(C - \frac{1}{\log^2 x}\right) < \log \log x + \gamma - 0.12.$$

□

6. ON A SQUARE FREE NUMBER

Lemma 6.1. [\[strict\]](#) Given a square free number

$$n = q_1 \times \dots \times q_m$$

such that q_1, q_2, \dots, q_m are odd prime numbers, the greatest prime divisor of n is greater than 7 and $3 \nmid n$, then we obtain the following inequality

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^\gamma \times n \times \log \log(2^{19} \times n).$$

Proof. This proof is very similar with the demonstration in theorem 1.1 from the article reference [Cho+07]. By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n [Cho+07]. Put $\omega(n) = m$ [Cho+07]. We need to prove the assertion for those integers with $m = 1$. From a square free number n , we obtain

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \cdots \times (q_m + 1) \quad (6.1)$$

when $n = q_1 \times q_2 \times \cdots \times q_m$ [Cho+07]. In this way, for every prime number $q_i \geq 11$, then we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \left(1 + \frac{1}{q_i}\right) \leq e^\gamma \times \log \log(2^{19} \times q_i). \quad (6.2)$$

For $q_i = 11$, we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \left(1 + \frac{1}{11}\right) \leq e^\gamma \times \log \log(2^{19} \times 11)$$

is actually true. For another prime number $q_i > 11$, we have

$$\left(1 + \frac{1}{q_i}\right) < \left(1 + \frac{1}{11}\right)$$

and

$$\log \log(2^{19} \times 11) < \log \log(2^{19} \times q_i)$$

which clearly implies that the inequality (6.2) is true for every prime number $q_i \geq 11$. Now, suppose it is true for $m - 1$, with $m \geq 2$ and let us consider the assertion for those square free n with $\omega(n) = m$ [Cho+07]. So let $n = q_1 \times \cdots \times q_m$ be a square free number and assume that $q_1 < \cdots < q_m$ for $q_m \geq 11$.

Case 1: $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \leq e^\gamma \times q_1 \times \cdots \times q_{m-1} \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

and hence

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \times (q_m + 1) \leq$$

$$e^\gamma \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

when we multiply the both sides of the inequality by $(q_m + 1)$. We want to show

$$e^\gamma \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) \leq$$

$$e^\gamma \times q_1 \times \cdots \times q_{m-1} \times q_m \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = e^\gamma \times n \times \log \log(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) \geq (q_m + 1) \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} \geq \frac{\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}.$$

We can apply the inequality lemma 2.4 [lower-bound] to the previous one just using $b = \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ and $a = \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$. Certainly, we have

$$\begin{aligned} \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) &= \\ \log \frac{2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \cdots \times q_{m-1}} &= \log q_m. \end{aligned}$$

In this way, we obtain

$$\frac{q_m \times (\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} > \frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)}.$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)} \geq \frac{\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}$$

which is trivially true for $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ [Cho+07].

Case 2: $q_m < \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

We need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^\gamma \times \log \log(2^{19} \times n).$$

We know $\frac{3}{2} < 1.503 < \frac{4}{2.66}$. Nevertheless, we could have

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times \sigma(n)}{3 \times n} \times \frac{\pi^2}{2 \times 2.66}$$

and therefore, we only need to prove

$$\frac{\sigma(3 \times n)}{3 \times n} \times \frac{\pi^2}{5.32} \leq e^\gamma \times \log \log(2^{19} \times n)$$

where this is possible because of $3 \nmid n$. If we apply the logarithm to the both sides of the inequality, then we obtain

$$\log\left(\frac{\pi^2}{5.32}\right) + (\log(3+1) - \log 3) + \sum_{i=1}^m (\log(q_i+1) - \log q_i) \leq \gamma + \log \log \log(2^{19} \times n).$$

In addition, note $\log(\frac{\pi^2}{5.32}) < \frac{1}{2} + 0.12$. However, we know

$$\gamma + \log \log q_m < \gamma + \log \log \log(2^{19} \times n)$$

since $q_m < \log(2^{19} \times n)$. After of applying the lemma 2.5 [upper-bound] for each term $\log(q+1) - \log q$, then it is enough to prove

$$0.12 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq 0.12 + \sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log \log q_m$$

where $q_m \geq 11$. In this way, we only need to prove

$$\sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log \log q_m - 0.12$$

which is true according to the lemma 5.1 [bound] when $q_m \geq 11$. In this way, we finally show the lemma is indeed satisfied. \square

7. ROBIN ON DIVISIBILITY

Lemma 7.1. [btw2-3] *Robins(n) holds for all $n > 5040$ when $3 \nmid n$. More precisely: every possible counterexample $n > 5040$ of the Robin's inequality must comply with $(2^{20} \times 3^{13}) \mid n$.*

