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RESEARCH ARTICLE

Two-Parameter Generalization of the Collatz Function
Characterization of Terminal Cycles and Empirical Results

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Abstract

This paper proposes a new two-parameter generalization $T_{b,s}(x)$ of the $\mathbb{Z} \to \mathbb{Z}$ Collatz function $T(x)$ and restates the eponymous conjecture in terms of the proposed function. The generalization obviates some of the conditions for emergence of terminal cycles for the Collatz $T(x)$ function over the integers. The stopping behavior of the $T_{b,s}(x)$ is qualitatively similar to that of the $T(x)$. The paper presents theoretical discussion of the generalization and computational results on the terminal cycles and stopping times of $T_{b,s}(x)$. The $\{1,2\}$ cycle of $T(x)$ is shown to be a case of coincidence of three independent cycle categories of $T_{b,s}(x)$.

Keywords: Collatz conjecture, $3x+1$ problem, iteration, convergence, terminal cycles

Mathematics Subject Classification: 11B50, 11B83, 26A18

1 Introduction

The Collatz or $3x+1$ conjecture, also known as the Syracuse, Kakutani, Ulam's or Hasse's problem [3] [7] is a long-standing open problem in mathematics with extensive bibliography [8] [9]. A wide array of methods have been applied to it with some success and it has been numerically verified for starting numbers up to 5.7646×10^{18} [10]. Korec [5], for example, has shown that the set of natural numbers which iterate to a lower number has an asymptotic density of 1. The stopping behavior of $T(n)$ has been discussed in some detail by Terras [12]. A recent work by Tao [11] has proven rigorously that almost all orbits of the problem have bounded values. A complete proof, however, still remains elusive.

In its $(3x+1)$ problem formulation the conjecture states that the function

$$T(n) = \begin{Bmatrix} n/2, & \text{if } n \equiv 0 (mod\, 2) \\ (3n+1)/2, & \text{if } n \equiv 1 (mod\, 2) \end{Bmatrix} \tag{1}$$

applied to any natural number n, always reaches 1 after a finite number of iterations. The stopping time $\sigma(n)$ is defined as the number of iterations it takes for $T(n)$ to reach 1 starting from n. An orbit of a number is the sequence of numbers resulting from an iterative application of $T(n)$ to a starting number n.

As pointed out by Chamberland [2], there are three possibilities for an orbit: convergence to the cycle $\{1,2\}$, convergence to another cycle, or divergence. Thus proving (or disproving) the Collatz conjecture is equivalent to proving the absence (or presence) of cycles other than $\{1,2\}$ and/or of divergent orbits. In this vein, it is important to study the conditions that govern convergence and the conditions that lead to the emergence of additional terminal cycles.

Our motivation for proposing a new parametric generalization is to seek an approach that would further the understanding of this problem via understanding the behavior of a problem superset, where terminal cycles readily appear at different, and relatively small, values of n. While many generalizations of the problem have been proposed [1] [2] [4] [6], our proposed parametric generalization $T_{b,s}(x)$ permits a simpler analysis of the behavior of $T(n)$ and its extension over all integers $T(x)$, and especially the convergence behavior and appearance of terminal cycles as a function of the two parameters, b and s. Further, it shows that analogs of all known terminal cycles for $T(x)$ are present for other $T_{b,s}(x)$ functions.

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2 The b,s-Collatz Generalization

A two-parameter generalization of the *Collatz function* over the integer numbers can be defined compactly as follows:

Definition 2.1. *We shall call a b,s-Collatz function the $\mathbb{Z} \to \mathbb{Z}$ function*

$$T_{b,s}(z) = \frac{z-x}{b} + (z+s+1) \times \theta_b(z) \tag{2}$$

where $x \equiv z \pmod{b}$ and the step function $\theta_b(0) = 0$ and $\theta_b(x \neq 0) = 1$. In this generalization, $b \in \mathbb{N}$ and $s \in \mathbb{Z}$.

