THE RIEMANN HYPOTHESIS

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Abstract. In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution. In 1915, Ramanujan proved that under the assumption of the Riemann Hypothesis, the inequality $\sigma(n) < e^{\gamma} \times n \times \log \log n$ holds for all sufficiently large n, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann Hypothesis is true. In 2002, Lagarias proved that if the inequality $\sigma(n) \leq H_n + exp(H_n) \times \log H_n$ holds for all $n \geq 1$, then the Riemann Hypothesis is true, where H_n is the n^{th} harmonic number. We prove the Robin's inequality is true for every integer $n > 5040$ that is not divisible by any prime $q_m \leq 47$. Besides, we demonstrate the Lagarias's inequality is true for every integer $n > 5040$ when $n = r \times q_m$ and the Lagarias's inequality is true for r, where $q_m \geq 47$ denotes the largest prime factor of n. We finally show the union of these results over the both inequalities that leave us to a proof of the Lagarias's inequality and therefore, the Riemann Hypothesis is true.

1. Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of n [2]:

$$\sum_{d \mid n} d.$$

such that $d \mid n$ means the integer d divides to n while $d \nmid n$ means the integer d does not divide to n. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log \log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and log is the natural logarithm. Let H_n be $\sum_{j=1}^{n} \frac{1}{j}$. Say $\mathsf{Lagarias}(n)$ holds provided

$$\sigma(n) \leq H_n + exp(H_n) \times \log H_n.$$

The importance of these properties is:

Theorem 1.1. *If* $\mathsf{Robins}(n)$ *holds for all* $n > 5040$*, then the Riemann Hypothesis is true* [6]*. If* $\mathsf{Lagarias}(n)$ *holds for all* $n \geq 1$*, then the Riemann Hypothesis is true* [5].

It is known that $\mathsf{Robins}(n)$ and $\mathsf{Lagarias}(n)$ hold for many classes of numbers n. We know this:

Lemma 1.2. *If* $\mathsf{Robins}(n)$ *holds for some* $n > 5040$*, then* $\mathsf{Lagarias}(n)$ *holds* [5].

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Here, it is some other results that we use:

Lemma 1.3. $\mathsf{Robins}(n)$ *holds for every* $n > 5040$ *that is not divisible by* 2 [2]. *In general, we know that if a positive integer* n *satisfies either* $\nu_2(n) \leq 19$, $\nu_3(n) \leq 12$ *or* $\nu_7(n) \leq 6$, *then* $\mathsf{Robins}(n)$ *holds, where* $\nu_p(n)$ *is the* p*-adic order of* n [4].

Our goal is to prove our main two theorems:

Theorem 1.4. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when a prime number* $q_m \nmid n$ *for* $q_m \leq 47$.

Theorem 1.5. *Let* $n > 5040$ *and* $n = r \times q_m$, *where* $q_m \geq 47$ *denotes the largest prime factor of* n. *We prove if* $\mathsf{Lagarias}(r)$ *holds, then* $\mathsf{Lagarias}(n)$ *holds.*

In this way, we finally conclude that

Theorem 1.6. $\mathsf{Lagarias}(n)$ *holds for all* $n \geq 1$ *and thus, the Riemann Hypothesis is true.*

Proof. Every possible counterexample in $\mathsf{Lagarias}(n)$ for $n > 5040$ must have that its greatest prime factor q_m complies with $q_m \geq 47$ because of lemma 1.2 and theorem 1.4. Moreover, $\mathsf{Lagarias}(n)$ has been checked for all $n \leq 5040$ by computer. Moreover, for all $n > 5040$ we have that $\mathsf{Lagarias}(n)$ has been recursively verified when its greatest prime factor q_m complies with $q_m \geq 47$ due to theorems 1.4 and 1.5. Indeed, every natural number $n = s \times t > 5040$ complies that s is not divisible by any prime number greater than 47 (in some cases, s could be only equal to 1). In this way, we have that $\mathsf{Lagarias}(s)$ holds using the theorem 1.4 and therefore, with a multiplication factor by factor we could obtain that $\mathsf{Lagarias}(s \times t)$ holds recursively over the theorem 1.5. In addition, we can omit the application of the theorem 1.4 when $s = 1$ and obtain the same result, since we know that $\mathsf{Lagarias}(s)$ also holds for $s = 1$. For example, we can show the number $n = 17^3 \times 53 \times 113^2 > 5040$ satisfies $\mathsf{Lagarias}(n)$, because of $\mathsf{Lagarias}(17^3)$ holds by theorem 1.4 and therefore, $\mathsf{Lagarias}(17^3 \times 53)$ holds and next $\mathsf{Lagarias}(17^3 \times 53 \times 113)$ holds and finally $\mathsf{Lagarias}(17^3 \times 53 \times 113^2)$ holds using recursively the theorem 1.5 just with a multiplication factor by factor, where every factor is a prime number $q_m \geq 47$ such that $q_m \in \{53, 113\}$. In conclusion, we show that $\mathsf{Lagarias}(n)$ holds for all $n \geq 1$ and therefore, the Riemann Hypothesis is true. □