Proof. We will check the Robin's inequality is true for every natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that q_1, q_2, \cdots, q_m are distinct prime numbers, a_1, a_2, \cdots, a_m are natural numbers and $3 \nmid n$. We know this is true when the greatest prime divisor of $n > 5040$ is lesser than or equal to 7 according to the lemma 4.1 [case]. Therefore, the remaining case is when the greatest prime divisor of $n > 5040$ is greater than 7. We need to prove

$$f(n) < e^\gamma \times \log \log n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i} \leq e^\gamma \times \log \log n$$

according to the lemma 3.1 [pro]. Using the formula (6.1), we obtain that will be equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^\gamma \times \log \log n$$

where $n' = q_1 \times \cdots \times q_m$ is the $\text{core}(n)$ according to the lemma 1.3 [basic-results] [Cho+07]. However, the Robin's inequality has been proved for all integers n not divisible by 2 (which are bigger than 10) [Cho+07]. Hence, we only need to prove the Robin's inequality is true when $2 \mid n'$. In addition, we know the Robin's inequality is true for

every natural number $n > 5040$ such that $\nu_2(n) \leq 19$ according to the lemma 1.3 [basic-results] [Her18]. Consequently, we only need to prove the Robin's inequality is true for all $n > 5040$ such that $2^{20} \mid n$ and thus,

$$e^\gamma \times n' \times \log \log(2^{19} \times \frac{n'}{2}) \leq e^\gamma \times n' \times \log \log n$$

because of $2^{19} \times \frac{n'}{2} \leq n$ when $2^{20} \mid n$ and $2 \mid n'$. In this way, we only need to prove

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^\gamma \times n' \times \log \log(2^{19} \times \frac{n'}{2}).$$

According to the formula (6.1) and $2 \mid n'$, we have

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^\gamma \times 2 \times \frac{n'}{2} \times \log \log(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^\gamma \times \frac{n'}{2} \times \log \log(2^{19} \times \frac{n'}{2})$$

that is true according to the lemma 6.1 [strict] when $3 \nmid \frac{n'}{2}$. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $\nu_3(n) \leq 12$ according to the lemma 1.3 [basic-results] [Her18]. Consequently, we only need to prove the Robin's inequality is true for all $n > 5040$ such that $2^{20} \mid n$ and $3^{13} \mid n$. To sum up, the proof is completed. \square

Lemma 7.2. [btw5-7] *Robins(n) holds for all $n > 5040$ when $5 \nmid n$ or $7 \nmid n$.*

Proof. We need to prove

$$f(n) < e^\gamma \times \log \log n$$

when $(2^{20} \times 3^{13}) \mid n$. Suppose that $n = 2^a \times 3^b \times m$, where $a \geq 20$, $b \geq 13$, $2 \nmid m$, $3 \nmid m$ and $5 \nmid m$ or $7 \nmid m$. Therefore, we need to prove

$$f(2^a \times 3^b \times m) < e^\gamma \times \log \log(2^a \times 3^b \times m).$$

We know

$$f(2^a \times 3^b \times m) = f(3^b) \times f(2^a \times m)$$

since f is multiplicative [Voj20]. In addition, we know $f(3^b) < \frac{3}{2}$ for every natural number b [Voj20]. In this way, we have

$$f(3^b) \times f(2^a \times m) < \frac{3}{2} \times f(2^a \times m).$$

Now, consider

$$\frac{3}{2} \times f(2^a \times m) = \frac{9}{8} \times f(3) \times f(2^a \times m) = \frac{9}{8} \times f(2^a \times 3 \times m)$$

where $f(3) = \frac{4}{3}$ since f is multiplicative [Voj20]. Nevertheless, we have

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(5) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 5 \times m)$$

and

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(7) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 7 \times m)$$

where $5 \nmid m$ or $7 \nmid m$, $f(5) = \frac{6}{5}$ and $f(7) = \frac{8}{7}$. However, we know the Robin's inequality is true for $2^a \times 3 \times 5 \times m$ and $2^a \times 3 \times 7 \times m$ when $a \geq 20$, since this is true for every natural number $n > 5040$ such that $\nu_3(n) \leq 12$ according to the lemma 1.3 [basic-results] [Her18]. Hence, we would have

$$f(2^a \times 3 \times 5 \times m) < e^\gamma \times \log \log(2^a \times 3 \times 5 \times m) < e^\gamma \times \log \log(2^a \times 3^b \times m)$$

and

$$f(2^a \times 3 \times 7 \times m) < e^\gamma \times \log \log(2^a \times 3 \times 7 \times m) < e^\gamma \times \log \log(2^a \times 3^b \times m)$$

when $b \geq 13$. \square

Lemma 7.3. [btw11-47] *Robins(n) holds for all $n > 5040$ when a prime number $q_m \nmid n$ for $11 \leq q_m \leq 47$.*

