NOT DENUMERABILITY OF RATIONAL NUMBERS

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ABSTRACT. The famous Cantor's demonstration of the Denumerability of the Rational Numbers is based on the wrong use of the term all and, in the tabular representation, on the wrong use of the limits.
In this paper it's first rigorously showed that the Cantor demonstration is erroneous. Then two direct proofs and two indirect proofs that Rational Numbers are not-denumerable are shown.
The direct proofs are the not bijectivity between ℕ and ℚ, and the not existence of a whatsoever successor operator in ℚ. In the first indirect proof will be showed that denumerability of Rational Numbers leads to a null Lebesgue measure of any interval of ℝ. In the second indirect proof will be demonstrated that the Power Set $\{x^n\}$, $n \in \mathbb{N}_0$; and the Trigonometric Set $\{\sin mx, \cos nx\}$, $m,n \in \mathbb{N}_0$; of L^2 are not equivalent.

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1. INTRODUCTION

In the article "Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen" [1] Cantor demonstrated that the algebraic real numbers are denumerable, that is, they are in a one-to-one correspondence with the Naturals, ordering them on the basis of the highness N:

> «If we turn to equation 1 [$a_0 \omega^n + a_1 \omega^{n-1} + \ldots + a_n$], satisfied by an algebraic number ω and fully determined thanks to the conditions we have imposed, we can call height of ω and indicate with N the sum of the absolute values of its coefficients plus $n-1$ where n is the grade of ω; it will therefore be, applying a now usual notation,
>
> $$N = n-1 + |a_0| + |a_1| + \ldots + |a_n|.$$
>
> The height N will therefore be a positive natural number determined for every algebraic real ω; on the other hand, for every positive natural value of N there exists only a finite number of algebraic real numbers of height N. Let ϕ be this number; then it will be, for example, $\phi(1)=1$, $\phi(2)=2$, $\phi(3)=4$.
>
> Thus it is possible to order the elements of the class (ω), that is all the algebraic real numbers, in the following manner: put in first place, as ω_1, the single number of height $N = 1$; this will be followed, in increasing order, by the $\phi(2) = 2$ algebraic real numbers of height $N = 2$, indicated by ω_2, ω_3; then, in increasing order, there will be the $\phi(3) = 4$ numbers of height $N = 3$; and in general after having, in this way, numbered and collocated in a determined position all the elements of ω up to a certain height $N = N_1$, we will follow them, always in increasing order, with the algebraic real numbers with height $N = N_1+1$.
>
> In this way we obtain the class ω of all the algebraic real numbers in the form
>
> $$\omega_1, \omega_2, \ldots, \omega_\nu$$
>
> and with this ordering, we can speak (without any element of class ω being omitted) of the ν-th algebraic real number.»

Although the argument of these pages concerns the Rational Numbers, it is opportune to highlight one point, which will be amplified below, regarding Cantor's demonstration. In the demonstration cited, Cantor never uses the concept of infinite or that of limit. He implicitly uses the principle of induction which is a fundamental part of the definition of Natural Numbers. He then makes a leap in logic by claiming to obtain *all* the Algebraic Real Numbers. This leap is not justified (or demonstrated) and is taken for granted. Given that what must be demonstrated is the nature of the set of Algebraic Real Numbers, the use of the term all, which implies their denumerability, within a demonstration which aims to prove this denumerability, makes the demonstration itself self-referencing and thus invalid. The term *all*[1] implies a passage to single or double limits which must be justified. As will be shown in this paper, the passage is necessarily to double limits and cannot be reduced to a single limit, so that the term all in Cantor's paper is not only unjustified but also wrong. A more accurate and mathematical analysis of the problem of the denumerability of rational numbers is therefore required.