The *b,s-Collatz function* definition is equivalent to:

$$T_{b,s}(z) = \left\{ \begin{array}{cc} z/b, & \text{if } 0 \equiv z(mod\, b) \\ ((b+1) \times z + (s+1) \times b - x)/b, & \text{if } 0 \neq x \equiv z(mod\, b) \end{array} \right\}, \tag{3}$$

Definition 2.2. *We shall call* ***terminal cycles*** *$L^i_{b,s}$ the non-trivial subsets of $\mathbb{Z}$ which are automorphic under $T_{b,s}(z)$ and have no non-trivial automorphic subsets.*

Definition 2.3. *We shall call* ***length*** *$\rho^i_{b,s}$ of $L^i_{b,s}$ the number of elements of $L^i_{b,s}$.*

Definition 2.4. *We shall call a* ***trivial cycle*** *any terminal cycle of length 1.*

Definition 2.5. *The* ***root*** *$\mu^i_{b,s}$ of a terminal cycle $L^i_{b,s}$ is its element with least absolute value. Since for any $T_{b,s}$, the sets $L^i_{b,s}$ are non-intersecting, they are uniquely identified by their least elements. For convenience, we will index $L^i_{b,s}$ in increasing order of their roots $\mu^i_{b,s}$.*

Definition 2.6. *We shall call a* ***parametric cycle*** *any terminal cycle with a shown analytical dependence $\mu^i_{b,s} = f(b,s)$.*

Definition 2.7. *The* ***stopping time*** *$\sigma_{b,s}(z)$ of z is the number of iterations of $T_{b,s}(z)$ to reach an element of any terminal cycle $L^i_{b,s}$.*

It can be easily shown by the setting $s = 0$ that $T_{2,0}(n)$, $n \in \mathbb{N}$ is the Collatz function in the $3n+1$ function formulation:

$$T_{2,0}(n) = \frac{n-x}{2} + (n+1) \times x, \tag{4}$$

which is equivalent to equation 1. Note that the definition of *stopping time* varies slightly from the definition of stopping time formulated for the $(3x+1)$ function in the Introduction. In our definition, stopping time for the $(3x+1)$ function is the time for iterations to reach 1 or 2 (not just 1).

With the above definitions, we can also restate the Collatz conjecture as follows: $T_{2,0}(n)$ *has only one terminal cycle, $L^1_{2,0} = \{1,2\}$ in $\mathbb{N}$.*

Also, $T_{b,0}(n)$, $n \in N$ gives:

$$T_{b,0}(n) \equiv T_b = \frac{n-x}{b} + (n+1)\theta_b(n), \tag{5}$$

which is equivalent to the *b-Collatz* generalization recently proposed by Gutierrez [4]:

$$T_{b,0}(n) \equiv T_b = \left\{ \begin{array}{cc} n/b, & \text{if } n(mod\, b) = 0 \\ ((b+1) \times n + b - x)/b, & \text{if } x = n(mod\, b) \neq 0 \end{array} \right\}. \tag{6}$$

The *b,s-Collatz* generalization yields series such as:

$$T_{2,2}(z) = \left\{ \begin{array}{ll} z/2 & \text{if } n \equiv 0\,(mod\, 2) \\ (3z+5)/2 & \text{if } z \equiv 0\,(mod\, 2) \end{array} \right\}, \tag{7}$$

$$T_{3,1} = \left\{ \begin{array}{ll} z/3 & \text{if } z \equiv 0\,(mod\, 3) \\ (4z+5)/3 & \text{if } z \equiv 1\,(mod\, 3) \\ (4z+4)/3 & \text{if } z \equiv 2\,(mod\, 3) \end{array} \right\}, \tag{8}$$

$$T_{2,-1}(n) = \left\{ \begin{array}{ll} n/2 & \text{if } n \equiv 0\,(mod\, 2) \\ (3n-1)/2 & \text{if } n \equiv 0\,(mod\, 2) \end{array} \right\}. \tag{9}$$