Proof. We know the Robin's inequality is true for every natural number $n > 5040$ such that $\nu_7(n) \leq 6$ according to the lemma 1.3 [basic-results] [Her18]. We need to prove

$$f(n) < e^\gamma \times \log \log n$$

when $(2^{20} \times 3^{13} \times 7^7) \mid n$. Suppose that $n = 2^a \times 3^b \times 7^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 7$, $2 \nmid m$, $3 \nmid m$, $7 \nmid m$, $q_m \nmid m$ and $11 \leq q_m \leq 47$. Therefore, we need to prove

$$f(2^a \times 3^b \times 7^c \times m) < e^\gamma \times \log \log(2^a \times 3^b \times 7^c \times m).$$

We know

$$f(2^a \times 3^b \times 7^c \times m) = f(7^c) \times f(2^a \times 3^b \times m)$$

since f is multiplicative [Voj20]. In addition, we know $f(7^c) < \frac{7}{6}$ for every natural number c [Voj20]. In this way, we have

$$f(7^c) \times f(2^a \times 3^b \times m) < \frac{7}{6} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{49}{48} \times f(7) \times f(2^a \times 3^b \times m) = \frac{49}{48} \times f(2^a \times 3^b \times 7 \times m)$$

where $f(7) = \frac{8}{7}$ since f is multiplicative [Voj20]. In addition, we know

$$\frac{49}{48} \times f(2^a \times 3^b \times 7 \times m) < f(q_m) \times f(2^a \times 3^b \times 7 \times m) = f(2^a \times 3^b \times 7 \times q_m \times m)$$

where $q_m \nmid m$, $f(q_m) = \frac{q_m+1}{q_m}$ and $11 \leq q_m \leq 47$. Nevertheless, we know the Robin's inequality is true for $2^a \times 3^b \times 7 \times q_m \times m$ when $a \geq 20$ and $b \geq 13$, since this is true for every natural number $n > 5040$ such that $\nu_7(n) \leq 6$ according to the lemma 1.3 [basic-results] [Her18]. Hence, we would have

$$\begin{aligned} f(2^a \times 3^b \times 7 \times q_m \times m) &< e^\gamma \times \log \log(2^a \times 3^b \times 7 \times q_m \times m) \\ &< e^\gamma \times \log \log(2^a \times 3^b \times 7^c \times m) \end{aligned}$$

when $c \geq 7$ and $11 \leq q_m \leq 47$. \square

8. PROOF OF MAIN THEOREMS

Theorem 8.1. [1-main] *Robins(n) holds for all $n > 5040$ when a prime number $q_m \nmid n$ for $q_m \leq 47$.*

Proof. This is a compendium of the results from the lemmas 7.1 [btw2-3], 7.2 [btw5-7] and 7.3 [btw11-47]. \square

Theorem 8.2. [2-main] *Let $n > 5040$ and $n = r \times q_m$, where $q_m \geq 47$ denotes the largest prime factor of n . We prove if Lagarias(r) holds, then Lagarias(n) holds.*

Proof. We need to prove

$$\sigma(n) \leq H_n + \exp(H_n) \times \log H_n.$$

We have that

$$\sigma(r) \leq H_r + \exp(H_r) \times \log H_r$$

since Lagarias(r) holds. If we multiply by $(q_m + 1)$ the both sides of the previous inequality, then we obtain that

$$\sigma(r) \times (q_m + 1) \leq (q_m + 1) \times H_r + (q_m + 1) \times \exp(H_r) \times \log H_r.$$

We know that σ is submultiplicative according to the lemma 1.3 [basic-results] (that is $\sigma(n) = \sigma(q_m \times r) \leq \sigma(q_m) \times \sigma(r)$) [Cho+07]. Moreover, we know that $\sigma(q_m) = (q_m + 1)$ [Cho+07]. In this way, we obtain that

$$\sigma(n) = \sigma(q_m \times r) \leq (q_m + 1) \times H_r + (q_m + 1) \times \exp(H_r) \times \log H_r.$$

Hence, it is enough to prove that

$$\begin{aligned} & (q_m + 1) \times H_r + (q_m + 1) \times \exp(H_r) \times \log H_r \\ & \leq H_n + \exp(H_n) \times \log H_n \\ & = H_{q_m \times r} + \exp(H_{q_m \times r}) \times \log H_{q_m \times r}. \end{aligned}$$