1. As it can easily be seen, as the argument of ϕ gradually increases, the value of ϕ_ν also increases, i.e. in mathematical symbols: for $\nu \to \infty$, $\phi_\nu \to \infty$. On the other hand, for $\nu \to \infty$ not only the value of ϕ_ν but also the number of ϕ_ν's tend to infinity. Thus it is no longer so evident that it is possible to put all the algebraic real numbers in bi-univocal correspondence with ℕ.

2. THE TABULAR DEMONSTRATION OF THE DENUMERABILITY OF THE RATIONAL NUMBERS

A famous visualisation of the theorem of the denumerability of rational numbers, derived from Cantor's theorem, and its "solution" makes use of a table. Here is one taken from "Elements of theory of functions and functional analysis" by [3, pp.5-6]:

2°. The sum of an arbitrary finite or denumerable set of denumerable sets is again a finite or denumerable set.

Proof. Let A_1, A_2, ... be denumerable sets. All their elements can be written in the form of the following infinite table:

a_{11}	a_{12}	a_{13}	a_{14}	...
a_{21}	a_{22}	a_{23}	a_{24}	...
a_{31}	a_{32}	a_{33}	a_{34}	...
a_{41}	a_{42}	a_{43}	a_{44}	...
....				

Figure 1.

where the elements of the set A_1 are listed in the first row, the elements of A_2 are listed in the second row, and so on. We now enumerate all these elements by the "diagonal method", i.e. we take a_{11} for the first element, a_{12} for the second, a_{21} for the third, and so forth, taking the elements in the order indicated by the arrows in the following table:

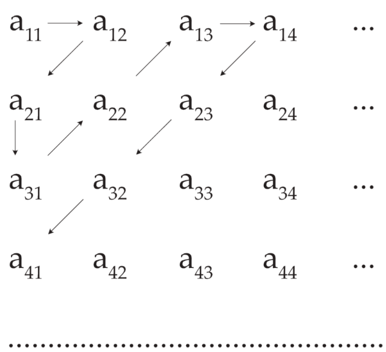


Figure 2.

It is clear that in this enumeration every element of each of the sets A_i receives a definite index, i.e. we shall have established a one-to-one correspondence between all the elements of all the $A_1, A_2,$ ... and the set of natural numbers. This completes the proof of our assertion.

3. THE FALLACIOUSNESS OF CANTOR'S DEMONSTRATION

Let us return to the previous theorem and rigorously analyse the demonstration. Let $A_1, A_2,$ ... be the sets of the theorem verifying the same conditions, i.e. that they are non-intersecting two by two. Call $a_{1\,1}, a_{1\,2}, a_{1\,3},$ ..., the elements of A_1; $a_{2\,1}, a_{2\,2}, a_{2\,3},$ ..., the elements of A_2; $a_{n\,1}, a_{n\,2}, a_{n\,3},$ ..., the elements of A_n; ..., the elements of A_n; etc. The sets A_n form a succession (a succession is denumerable by definition) $\{A_n\}$ with $n \in \mathbb{N}$. Also the elements of the sets A_n form successions $\{a_m\}_n$, $m \in \mathbb{N}$, $\forall n \in \mathbb{N}$. Cantor's theorem claims that modifying the order in which the elements of the various A_n are ordered it is possible to number all of them. If the new order is described rigorously we can see if it is possible to put it in bijective correspondence with $\mathbb{N}$ or not (see Figure 2).

If we consider the elements of the diagonal and call D_1 the set formed by the elements of the *i*-th diagonal, we have:

$$
\begin{aligned}
&D_1 \equiv \{a_{1\,1}\};\\
&D_2 \equiv \{a_{2\,1},\ a_{2\,2}\};\\
&\dots\\
&D_n \equiv \{a_{1\,n},\ a_{2\,n-1},\ \dots,\ a_{n-1\,2},\ a_{n\,1}\};\\
&\dots
\end{aligned}
$$

this can be rewritten as:

$$
\begin{aligned}
&D_1 \equiv \{d_{1\,1}\};\\
&D_2 \equiv \{d_{2\,1},\ d_{2\,2}\};\\
&\dots\\
&D_n \equiv \{d_{n\,1},\ d_{n\,2},\ \dots,\ d_{n\,n-1},\ d_{n\,n}\};\\
&\dots
\end{aligned}
$$

since D_i has i elements $\forall i$.