3 Properties of the b,s-Collatz Function

Theorem 3.1.

$$T_{b,s}(-z) = -T_{b,-1-s}(z). \tag{10}$$

Proof.

$$T_{b,s}(-z) = \frac{-z-b+x}{b} + (-z+s+1) \times \theta_b(-z) =$$

$$\frac{-z+x}{b} + (-z+(s+1)-1) = -(\frac{z-x}{b} + z + (-1-s)+1) = -T_{b,-1-s}(z).$$

□

It follows from this theorem that we can restrict our investigation to the *b,s-Collatz* series for $s \geq 0$ without loss of generality, as all series for $s < 0$ can be derived from them.

Theorem 3.2. *Let $b,k,s,t \in N$ and $a \in \mathbb{Z}$ such that $a \pmod b \neq 0$, $b > t, s, > 0$, $s \mid t$, $k \geq t/s$ and $z = a \times b^k - t$. Then*

$$T^k_{b,s}(z) = a \times (b+1)^{t/s}. \tag{11}$$

Proof. We note that $b - t \equiv z \pmod b$. Since $s \mid t$ we can write $t = m \times s$, $m \in \mathbb{N}$ Then,

$$T^1_{b,s}(z) = \frac{(a \times b^k - t) - (b-t)}{b} + a \times b^k - t + s + 1 = a \times b^{k-m} \times (b+1)^m - t + s$$

$$T^m_{b,s}(z) = a \times b^{k-m} \times (b+1)^m - t + m \times s, = a \times b^{k-m} \times (b+1)^m.$$

For the $m < j < k, \theta^j_b(T_{b,s}(z) = 0$, yielding

$$T^j_{b,s}(z) = a \times b^{k-j} \times (b+1)^{t/s},$$

such that for $j = k$,

$$T^k_{b,s}(z) = a \times (b+1)^{t/s},$$

□

Corollary 3.1. *If $\sigma_{b,s}(z)$ is the stopping time for $(T_{b,s}(z)$ and $b,k.s,t \in N$ and $a \in Z$ such that $a \pmod b \neq 0$, $b > t, s, > 0$, $s \mid t$, $k \geq t/s$ and $z = a \times b^k - t$. Then*

$$\sigma_{b,s}(ab^k - t) = \sigma_{b,s}(a(b+1)^{t/s}) + k. \tag{12}$$

Theorem 3.3. *Let $b,k,t \in N$ and $a \in Z$ such that $a \pmod b \neq 0$, $b > t > 0$, and $z = a \times b^k - t$. Then*

$$T^k_{b,0}(z) = a \times (b+1)^k. \tag{13}$$

Proof. We note that $b - t \equiv n \pmod b$. Then,

$$T^1_{b,0}(n) = \frac{(a \times b^k - t) - (b-t)}{b} + a \times b^k - t + 1 = a \times b^{k-1} \times (b+1) - t$$

$$T^k_{b,0}(n) = a \times b^{k-k} \times (b+1)^k - t = a \times (b+1)^k - t.$$

□

Corollary 3.2. *If $\sigma_{b,s}(z)$ is the stopping time for $(T_{b,s}(z), n \in \mathbb{N}$ and $z = a \times b^k - t$, then*

$$\sigma_{b,s}(ab^k - t) = \sigma_{b,s}(a(b+1)^k) + k. \tag{14}$$

Theorem 3.4. *For any $T_{b,s}(x)$, where $x, b \in \mathbb{N}, s \in \mathbb{Z}$, $0 < x < b$, and $s \leq -1$,*

$$T_{b,s}(x) = x + s + 1. \tag{15}$$

Proof. From $0 < x < b$ follows that $0 \neq x \pmod{()b}$ and $\theta_b x = 1$. Then from equation 2, $T_{b,s}(x) = x + s + 1$. □

