# A strengthened form of the strong Goldbach conjecture

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**Abstract.** This paper shows that a strengthened form of the strong Goldbach conjecture as well as its negation are true. The paper thus constitutes an antinomy within ZFC.

**Notations.** Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

**Theorem.** *Both SSGB and the negation ¬SSGB hold.*

*Proof.* We define the set $S_g$ := { (pk, mk, qk) | k, m $\in \mathbb{N}$; p, q $\in \mathbb{P}_3$, p < q; m = (p + q) / 2 }.

SSGB is equivalent to saying that every integer x ≥ 4 is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers x ≥ 4 appear as m in a middle component mk of $S_g$.

There are two possibilities for $S_g$, exactly one of which must occur: Either there is an n $\in \mathbb{N}_4$ in addition to all the numbers m defined in $S_g$ or there is not. The latter corresponds to SSGB and the former corresponds to the negation ¬SSGB.

The set $S_g$ has the following property: The whole range of $\mathbb{N}_3$ can be expressed by the triple components of $S_g$, since every integer x ≥ 3 can be written as some pk with k = 1 when x is prime, as some pk with k ≠ 1 when x is composite and not a power of 2, or as (3 + 5)k / 2 when x is a power of 2; p $\in \mathbb{P}_3$, k $\in \mathbb{N}$.

We can split $S_g$ into two complementary subsets: For any y $\in \mathbb{N}_3$, $S_g = S_g+(y) \cup S_g-(y)$, with

$S_g+(y)$ = { (pk', mk', qk') $\in S_g$ | $\exists$ k $\in \mathbb{N}$   pk' = yk $\vee$ mk' = yk $\vee$ qk' = yk } and

$S_g-(y)$ = { (pk', mk', qk') $\in S_g$ | $\forall$ k $\in \mathbb{N}$   pk' ≠ yk $\wedge$ mk' ≠ yk $\wedge$ qk' ≠ yk }.

In the case of ¬SSGB, there is at least one n $\in \mathbb{N}_4$ different from all the numbers m that are defined in $S_g$. In the case of SSGB, there is no such n. The following steps work regardless of the choice of n if there is more than one n.

According to the above three types of expression by $S_g$ triple components, for n we have

**(C)** $\forall$ k $\in \mathbb{N}$   $\exists$ (pk', mk', qk') $\in S_g$     nk = pk' $\vee$ nk = mk' = 4k'.

Moreover, due to the definition of $S_g$, we have

**(M)** $\nexists$ p, q $\in \mathbb{P}_3$, p < q    n = (p + q) / 2.

Because the properties (C) and (M) hold for any n given by ¬SSGB, under the assumption ¬SSGB the set $S_g$ can be written as the union of the following triples, which would otherwise be impossible.

**(i)** $S_g$ triples of the form (pk' = nk, mk', qk') with k' = k in case n is prime, due to (C)

**(ii)** $S_g$ triples of the form (pk' = nk, mk', qk') with k' ≠ k in case n is composite and not a power of 2, due to (C)

**(iii)** $S_g$ triples of the form (3k', 4k' = nk, 5k') in case n is a power of 2, due to (C)

**(iv)** all remaining $S_g$ triples of the form (pk' = nk, mk', qk'), (pk', mk' = nk, qk') or (pk', mk', qk' = nk)

and

**(v)** $S_g$ triples of the form (pk' ≠ nk, mk' ≠ nk, qk' ≠ nk), i.e. those $S_g$ triples where none of the nk's equals a component.

Let $S_g$+ be shorthand for $S_g$+(n) and let $S_g$- be shorthand for $S_g$-(n). Then, as $S_g$+ denotes the union of the triples of types (i) to (iv) and $S_g$- denotes the union of the triples of type (v), we can state

¬SSGB => (($S_g$ = $S_g$+ ∪ $S_g$-) or ¬(C) or ¬(M)).

Since (C) and (M) are true, we get

**(*)** ¬SSGB => $S_g$ = $S_g$+ ∪ $S_g$-.

Therefore,

∀ S (¬SSGB => $S_g$+ ∪ $S_g$- = S) <=> (¬SSGB => $S_g$ = S).

Since under ¬SSGB $S_g$ equals $S_g$+ ∪ $S_g$-, we obtain

∀ S (¬SSGB => $S_g$+ ∪ $S_g$- = S) <=> ($S_g$+ ∪ $S_g$- = S)

<=>

**(NG)** ∀ S (¬SSGB => $S_g$ = S) <=> ($S_g$ = S),

which is equivalent to ¬SSGB, because (NG) is true if ¬SSGB is true, and false if SSGB is true. So, ¬SSGB is proven.

Now, let us return to the step (*) above. There, $S_g+ \cup S_g-$ is independent of n since for every n it equals $S_g$. So, based on (*) we have

$\forall y \in \mathbb{N}_3 \ \ \neg SSGB \Rightarrow S_g = S_g+(y) \cup S_g-(y)$.

Since $\neg$SSGB is true, we also have

$\forall y \in \mathbb{N}_3 \ \ SSGB \Rightarrow S_g = S_g+(y) \cup S_g-(y)$.

Therefore,

$\forall y \in \mathbb{N}_3 \ \ (\forall S \ \ (SSGB \Rightarrow S_g+(y) \cup S_g-(y) = S) \Leftrightarrow (SSGB \Rightarrow S_g = S))$.

Since under SSGB $S_g$ equals $S_g+(y) \cup S_g-(y)$ for every $y \in \mathbb{N}_3$, we obtain

$\forall y \in \mathbb{N}_3 \ \ (\forall S \ \ (SSGB \Rightarrow S_g+(y) \cup S_g-(y) = S) \Leftrightarrow (S_g+(y) \cup S_g-(y) = S))$

$\Leftrightarrow$

$\forall S \ \ (SSGB \Rightarrow S_g = S) \Leftrightarrow (S_g = S)$,

which is equivalent to SSGB.

So, we have also shown SSGB. □

**Note.** The above splitting of all the $S_g$ triples into two complementary subsets $S_g+$ and $S_g-$ is independent of our information about $S_g$ and it is also independent of the property behind n. The splitting works solely on the basis of the existence of n.

The main reason for the antinomy above is the formula (NG) which in classical logic is equivalent to $\neg$SSGB, whereas an intuitionistic interpretation would say: If the set $S_g$ under the assumption of an $n \geq 4$ additional to all m equals $S_g$ without this assumption, then that n does not exist, and therefore SSGB holds.