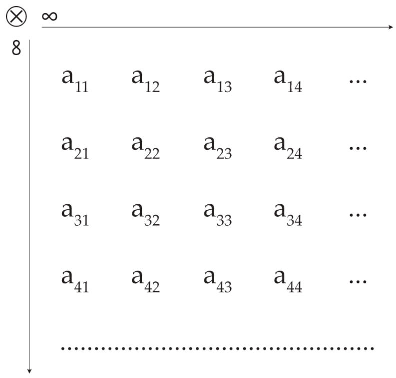
The succession of these D_n defines the succession in Cantor's diagonal procedure:
$a_{1\,1},\ a_{1\,2},\ a_{2\,1},\ \dots,\ a_{1\,n},\ a_{2\,n-1},\ \dots,\ a_{n-1\,2},\ a_{n\,1},\ \dots$
But for $n \to \infty$, $\{D_n\} \leftrightarrow \mathbb{N}$; and

$$
\begin{aligned}
D_n &\equiv \{a_{1\,n},\ a_{2\,n-1},\ \dots,\ a_{n-1\,2},\ a_{n\,1}\}\\
&\equiv \{d_{n\,1},\ d_{n\,2},\ \dots,\ d_{n\,n-1},\ d_{n\,n}\}\\
&\equiv \{d_m\}_n \to N
\end{aligned}
$$

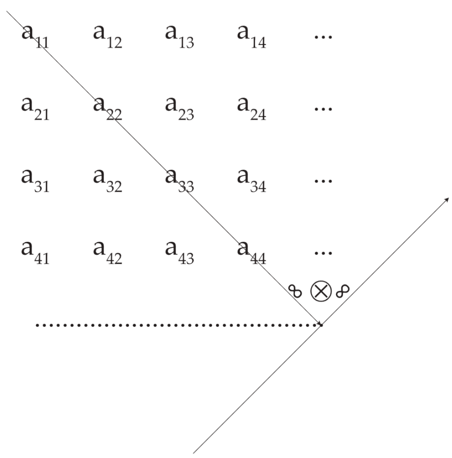
for $m \in (1,n)$ and $n \to \infty$.

Therefore for $n \to \infty$ we have that $\{\{a_m\}_n\} \to \infty$ as $\mathbb{N}\otimes\mathbb{N}$. That is, as one proceeds along the diagonals these become greater and greater and, at the limit, infinite. Since also the number of the diagonals D_n tend to infinite at the limit (because we are speaking of limits), the rearrangement of Cantor does not offer any real advantage but to hide what was evident.

In his demonstration, Cantor seems to forget the previous diagonals as he proceeds. But this results in simply demonstrating that, at the limit for infinite n, the diagonal D_n tends to "possess" infinite elements and that what is in *one-to-one* correlation with $\mathbb{N}$ is not the set of rational numbers, but only, at the limit, this last diagonal, ignoring all the previous ones which are, always at the limit, infinite in number. Heuristically speaking, we have gone from:



to:



4. NOT DENUMERABILITY PROOFS

Direct proofs of not-denumerability of Rational Numbers already exist in mathematics thought they are not recognised as such. For example the not bijectivity between ℕ and ℚ and the density of the Rational Numbers and its relation to the not-existence of successor operator.

Other proofs are indirect such as the problem of the Lebesgue measure of Dirichlet Function and the not-equivalence of Complete Sets in L^2.