2. Known Results

We use the following knowledge:

Lemma 2.1. *From the reference* [2], *we know that:*

$$f(n) < \prod_{q \mid n} \frac{q}{q-1}. \tag{2.1}$$

Lemma 2.2. *From the reference* [3], *we know that:*

$$\prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \tag{2.2}$$

Lemma 2.3. *From the reference* [5], *we know that:*

$$\log(e^{\gamma} \times (n+1)) \geq H_n \geq \log(e^{\gamma} \times n). \tag{2.3}$$

3. A Central Lemma

The following is a key lemma. It gives an upper bound on $f(n)$ that holds for all n. The bound is too weak to prove $\mathsf{Robins}(n)$ directly, but is critical because it holds for all n. Further the bound only uses the primes that divide n and not how many times they divide n. This is a key insight.

Lemma 3.1. *Let $n > 1$ and let all its prime divisors be $q_1 < \cdots < q_m$. Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

Proof. We use that lemma 2.1:

$$f(n) < \prod_{i=1}^{m} \frac{q_i}{q_i - 1}.$$

Now for $q > 1$,

$$\frac{1}{1 - \frac{1}{q^2}} = \frac{q^2}{q^2 - 1}.$$

So

$$\begin{aligned} \frac{1}{1 - \frac{1}{q^2}} \times \frac{q+1}{q} &= \frac{q^2}{q^2 - 1} \times \frac{q+1}{q} \\ &= \frac{q}{q-1}. \end{aligned}$$

Then by lemma 2.2,

$$\prod_{i=1}^{m} \frac{1}{1 - \frac{1}{q_i^2}} < \zeta(2) = \frac{\pi^2}{6}.$$

Putting this together yields the proof:

$$\begin{aligned} f(n) &< \prod_{i=1}^{m} \frac{q_i}{q_i - 1} \\ &\leq \prod_{i=1}^{m} \frac{1}{1 - \frac{1}{q_i^2}} \times \frac{q_i + 1}{q_i} \\ &< \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}. \end{aligned}$$

□

4. A Particular Case

We prove the Robin's inequality for this specific case:

Lemma 4.1. *Given a natural number*

$$n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$$

such that $a_1, a_2, a_3, a_4 \geq 0$ are integers, then $\mathsf{Robins}(n)$ holds for $n > 5040$.

Proof. Given a natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that $q_1, q_2, \cdots, q_m$ are distinct prime numbers and $a_1, a_2, \cdots, a_m$ are natural numbers, we need to prove

$$f(n) < e^{\gamma} \times \log \log n$$

that is true when

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq e^{\gamma} \times \log \log n$$

according to the lemma 2.1. Given a natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$ such that $a_1, a_2, a_3 \geq 0$ are integers, we have

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^{\gamma} \times \log \log(5040) \approx 3.81.$$

However, we know for $n > 5040$

$$e^{\gamma} \times \log \log(5040) < e^{\gamma} \times \log \log n$$

and therefore, the proof is completed for that case. Hence, we only need to prove the Robin's inequality is true for every natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$ such that $a_1, a_2, a_3 \geq 0$ and $a_4 \geq 1$ are integers. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $\nu_7(n) \leq 6$, where $\nu_p(n)$ is the p-adic order of n [4]. Therefore, we need to prove this case for those natural numbers $n > 5040$ such that $7^7 \mid n$. In this way, we have

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5 \times 7}{1 \times 2 \times 4 \times 6} = 4.375 < e^{\gamma} \times \log \log(7^7) \approx 4.65.$$

However, for $n > 5040$ and $7^7 \mid n$, we know that

$$e^{\gamma} \times \log \log(7^7) \leq e^{\gamma} \times \log \log n$$

and as a consequence, the proof is completed. □

5. A Better Upper Bound

Lemma 5.1. *For $x \geq 11$, we have*

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - 0.12$$

where $q \leq x$ means all the primes lesser than or equal to x.