4 Terminal Cycles of the b.s-Collatz Function

Theorem 4.1. *Any $T_{b,s}$ has a trivial cycle $L^0_{b,s} = \{0\}$.*

Proof. $\theta_b(0) = 0$ by definition, so $T_{b,s}(0) = 0/b = 0$. □

Theorem 4.2. *$T_{b,0}$ has $(n-1)$ trivial cycles $L^{-x}_{b,0} = \{-x\}$ for $0 < x < b$.*

Proof. We note that $-x \equiv b - x \pmod b$. Then from equation 2

$$T_{b,0}(-x) = \frac{-x-(b-x)}{b} + (-x+1) = -x.$$

□

Theorem 4.3. *$T_{b,s}(z), s > 0$ has b-1 trivial cycles $L^{-s}(-x)_{b,s} = \{-b \times s - x\}, 0 < x < b$.*

Proof. We note that $b - x \equiv -b \times s - x \pmod b$. Then from equation 2:

$$T_{b,s}(-b \times s - x) = \frac{-b \times s - x - (b-x)}{b} + -b \times s - x + s + 1 =$$
$$= -s - 1 + -b \times s - x + s + 1 = -b \times s - x.$$

□

Theorem 4.4. *$T_{b,0}(x)$ has a parametric cycle $L^1_{b,0} = \{1, ..., b\}$.*

Proof. From equation 2: $T_{b,0}(x) = x + 1$ for $0 < x < b$, so $T_{b,0}(1) = 2$ can be iterated until the result is b. Then $T_{b,0}(b) = 1$, which completes the cycle. □

Theorem 4.5. *$T_{b,b-2}(x)$ has a parametric cycle $L^1_{b,b-2} = \{1, b\}$.*

Proof. From equation 2: $T_{b,b-2}(1) = 1 + b - 2 + 1 = b$. Then $T_{b,b-2}(b) = b/b = 1$. □

Theorem 4.6. *$T_{2k+1,1}(x)$ has a parametric cycle $L^1_{2k+1,1} = \{1, ..., 2k+1\}$.*

Proof. $T_{2k+1,1}(x) = x + 2$ for $0 < x < b$, so each iteration adds 2 until the result is 2k+1. Then $T_{2k+1,0}(2k+1) = 1$, which completes the cycle. □

Theorem 4.7. *$T_{2,s}(x)$ has a parametric cycle $L^{2s+1}_{2,s} = \{2s+1, 4s+2\}$.*

Proof. From equation 2: $T_{2,s}(2s+1) = \frac{2s+1-1}{2} + 2s + 1 + s + 1 = 4s + 2$. Then $T_{2,s}(4s+2) = 2s + 1$. □

Theorem 4.8. *$T_{2,s}(x)$ has a parametric cycle $L^{-10s-5}_{2,s} = \{-10s-5, -14s-7, -20s-10\}$.*

Proof. We prove this by successive iterations of equation 2.

$$T_{2,s}(-10s-5) = \frac{-10s-5-1}{2} - 10s - 5 + s + 1 = -14s - 7,$$
$$T_{2,s}(-14s-7) = \frac{-14s-7-1}{2} - 14s - 7 + s + 1 = -20s - 10,$$
$$T_{2,s}(-20s-10) = \frac{-20s-10}{2} = -10s - 5.$$

□

Theorem 4.9. *$T_{2,s}(x)$ has a parametric cycle*

$$L^{-34s-17}_{2,s} = \{-34s-17, -50s-25, -74s-37, -110s-55, -164s-82, -82s-41,$$
$$-122s-61, -182s-91, -272s-136, -136s-68, -68s-34\}.$$

Proof. We prove this by successive iterations of equation 2. For the sake of brevity we omit the long and straightforward arithmetic proof similar to the proof above. □

Tables 1 and 2 summarize the above results for trivial and parametric cycles. The above theorems also allow us to again offer a restatement of the Collatz conjecture as follows: *The function $T_{2,0}$ has no non-parametric terminal cycles.*

We note here that the $\{1, 2\}$ cycle of the $(3x+1)$ problem belongs to the first, third, and fourth categories of parametric cycles in Table 2, and can thus be considered a triply degenerate parametric cycle. This may be a clue to the absence of other terminal cycles in the problem.