4.1. Not bijectivity between Natural and Rational Numbers

A bijective correspondence is represented mathematically by a function which can be inverted. In the present case, in order to represent the succession of denumerable "Cantorian" terms it is necessary to use a function to link the index of the succession, *n*, with the indices of the table, *i* and *j*. That is, we must be able to write $n = f(i,j)$. But this function, whatever it may be, cannot be inverted in that it is a function of two variables and to resolve it a further equation must be placed in system with the first, $n = g(i,j)$. From this we deduce that $f(i,j) = g(i,j)$, which defines an implicit function of *i* and *j*, so that we will have $j = h(i)$ and $n = f(i,h(i))$. The condition necessary for *n* to be linked to *i* and *j* by a function which can be inverted is, therefore, that $j = h(i)$. This is exactly what happens when considering particular successions in the elements of the table, such as the diagonals, etc. If *n* is considered a parameter, the equation $f(i,j) = n$ is an equation with two unknowns and, as above, its solution requires a further associated equation.

4.2. Density of Rational Numbers and Denumerability

The Natural Numbers are defined by the rules given by Giuseppe Peano. In the following summary of the original version given in "Formulario Mathematico" [4, page 27], we use modern notation rather than the original symbols:

Natural number:

- $\mathbb{N}_0$ stands for **natural number**.
- **0** stands for **zero**.
- + stands for **plus**. If **a** is a natural number, **a+** indicates the **number following a**.

Assume the three concepts $\mathbb{N}_0$, **0**, +, to be primitive concepts by which we define every arithmetical symbol.

We determine the meaning of the undefined symbols $\mathbb{N}_0$, **0**, +, by the following system of primitive propositions:

$\mathbb{N}_0$ is a class.
$\mathbf{0} \in \mathbb{N}_0$.
If $\mathbf{a} \in \aleph_0$ then $\mathbf{a}+ \in \mathbb{N}_0$.
If **s** is a class and $\mathbf{0} \in \mathbf{s}$; and if $\forall a \in s,\ a+ \in s$; then $\mathbb{N}_0 \in \mathbf{s}$.
If $\mathbf{a}, \mathbf{b} \in \mathbb{N}_0$ then $\mathbf{a}+ = \mathbf{b}+$ implies $\mathbf{a} = \mathbf{b}$.
If $\mathbf{a} \in \mathbb{N}_0$ then $\mathbf{a}+ \neq \mathbf{0}$.

This definition identifies the class of Natural Numbers and is characteristically linked to the principle of induction.

In the case of Rational Numbers (ℚ), however, we are not able even to define an operator of succession. If we consider the table ordered by rows, that is so that the second row is considered successive to the first and so on, we find, precisely because of the definition of a natural number and the construction of the table, that any element of the table has successors only in its own row. In other words, it is not possible to pass from one row to the successive applying the operator of succession. For example, no first element of a row has precedents, which is implicitly defined by the successor. The same holds for columns.

The not-existence of a successor is particularly evident in the usual representation of the Rational Numbers as points on a line (hereafter called *arithmetic representation*). The mathematical proof of this statement is given by the a well known:

Theorem 4.1. If $a, b \in \mathbb{Q}$ then $\exists\, c,\ c \in \mathbb{Q}$ so that $a < c < b$.

We are not able to find a successor because from the theorem follows: $\forall\, a \in \mathbb{Q}\ \exists\, c,\ c \in \mathbb{Q}$ *so that* $a < c < a+_u$, that is absurd (an antinomy) because $a+_u$ is the successor of *a* by definition. In particular, we may choose $c = \dfrac{a + (a+_u)}{2}$ and since $a \in \mathbb{Q}$ then $a+_u \in \mathbb{Q}$ *and* $c \in \mathbb{Q}$; $a < c < a+_u$. Here the + is the usual sum operator while $+_u$ is the successor operator.