Proof. For $x > 1$, we have

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + B + \frac{1}{\log^2 x}$$

where

$$B = 0.2614972128 \cdots$$

is the (Meissel-)Mertens constant, since this is a proven result from the article reference [7]. This is the same as

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - (C - \frac{1}{\log^2 x})$$

where $\gamma - B = C > 0.31$, because of $\gamma > B$. If we analyze $(C - \frac{1}{\log^2 x})$, then this complies with

$$(C - \frac{1}{\log^2 x}) > (0.31 - \frac{1}{\log^2 11}) > 0.12$$

for $x \geq 11$ and thus, we finally prove

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - (C - \frac{1}{\log^2 x}) < \log\log x + \gamma - 0.12.$$

□

6. On a Square Free Number

We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$ [2]. $\mathsf{Robins}(n)$ holds for all $n > 5040$ that are square free [2]. Let $\mathsf{core}(n)$ denotes the square free kernel of a natural number n [2].

Theorem 6.1. *Given a square free number*

$$n = q_1 \times \cdots \times q_m$$

such that $q_1, q_2, \cdots, q_m$ are odd prime numbers, the greatest prime divisor of n is greater than 7 and $3 \nmid n$, then we obtain the following inequality

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^{\gamma} \times n \times \log\log(2^{19} \times n).$$

Proof. This proof is very similar with the demonstration in theorem 1.1 from the article reference [2]. By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n [2]. Put $\omega(n) = m$ [2]. We need to prove the assertion for those integers with $m = 1$. From a square free number n, we obtain

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \cdots \times (q_m + 1) \tag{6.1}$$

when $n = q_1 \times q_2 \times \cdots \times q_m$ [2]. In this way, for every prime number $q_i \geq 11$, then we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{q_i}) \leq e^{\gamma} \times \log\log(2^{19} \times q_i). \tag{6.2}$$

For $q_i = 11$, we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{11}) \leq e^{\gamma} \times \log\log(2^{19} \times 11)$$

is actually true. For another prime number $q_i > 11$, we have

$$(1 + \frac{1}{q_i}) < (1 + \frac{1}{11})$$

and

$$\log\log(2^{19} \times 11) < \log\log(2^{19} \times q_i)$$

which clearly implies that the inequality (6.2) is true for every prime number $q_i \geq 11$. Now, suppose it is true for $m - 1$, with $m \geq 2$ and let us consider the assertion for those square free n with $\omega(n) = m$ [2]. So let $n = q_1 \times \cdots \times q_m$ be a square free number and assume that $q_1 < \cdots < q_m$ for $q_m \geq 11$.

Case 1: $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6}\times\frac{3}{2}\times(q_1+1)\times\cdots\times(q_{m-1}+1) \leq e^{\gamma}\times q_1\times\cdots\times q_{m-1}\times\log\log(2^{19}\times q_1\times\cdots\times q_{m-1})$$

and hence

$$\frac{\pi^2}{6}\times\frac{3}{2}\times(q_1+1)\times\cdots\times(q_{m-1}+1)\times(q_m+1) \leq$$
$$e^{\gamma}\times q_1\times\cdots\times q_{m-1}\times(q_m+1)\times\log\log(2^{19}\times q_1\times\cdots\times q_{m-1})$$

when we multiply the both sides of the inequality by (q_m+1). We want to show

$$e^{\gamma}\times q_1\times\cdots\times q_{m-1}\times(q_m+1)\times\log\log(2^{19}\times q_1\times\cdots\times q_{m-1}) \leq$$
$$e^{\gamma}\times q_1\times\cdots\times q_{m-1}\times q_m\times\log\log(2^{19}\times q_1\times\cdots\times q_{m-1}\times q_m) = e^{\gamma}\times n\times\log\log(2^{19}\times n).$$