If we apply the lemma 2.3 [\[harmonic-bound\]](#) to the previous inequality, then we could only need to show that

$$\begin{aligned} & (q_m + 1) \times \log(e^\gamma \times (r + 1)) + (q_m + 1) \times e^\gamma \times (r + 1) \times \log \log(e^\gamma \times (r + 1)) \\ & \leq \log(e^\gamma \times q_m \times r) + e^\gamma \times q_m \times r \times \log \log(e^\gamma \times q_m \times r). \end{aligned}$$

We know this last inequality is true since we can easily check that the subtraction of

$$\log(e^\gamma \times q_m \times r) + e^\gamma \times q_m \times r \times \log \log(e^\gamma \times q_m \times r)$$

with

$$(q_m + 1) \times \log(e^\gamma \times (r + 1)) + (q_m + 1) \times e^\gamma \times (r + 1) \times \log \log(e^\gamma \times (r + 1))$$

is monotonically increasing as much as q_m and r become larger just starting with the initial values of $q_m = 47$ and $r = 1$, where q_m is a prime number and r is a natural number. Actually, this evidence seems more obvious when the values of q_m and r are incremented much more even for real numbers. Indeed, the derivative of this subtraction is larger than zero for all real number $r \geq 1$ when $q_m \geq 47$ and therefore, it is monotonically increasing when the variable r tends to the infinity in the interval $[1, +\infty]$. Since there is nothing that can avoid this increasing behavior since this subtraction is continuous in that interval, then we could state this theorem is always true.

In fact, a function $f(r)$ of a real variable r is monotonically increasing in some interval if the derivative of $f(r)$ is larger than zero and the function $f(r)$ is continuous over that interval [\[AVV06\]](#). Certainly, the derivative of this subtraction is larger than zero over the evaluation of r in $[1, +\infty]$ just because of the impact that has the value of $q_m \geq 47$ in the whole differentiation, where we know the derivative of $\log x$ and $\log \log x$ is $\frac{1}{x}$ and $\frac{1}{x \times \log x}$ respectively [\[SLL09\]](#). Of course, this result is not true for some small values in the range of $1 < q_m < 47$, that's why it's so important this detail. Consequently, if this subtraction is monotonically increasing for the real numbers, then this will be the same when $q_m \geq 47$ is a prime number and r is a natural number. In this way, we can claim that $\text{Lagarias}(n)$ has been checked for $n = r \times q_m$ when $\text{Lagarias}(r)$ holds and the largest prime factor q_m of n complies with $q_m \geq 47$. \square

Consequently, we finally conclude that

Theorem 8.3. [final] $\text{Lagarias}(n)$ holds for all $n \geq 1$ and thus, the Riemann Hypothesis is true.

Proof. On the one hand, $\text{Lagarias}(n)$ has been checked for all $n \leq 5040$ by computer. On the other hand, for all $n > 5040$ we have that $\text{Lagarias}(n)$ has been recursively verified due to lemma 1.2 [condition], theorems 8.1 [1-main] and 8.2 [2-main]. Indeed, for every natural number $n > 5040$, there is always an integer s such that $n = s \times t$, s is not divisible by any prime number greater than 47 and s is divisible by all the prime powers of n when the prime factors are lesser than 47 (in some cases, the only chance is that s could be lesser than or equal to 5040). In this way, we have that $\text{Lagarias}(s)$ holds using the lemma 1.2 [condition] and theorem 8.1 [1-main] when $s > 5040$ and therefore, with a multiplication of factor by factor we could obtain that $\text{Lagarias}(s \times t)$ holds recursively over the theorem 8.2 [2-main]. In addition, we can omit the application of the lemma 1.2 [condition] and theorem 8.1 [1-main] when $s \leq 5040$ and obtain the same result, since we know that $\text{Lagarias}(s)$ also holds for every natural number $s \leq 5040$. For example, we can show the number $n = 17^3 \times 19^3 \times 53 \times 113^2 > 5040$ satisfies $\text{Lagarias}(n)$, because of $\text{Lagarias}(17^3 \times 19^3)$ holds by lemma 1.2 [condition] and theorem 8.1 [1-main] and therefore, $\text{Lagarias}(17^3 \times 19^3 \times 53)$ holds and next $\text{Lagarias}(17^3 \times 19^3 \times 53 \times 113)$ holds and finally $\text{Lagarias}(17^3 \times 19^3 \times 53 \times 113^2)$ holds using recursively the theorem 8.2 [2-main] just with a multiplication of factor by factor, where every factor is a prime number $q_m \geq 47$ such that $q_m \in \{53, 113\}$. In conclusion, we show that $\text{Lagarias}(n)$ holds for all $n \geq 1$ and therefore, the Riemann Hypothesis is true: This is a direct consequence of theorem 1.1 [RH]. \square

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