To avoid any remaining doubt, due to the visual impact of Cantor's demonstration and not to its formal value, here it will be shown how the property of not-existence of a successor of a Rational Number is also found in the tabular representation. In the usual representation of Rational Numbers, the demonstration of the not-existence of a successor is given by the previous theorem Theorem 4.1. We will show here how the same theorem is valid in the tabular representation of the Rational Numbers. In the tabular representation, the order is imposed on the indices. The second index is ordered, in the usual way, by its entire value along a row; the first index does not vary along a row. It should be remembered that the second index is the denominator of the fraction representing the rational number. In the tabular representation, therefore, the order of the rational numbers along a row is completely different from that in the *arithmetic representation*. The first index is ordered down columns (while the second one along rows); it describes the numerator of the fraction representing the rational number.

Consider a generic term of the table *(m, n); m, n* ∈ *N*. This term, together with all the terms of the general form *(ml, nl)*, represents the rational number $\frac{m}{n}$. Let us consider, together with *(ml, nl), l* ∈ *N*, the adjacent terms *(ml, nl+1)* which represent the rational number $\frac{m}{n}\left(1-\frac{1}{nl+1}\right)$. When $l \to \infty$, the terms $\frac{m}{n}\left(1-\frac{1}{nl+1}\right)$ tend to $\frac{m}{n}$ i.e. *(m, n),* ∀ *m, n*[2] (see Figure 1).

Thus we see that the theorem discussed above is also valid in the tabular representation. In the proof given, the doubly infinite degree of freedom of the ordered pairs is in no way reduced. Many other interesting relations between the tabular and arithmetic representations of rational numbers can be found. Their interest lie in the possibility they offer to investigate the property of rational numbers from different (and equivalent) points of view.

4.3. Lebesgue measure of Dirichlet Function

Let us consider the Dirichlet function:

$$d(x) = \begin{cases} 1, & \text{for x = Rational Number} \\ 0, & \text{for x = Irrational Number} \end{cases}$$

This function is Lebesgue integrable in every set of points *E*. If E_R is the subset of *E* of the Rational Numbers and E_I the subset complement of E_R composed by the Irrational Numbers, its Lebesgue integral is (**considering the Rational Numbers Set denumerable**):

$$\int_E d(x)dx = \int_{E_R} d(x)dx + \int_{E_I} d(x)dx = \int_{E_R} d(x)dx = \mu(E_R) = 0$$

because $d(x) \equiv 0$ for $x \in E_I$ and $\mu(E_R) = 0$ for every set of Natural Numbers is denumerable and so its Lebesgue measure is zero.

Now let us consider the following theorem:

Theorem 4.2. Between two Irrational Numbers there is at least one Rational Number

Proof: Let us consider two irrationals i_1 and i_2, with $i_1 < i_2$, and two rationals r_1 and r_2 with $r_1 < i_2$ and $r_2 > i_1$.

If it would be $r_2 \le r_1$ then: $i_1 < r_2 \le r_1 < i_2$ and the theorem would follow from the arbitrariness of i_1 and i_2.

Let us consider the case $r_1 < r_2$, if one or both of r_1 and r_2 lies between i_1 and i_2 the theorem is demonstrated for the same reason above, the arbitrariness of i_1 and i_2.

Let's consider the case $r_1 < i_1 < i_2 < r_2$. For the density property of Rationals $\frac{r_1+r_2}{2}$ is also a rational greater than r_1 and smaller than r_2, call it R_1, $R_1 = \frac{r_1+r_2}{2}$. We have: $r_2 - R_1 = R_1 - r_1 = \frac{r_2-r_1}{2}$. We can have three cases:

1) $r_1 < i_1 < R_1 < i_2 < r_2$, we fall back in the second case and the theorem is demonstrated;

2) $r_1 < R_1 < i_1 < i_2 < r_2$, if so let's call it r_{1_1}, $r_{1_1} \equiv R_1$, so that we have: $r_1 < r_{1_1} < i_1 < i_2 < r_2$; we can find a second number, let's call it R_2, such that $R_2 = \frac{r_2 + r_{1_1}}{2}$; we have:

$$r_2 - R_2 = R_2 - r_{1_1} = \frac{r_2 - r_{1_1}}{2} \equiv \frac{r_2 - R_1}{2} = \frac{\frac{r_2-r_1}{2}}{2} = \frac{r_2-r_1}{2^2};$$