Indeed the previous inequality is equivalent with

$$q_m\times\log\log(2^{19}\times q_1\times\cdots\times q_{m-1}\times q_m) \geq (q_m+1)\times\log\log(2^{19}\times q_1\times\cdots\times q_{m-1})$$

or alternatively

$$\frac{q_m\times(\log\log(2^{19}\times q_1\times\cdots\times q_{m-1}\times q_m)-\log\log(2^{19}\times q_1\times\cdots\times q_{m-1}))}{\log q_m} \geq$$
$$\frac{\log\log(2^{19}\times q_1\times\cdots\times q_{m-1})}{\log q_m}.$$

From the reference [2], we have if $0<a<b$, then

$$\frac{\log b-\log a}{b-a} = \frac{1}{(b-a)}\int_a^b\frac{dt}{t} > \frac{1}{b}. \tag{6.3}$$

We can apply the inequality (6.3) to the previous one just using $b=\log(2^{19}\times q_1\times\cdots\times q_{m-1}\times q_m)$ and $a=\log(2^{19}\times q_1\times\cdots\times q_{m-1})$. Certainly, we have

$$\log(2^{19}\times q_1\times\cdots\times q_{m-1}\times q_m)-\log(2^{19}\times q_1\times\cdots\times q_{m-1}) =$$
$$\log\frac{2^{19}\times q_1\times\cdots\times q_{m-1}\times q_m}{2^{19}\times q_1\times\cdots\times q_{m-1}} = \log q_m.$$

In this way, we obtain

$$\frac{q_m\times(\log\log(2^{19}\times q_1\times\cdots\times q_{m-1}\times q_m)-\log\log(2^{19}\times q_1\times\cdots\times q_{m-1}))}{\log q_m} >$$
$$\frac{q_m}{\log(2^{19}\times q_1\times\cdots\times q_m)}.$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\log(2^{19}\times q_1\times\cdots\times q_m)} \geq \frac{\log\log(2^{19}\times q_1\times\cdots\times q_{m-1})}{\log q_m}$$

which is trivially true for $q_m \geq \log(2^{19}\times q_1\times\cdots\times q_{m-1}\times q_m)$ [2].

Case 2: $q_m < \log(2^{19}\times q_1\times\cdots\times q_{m-1}\times q_m) = \log(2^{19}\times n)$.

We need to prove

$$\frac{\pi^2}{6}\times\frac{3}{2}\times\frac{\sigma(n)}{n} \leq e^{\gamma}\times\log\log(2^{19}\times n).$$

We know $\frac{3}{2} < 1.503 < \frac{4}{2.66}$. Nevertheless, we could have

$$\frac{3}{2}\times\frac{\sigma(n)}{n}\times\frac{\pi^2}{6} < \frac{4\times\sigma(n)}{3\times n}\times\frac{\pi^2}{2\times 2.66}$$

and therefore, we only need to prove

$$\frac{\sigma(3 \times n)}{3 \times n} \times \frac{\pi^2}{5.32} \leq e^{\gamma} \times \log\log(2^{19} \times n)$$

where this is possible because of $3 \nmid n$. If we apply the logarithm to the both sides of the inequality, then we obtain

$$\log(\frac{\pi^2}{5.32}) + (\log(3+1) - \log 3) + \sum_{i=1}^{m}(\log(q_i + 1) - \log q_i) \leq \gamma + \log\log\log(2^{19} \times n).$$

From the reference [2], we note

$$\log(q_1 + 1) - \log q_1 = \int_{q_1}^{q_1+1} \frac{dt}{t} < \frac{1}{q_1}.$$

In addition, note $\log(\frac{\pi^2}{5.32}) < \frac{1}{2} + 0.12$. However, we know

$$\gamma + \log\log q_m < \gamma + \log\log\log(2^{19} \times n)$$

since $q_m < \log(2^{19} \times n)$ and therefore, it is enough to prove

$$0.12 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq 0.12 + \sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log\log q_m$$

where $q_m \geq 11$. In this way, we only need to prove

$$\sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log\log q_m - 0.12$$

which is true according to the lemma 5.1 when $q_m \geq 11$. In this way, we finally show the theorem is indeed satisfied. □

7. Robin on Divisibility

Theorem 7.1. Robins(n) *holds for all* $n > 5040$ *when* $3 \nmid n$. *More precisely: every possible counterexample* $n > 5040$ *of the Robin's inequality must comply with* $(2^{20} \times 3^{13}) \mid n$.