Table 1. Trivial cycles ($\rho^i_{b,s} = 1$)

b	s	$\mu^i_{b,s}$
$b > 1$	$-\infty \le s \le \infty$	0
$b > 1$	$s \ge 0$	$-b \times s - x, 0 < x < b$

Table 2. Non-trivial parametric cycles for $s > 0$

b	s	$\mu^i_{b,s}$ ($\rho^i_{b,s}$)		
2	s	$-17 \times (2s+1)(11)$	$-5 \times (2s+1)(3)$	$(2s+1)(2)$
b	0	$1(b)$		
$2k+1$	1	$1(k)$		
b	$b-2$	$1(2)$		

5 Non-parametric Cycles and Convergence Behavior: Empirical Results

We have calculated the terminal cycles for $0 \le b \le 5$ and $0 \le s \le 5$ for $-10^6 < z < 10^6$. For the sake of brevity, we are presenting only the non-parametric terminal cycles in Table 3. It is notable, that in addition to the Collatz function $T_{2,0}$, there are other cases without non-parametric cycles, i.e. $T_{2,1}$ and $T_{2,4}$. Also, as we would expect, with increasing values of b we see larger values of $\mu^i_{b,s}$ in both the positive $(z > 0)$ and the negative $(z < 0)$ branches. A better understanding of the conditions for appearance of non-parametric cycles may prove useful in proving (or disproving) the Collatz conjecture.

Table 3. Non-parametric terminal cycles for $0 \le b, s \le 5$

b	s			$\mu^i_{b,s}$ ($\rho^i_{b,s}$)			
2	0						
2	1						
2	2	1(3)		19(5)	23(5)	187(27)	347(27)
2	3	5(4)					
2	4						
2	5	-19(4)	1(6)		13(14)		
3	0	-22(5)		7(9)			
3	1	-16(11)		239(24)			
3	2	-104(5)	-100(5)	-97(5)	-88(10)	28(13)	
3	3	-146(5)	2(2)	7(3)	23(4)	50(9)	734(24)
3	4	-1931(82)	-182(5)	-175(5)	1(12)	29(4)	781(24)
3	5	-224(5)	-43(6)	1(14)	37(4)	71(9)	1036(24)
4	0	-18(15)	23(14)				
4	1	1(6)	6(5)	7(61)			
4	2	-49(8)	5(4)				
4	3	1(5)	3(3)	9(15)	1995(36)		
4	4	-1206(29)	7(15)	17(5)	39(6)	267(7)	2327(36)
4	5	-73(16)	2(2)	3(32)	29(19)	327(7)	
5	0	-57(10)					
5	1	6(15)					
5	2	-324(10)	-53(31)	1(25)			

5	3	17(7)	37(36)	83(9)		
5	4	-589(10)	1(34)			
5	5	-7462(59)	-6199(59)	1(14)	51(8)	77(18)

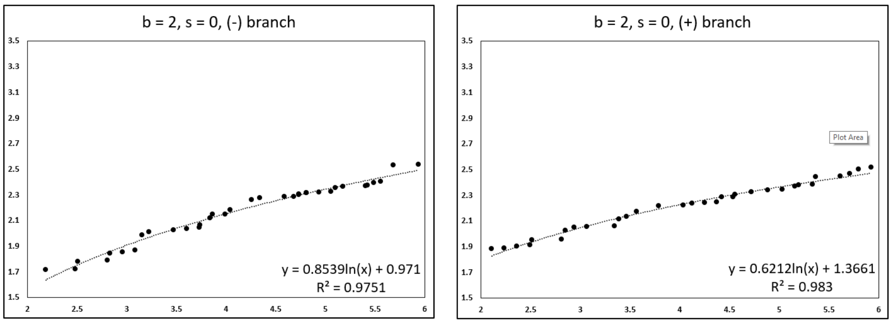

Figure 1. Convergence envelopes for b = 2 , s = 0, (-) and (+) branches..