3) $r_1 < i_1 < i_2 < R_1 < r_2$, if so let's call it r_{2_1}, $r_{2_1} \equiv R_1$, (r_2 to remember that it is greater than i_2) so that we have: $r_1 < i_1 < i_2 < r_{2_1} < r_2$; we can find a second number, let's call it R_2, such that $R_2 = \frac{r_{2_1}+r_1}{2}$; we have:

$$R_2 - r_1 = r_{2_1} - R_2 = \frac{r_{2_1} - r_1}{2} \equiv \frac{R_1 - r_1}{2} = \frac{\frac{r_2-r_1}{2}}{2} = \frac{r_2-r_1}{2^2}.$$

We can repeat this reasoning again with R_3, R_4, ..., R_n. If in any of these passages an R_m, $m < n$ falls between i_1 and i_2 the theorem is proven. Let us suppose that this is not the case, then we have at the n-th passage again three cases:

1) $r_1 < i_1 < R_n < i_2 < r_2$, we fall back in the second case and the theorem is demonstrated;

2. Other choices can be made, for example *(ml, nl)* and *(ml+1, nl)* that represent *m/n* and *(m/n)+(1/nl)* respectively.

2) $r_1 < r_{1_1} < r_{1_2} < \ldots < r_{1_p} < R_n < i_1 < i_2 < r_{2_q} < \ldots < r_{2_2} < r_{2_1} < r_2$ with p + q = n - 1, if so let's call it $r_{1_{p+1}}$, $r_{1_{p+1}} \equiv R_n$ so that we have:

$r_1 < r_{1_1} < r_{1_2} < \ldots < r_{1_p} < r_{1_{p+1}} < i_1 < i_2 < r_{2_q} < \ldots < r_{2_2} < r_{2_1} < r_2$;

then we have:

$r_{2_q} - r_{1_{p+1}} = r_{1_{p+1}} - r_{1_p} = \dfrac{r_2 - r_1}{2^n}$;

3) $r_1 < r_{1_1} < r_{1_2} < \ldots < r_{1_p} < i_1 < i_2 < R_n < r_{2_q} < \ldots < r_{2_2} < r_{2_1} < r_2$ with p + q = n - 1, if so let's call it $r_{2_{q+1}}$, $r_{2_{q+1}} \equiv R_n$

so that we have:

$r_1 < r_{1_1} < r_{1_2} < \ldots < r_{1_p} < i_1 < i_2 < r_{2_{q+1}} < r_{2_q} < \ldots < r_{2_2} < r_{2_1} < r_2$;

and we have: $r_{2_q} - r_{2_{q+1}} = r_{2_{q+1}} - r_{1_p} = \dfrac{r_2 - r_1}{2^n}$.

So in both the last two cases we have: $r_{2_q} - R_n = R_n - r_{1_p} = \dfrac{r_2 - r_1}{2^n} > i_2 - i_1$.

For $n \to \infty$, $i_2 - i_1 \leq \lim_{n\to\infty} r_{2_q} - R_n = \lim_{n\to\infty} R_n - r_{1_p} = \lim_{n\to\infty} \dfrac{r_2 - r_1}{2^n} = 0$ so that it must be:

$i_2 - i_1 \leq \lim_{n\to\infty} \dfrac{r_2 - r_1}{2^n} = 0,\ for\ n \to \infty$ i.e. $i_1 = i_2$, that contradicts the hypothesis.

The proof is complete.

So, since for each Irrational Number there is at least a Rational one *$\mu(E_I) \leq \mu(E_R)$*. So *$\mu(E) = \mu(E_R) + \mu(E_I) = 0$* that is a strange result.