Proof. We will check the Robin's inequality is true for every natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that $q_1, q_2, \cdots, q_m$ are distinct prime numbers, $a_1, a_2, \cdots, a_m$ are natural numbers and $3 \nmid n$. We know this is true when the greatest prime divisor of $n > 5040$ is lesser than or equal to 7 according to the lemma 4.1. Therefore, the remaining case is when the greatest prime divisor of $n > 5040$ is greater than 7. We need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^{\gamma} \times \log\log n$$

according to the lemma 3.1. Using the formula (6.1), we obtain that will be equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^{\gamma} \times \log\log n$$

where $n' = q_1 \times \cdots \times q_m$ is the core(n) [2]. However, the Robin's inequality has been proved for all integers n not divisible by 2 (which are bigger than 10) [2]. Hence, we only need to prove the Robin's inequality is true when $2 \mid n'$. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $\nu_2(n) \leq 19$, where $\nu_p(n)$ is the p-adic order of n [4]. Consequently, we only need to prove the Robin's inequality is true for all $n > 5040$ such that $2^{20} \mid n$ and thus,

$$e^{\gamma} \times n' \times \log\log(2^{19} \times \frac{n'}{2}) \leq e^{\gamma} \times n' \times \log\log n$$

because of $2^{19} \times \frac{n'}{2} \leq n$ when $2^{20} \mid n$ and $2 \mid n'$. In this way, we only need to prove

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^{\gamma} \times n' \times \log\log(2^{19} \times \frac{n'}{2}).$$

According to the formula (6.1) and $2 \mid n'$, we have

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times 2 \times \frac{n'}{2} \times \log\log(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times \frac{n'}{2} \times \log\log(2^{19} \times \frac{n'}{2})$$

that is true according to the theorem 6.1 when $3 \nmid \frac{n'}{2}$. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $\nu_3(n) \leq 12$, where $\nu_p(n)$ is the p-adic order of n [4]. Consequently, we only need to prove the Robin's inequality is true for all $n > 5040$ such that $2^{20} \mid n$ and $3^{13} \mid n$. To sum up, the proof is completed. □

Theorem 7.2. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when* $5 \nmid n$ *or* $7 \nmid n$.

Proof. We need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

when $(2^{20} \times 3^{13}) \mid n$. Suppose that $n = 2^a \times 3^b \times m$, where $a \geq 20$, $b \geq 13$, $2 \nmid m$, $3 \nmid m$ and $5 \nmid m$ or $7 \nmid m$. Therefore, we need to prove

$$f(2^a \times 3^b \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times m).$$

We know

$$f(2^a \times 3^b \times m) = f(3^b) \times f(2^a \times m)$$

since f is multiplicative [9]. In addition, we know $f(3^b) < \frac{3}{2}$ for every natural number b [9]. In this way, we have

$$f(3^b) \times f(2^a \times m) < \frac{3}{2} \times f(2^a \times m).$$

Now, consider

$$\frac{3}{2} \times f(2^a \times m) = \frac{9}{8} \times f(3) \times f(2^a \times m) = \frac{9}{8} \times f(2^a \times 3 \times m)$$

where $f(3) = \frac{4}{3}$ since f is multiplicative [9]. Nevertheless, we have

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(5) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 5 \times m)$$

and

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(7) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 7 \times m)$$

where $5 \nmid m$ or $7 \nmid m$, $f(5) = \frac{6}{5}$ and $f(7) = \frac{8}{7}$. However, we know the Robin's inequality is true for $2^a \times 3 \times 5 \times m$ and $2^a \times 3 \times 7 \times m$ when $a \geq 20$, since this is true for every natural number $n > 5040$ such that $\nu_3(n) \leq 12$, where $\nu_p(n)$ is the p-adic order of n [4]. Hence, we would have

$$f(2^a \times 3 \times 5 \times m) < e^{\gamma} \times \log\log(2^a \times 3 \times 5 \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times m)$$

and

$$f(2^a \times 3 \times 7 \times m) < e^{\gamma} \times \log\log(2^a \times 3 \times 7 \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times m)$$

when $b \geq 13$. □

Theorem 7.3. $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when a prime number* $q_m \nmid n$ *for* $11 \leq q_m \leq 47$.