We also calculated the stopping times $\sigma_{b,s}(z)$ for $-10^6 < z < 10^6$ and selected increasing values of b and s to investigate the convergence behavior of the b,s-Collatz function. For the sake of brevity, we present the results for the negative and positive branches of $\sigma_{b,0}(z)$ for $b = 2, 10$ in Figures 1 and 2. The data presented is limited to the "convergence envelope," i.e. to the stopping times at points z_i, such that $\sigma_{b,0}(z_i) > \sigma_{b,0}(z)$ for any $z < z_i$. The graphs show $log_{10}(\sigma_{b,0}(z_i))$ as a function of $log_{10}(z_i)$ for $10^2 < |z| < 10^6$. The dotted lines show the best fit to a function of the form $f(x) = a \times log(x) + c$. The choice of fitting function is based on the behavior of $\sigma_{2,0}(z)$ which perfectly fits to it for sufficiently large numbers. The positive and negative branches for each value of b show roughly symmetric behavior. As expected, the stopping times grow with increasing b, but the growth appears to be sub-linear, even though the parametric cycle $L^1_{b,0}$ (see Theorem 4.4) grows linearly with b.

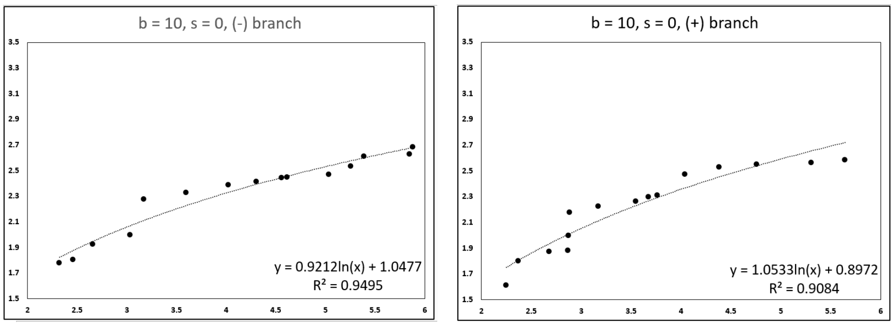

Figure 2. Convergence envelopes for b = 10 , s = 0, (-) and (+) branches.

6 Conclusion

We have discussed a new two-parameter generalization $T_{b,s}(x)$, where $x \in \mathbb{Z}$, $b \in \mathbb{N}$ and$s \in \mathbb{Z}$ for the Collatz function $T(n), n \in \mathbb{N}$, such that $T_{2,0}(n) \equiv T(n)$. $T_{b,s}(x)$ obviates the parametric behavior of all the known terminal cycles of the Collatz function $T(x), x \in \mathbb{Z}$. The terminal cycle $1,2$ for the $T(n)$ is shown to be a case of three distinct categories of terminal cycles of $T_{b,s}(x)$. All functions $T_{b,s}(x)$ appear to produce bounded orbits. The presence of non-parametric cycles for some (b,s) values leaves open the possibility of a non-parametric cycle for large $|z|$ values for the $T(n)$. The convergence of $T_{b,s}$ shows a consistent logarithmic behavior with growth of the b, apparently independent of the number of terminal cycles. We have also proposed a re-definition of the term "trivial cycle" as applied to the cycle $\{1,2\}$ as it is shown to correspond to three distinct categories of cycles of $T_{b,s}$ which happen to coincide for $b = 2, s = 0$. We propose the term "parametric cycle" for $\{1,2\}$, restricting "trivial cycle" to the single-point "cycles", such as $\{0\}$ and $\{-1\}$ for the $T(x)$ function.

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Competing Interests

The author has no competing interests.

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