But considering the Rational Numbers Set not denumerable, *$\mu(E_R) \neq 0$*[3] and so the Lebesgue integral of *d(x)* is not null too. This is reasonable because the *Rational Numbers Set* is dense in the *Real Number Set* so that in every neighbourhood of a *Real Number* (and so also of every *Irrational Number*) there are infinite *Rational Numbers*[4].

4.4. Not Equivalence of Power and Trigonometric Sets in L^2

Among the complete set in L^2 there are the Trigonometric Function Set:

1, sin(x), cos(x), sin(2x), cos(2x), …; $x \in [0, 2\pi)$;

with *Weight Function p(x)=1*;

written synthetically as:

{sin(mx), cos(nx)}; $m, n \in \mathbb{N}_0$, $x \in [0, 2\pi)$;

with *Weight Function p(x)=1*;

and the *Power Set*:

1, x, x^2, …; $x \in \forall(a, b)$; $a, b \in \mathbb{R}$;

valid with every *Weight Function*;

written synthetically as:

$\{x^n\}$; $n \in \mathbb{N}_0$, $x \in \forall(a, b)$; $a, b \in \mathbb{R}$;

valid with every *Weight Function*;

The first is an *($\mathbb{N}$, $\mathbb{N}$) set* while the second is an *($\mathbb{N}$) set*. Actually they are considered equivalent because the first has the power of *Rational Numbers Set* that is equal to the power of *Natural Numbers Set*, given that *Rational Numbers Set* is considered denumerable. We will verify hereafter if this is the case.

Let us consider the two *Complete Sets* above in the same range:

{sin(mx), cos(nx)}; $m, n \in \mathbb{N}_0$, $x \in [0, 2\pi)$;

with *Weight Function p(x)=1*;

$\{x^n\}$; $n \in \mathbb{N}_0$, $x \in [0, 2\pi)$;

with *Weight Function p(x)=1*;

---

3. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is no countable (finite or denumerable) collections of sets $B_m \in E_R$, E_R an interval of $\mathbb{R}$, such that B_m contains only one Rational Number r_m. So the Lebesgue measure of any interval of $\mathbb{Q}$ (as a subset of $\mathbb{R}$) cannot be proven to be null.

4. If the demonstration of the isomorphism between the number of functions of *Square Integrable Functions Space* and *Rational Numbers* still holds, then *Hilbert Space* (that is countable by definition) is no more isomorphic to *Square Integrable Functions Space*. This is a subject that needs further studies.

If both of them are complete in the same range, every function $f \in L^2_{[0,2\pi]}$ can be expressed as a sum of functions of the two sets:

$$f_\phi = \sum_{i=0}^{\infty} a_i x^i; \quad i \in \mathbb{N},\ x \in [0, 2\pi),\ a_i \in \mathbb{C};$$

$$f_\psi = \sum_{m,n=0}^{\infty} (b_m sin(mx) + c_n cos(nx));$$

$m, n \in \mathbb{N}$; $x \in [0, 2\pi)$; $b_m, c_n \in \mathbb{C}$;

The subscripts ϕ, ψ are for subsequent use, as we'll see.

Let us call $|\phi_i>$ the vector related to x^i of the base $\{x^i\}$ and $|\psi_{1,m}>$ the vector related to $sin(mx)$, $|\psi_{2,n}>$ the vector related to $cos(nx)$ (we use two indices m and n to remark that they are independent indices).