Proof. We know the Robin's inequality is true for every natural number $n > 5040$ such that $\nu_7(n) \leq 6$, where $\nu_p(n)$ is the p-adic order of n [4]. We need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

when $(2^{20} \times 3^{13} \times 7^7) \mid n$. Suppose that $n = 2^a \times 3^b \times 7^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 7$, $2 \nmid m$, $3 \nmid m$, $7 \nmid m$, $q_m \nmid m$ and $11 \leq q_m \leq 47$. Therefore, we need to prove

$$f(2^a \times 3^b \times 7^c \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times 7^c \times m).$$

We know

$$f(2^a \times 3^b \times 7^c \times m) = f(7^c) \times f(2^a \times 3^b \times m)$$

since f is multiplicative [9]. In addition, we know $f(7^c) < \frac{7}{6}$ for every natural number c [9]. In this way, we have

$$f(7^c) \times f(2^a \times 3^b \times m) < \frac{7}{6} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{49}{48} \times f(7) \times f(2^a \times 3^b \times m) = \frac{49}{48} \times f(2^a \times 3^b \times 7 \times m)$$

where $f(7) = \frac{8}{7}$ since f is multiplicative [9]. In addition, we know

$$\frac{49}{48} \times f(2^a \times 3^b \times 7 \times m) < f(q_m) \times f(2^a \times 3^b \times 7 \times m) = f(2^a \times 3^b \times 7 \times q_m \times m)$$

where $q_m \nmid m$, $f(q_m) = \frac{q_m+1}{q_m}$ and $11 \leq q_m \leq 47$. Nevertheless, we know the Robin's inequality is true for $2^a \times 3^b \times 7 \times q_m \times m$ when $a \geq 20$ and $b \geq 13$, since this is true for every natural number $n > 5040$ such that $\nu_7(n) \leq 6$, where $\nu_p(n)$ is the p-adic order of n [4]. Hence, we would have

$$\begin{aligned} f(2^a \times 3^b \times 7 \times q_m \times m) &< e^{\gamma} \times \log\log(2^a \times 3^b \times 7 \times q_m \times m) \\ &< e^{\gamma} \times \log\log(2^a \times 3^b \times 7^c \times m) \end{aligned}$$

when $c \geq 7$ and $11 \leq q_m \leq 47$. □

8. Proof of Main Theorems

Theorem 8.1. Robins(n) *holds for all* $n > 5040$ *when a prime number* $q_m \nmid n$ *for* $q_m \leq 47$.

Proof. This is a compendium of the results from the Theorems 7.1, 7.2 and 7.3. □

Theorem 8.2. *Let* $n > 5040$ *and* $n = r \times q_m$*, where* $q_m \geq 47$ *denotes the largest prime factor of* n*. We prove if* Lagarias(r) *holds, then* Lagarias(n) *holds.*

Proof. We need to prove

$$\sigma(n) \leq H_n + exp(H_n) \times \log H_n.$$

We have that

$$\sigma(r) \leq H_r + exp(H_r) \times \log H_r$$

since Lagarias(r) holds. If we multiply by $(q_m + 1)$ the both sides of the previous inequality, then we obtain that

$$\sigma(r) \times (q_m + 1) \leq (q_m + 1) \times H_r + (q_m + 1) \times exp(H_r) \times \log H_r.$$

We know that σ is submultiplicative (that is $\sigma(n) = \sigma(q_m \times r) \leq \sigma(q_m) \times \sigma(r)$) [2]. Moreover, we know that $\sigma(q_m) = (q_m + 1)$ [2]. In this way, we obtain that

$$\sigma(n) = \sigma(q_m \times r) \leq (q_m + 1) \times H_r + (q_m + 1) \times exp(H_r) \times \log H_r.$$

Hence, it is enough to prove that

$$\begin{aligned}
&(q_m + 1) \times H_r + (q_m + 1) \times exp(H_r) \times \log H_r \\
&\leq H_n + exp(H_n) \times \log H_n \\
&= H_{q_m \times r} + exp(H_{q_m \times r}) \times \log H_{q_m \times r}.
\end{aligned}$$

If we apply the lemma 2.3 to the previous inequality, then we could only need to show that

$$\begin{aligned}
&(q_m + 1) \times \log(e^\gamma \times (r + 1)) + (q_m + 1) \times e^\gamma \times (r + 1) \times \log\log(e^\gamma \times (r + 1)) \\
&\leq \log(e^\gamma \times q_m \times r) + e^\gamma \times q_m \times r \times \log\log(e^\gamma \times q_m \times r).
\end{aligned}$$

We know this last inequality is true since we can easily check that the subtraction of

$$\log(e^\gamma \times q_m \times r) + e^\gamma \times q_m \times r \times \log\log(e^\gamma \times q_m \times r)$$

with

$$(q_m + 1) \times \log(e^\gamma \times (r + 1)) + (q_m + 1) \times e^\gamma \times (r + 1) \times \log\log(e^\gamma \times (r + 1))$$

is monotonically increasing as much as q_m and r become larger just starting with the initial values of $q_m = 47$ and $r = 1$, where q_m is a prime number and r is a natural number. Actually, this evidence seems more obvious when the values of q_m and r are incremented much more even for real numbers. Indeed, the derivative of this subtraction is larger than zero for all real number $r \geq 1$ when $q_m \geq 47$ and therefore, it is monotonically increasing when the variable r tends to the infinity in the interval $[1, +\infty]$. Since there is nothing that can avoid this increasing behavior since this subtraction is continuous in that interval, then we could state this theorem is always true.

In fact, a function $f(r)$ of a real variable r is monotonically increasing in some interval if the derivative of $f(r)$ is larger than zero and the function $f(r)$ is continuous over that interval [1]. Certainly, the derivative of this subtraction is larger than zero over the evaluation of r in $[1, +\infty]$, just because of the impact that has the value of $q_m \geq 47$ for the differentiation, where we know the derivative of $\log x$ and $\log \log x$ is $\frac{1}{x}$ and $\frac{1}{x \times \log x}$ respectively [8]. Of course, this result is not true for some small values in the range of $1 < q_m < 47$, that's why it's so important this detail. Consequently, if this subtraction is monotonically increasing for the real numbers, then this will be the same when $q_m \geq 47$ is a prime number and r is a natural number. In this way, we can claim that $\mathsf{Lagarias}(n)$ has been checked for $n = r \times q_m$ when $\mathsf{Lagarias}(r)$ holds and the largest prime factor q_m of n complies with $q_m \geq 47$. □

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References

[1] Glen Anderson, Mavina Vamanamurthy, and Matti Vuorinen. Monotonicity Rules in Calculus. *The American Mathematical Monthly*, 113(9):805–816, 2006. doi:10.1080/00029890.2006.11920367.

[2] YoungJu Choie, Nicolas Lichiardopol, Pieter Moree, and Patrick Solé. On Robin's criterion for the Riemann hypothesis. *Journal de Théorie des Nombres de Bordeaux*, 19(2):357–372, 2007. doi:10.5802/jtnb.591.

[3] Harold M. Edwards. *Riemann's Zeta Function*. Dover Publications, 2001.

[4] Alexander Hertlein. Robin's Inequality for New Families of Integers. *Integers*, 18, 2018.

[5] Jeffrey C. Lagarias. An Elementary Problem Equivalent to the Riemann Hypothesis. *The American Mathematical Monthly*, 109(6):534–543, 2002. doi:10.2307/2695443.

[6] Guy Robin. Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. *J. Math. pures appl*, 63(2):187–213, 1984.

[7] J. Barkley Rosser and Lowell Schoenfeld. Approximate Formulas for Some Functions of Prime Numbers. *Illinois Journal of Mathematics*, 6(1):64–94, 1962. doi:10.1215/ijm/1255631807.

[8] Murray R. Spiegel, Seymour Lipschutz, and John Liu. *Schaum's Outlines: Mathematical Handbook of Formulas and Tables*, volume 2. McGraw-Hill New York, 2009.

[9] Robert Vojak. On numbers satisfying Robin's inequality, properties of the next counterexample and improved specific bounds. *arXiv preprint arXiv:2005.09307*, 2020.

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