Let us start with a generic f_ϕ of the $\{\phi\}$ space and let us project it on the ψ basis:

$$\sum_{m,n=0}^{\infty} \left(|\psi_{1,m}><\psi_{1,m}| + |\psi_{2,n}><\psi_{2,n}| \right) |f_\phi> =$$

$$\sum_{m,n=0}^{\infty} \Big(|\psi_{1,m}><\psi_{1,m}|f_\phi> + |\psi_{2,n}><\psi_{2,n}|f_\phi> \Big) =$$

$$\sum_{m,n=0}^{\infty} \left(|\psi_{1,m}> \sum_{i=0}^{\infty} a_i <\psi_{1,m}|\phi_i> + |\psi_{2,n}> \sum_{i=0}^{\infty} a_i <\psi_{2,n}|\phi_i> \right) =$$

This is a vector in the $\{\psi\}$ space:

$$f_\psi = \sum_{m,n=0}^{\infty} (b_m|\psi_{1,m}> + c_n|\psi_{2,n}>);$$

with:

$$\begin{cases} b_m = \sum_{i=0}^{\infty} a_i <\psi_{1,m}|\phi_i> \\ c_n = \sum_{i=0}^{\infty} a_i <\psi_{2,n}|\phi_i> \end{cases}$$

that can be written also as:

$$\begin{cases} b_m = <\psi_{1,m}| \sum_{i=0}^{\infty} a_i\phi_i> \\ c_n = <\psi_{2,n}| \sum_{i=0}^{\infty} a_i\phi_i> \end{cases}$$

For each $f_\phi \equiv \{a_i\}$ there is one and only one $b_m = <\psi_{1,m}| \sum_{i=0}^{\infty} a_i\phi_i>$ and one and only one $c_n = <\psi_{2,n}| \sum_{i=0}^{\infty} a_i\phi_i>$ i.e. one and only one $f_\psi \equiv \{b_m, c_n\}$.

Now let us project a generic f_ψ of the $\{\psi\}$ space on the ϕ basis:

$$\sum_{i=0}^{\infty} \left(|\phi_i><\phi_i| \right) |f_\psi> =$$

$$\sum_{i=0}^{\infty} \left(|\phi_i><\phi_i| \right) \left(\sum_{m,n=0}^{\infty} (b_m|\psi_{1,m}> + c_n|\psi_{2,n}>) \right) =$$

$$\sum_{i=0}^{\infty} |\phi_i> \left(\sum_{m=0}^{\infty} b_m < \phi_i | \psi_{1,m} > + \sum_{n=0}^{\infty} c_n < \phi_i | \psi_{2,n} > \right)$$

This is a vector in the *{ϕ}* space:

$$f_\phi = \sum_{i=0}^{\infty} a_i |\phi_i> ;$$

with:

$$a_i = \sum_{m=0}^{\infty} b_m < \phi_i | \psi_{1,m} > + \sum_{n=0}^{\infty} c_n < \phi_i | \psi_{2,n} >$$

that can be written also as:

$$a_i = < \phi_i | \sum_{m=0}^{\infty} b_m \psi_{1,m} > + < \phi_i | \sum_{n=0}^{\infty} c_n \psi_{2,n} >$$

Now, differently from above, we have an infinite choice of elements $< \phi_i | \sum_{m=0}^{\infty} b_m \psi_{1,m} >$ or $< \phi_i | \sum_{n=0}^{\infty} c_n \psi_{2,n} >$. For example we can choose, for each a_i, one $< \phi_i | \sum_{m=0}^{\infty} b_m \psi_{1,m} >$, i.e. $\{b_m\}$, $m \in \mathbb{N}_0$, and then we'll have:

$$< \phi_i | \sum_{n=0}^{\infty} c_n \psi_{2,n} > = a_i - < \phi_i | \sum_{m=0}^{\infty} b_m \psi_{1,m} > .$$

Now for each $f_\phi \equiv \{a_i\}$ there is an infinite choice of $f_\psi \equiv \{b_m, c_n\}$. There are infinite vectors of the *{ψ}* space that have the same projection on the *{ϕ}* space.

5. THE CURRENT DEFINITION OF INFINITE AND HIS PARADOX

At this point it only remains to clarify the thorny question posed by Cantor's diagonal representation: How is it possible that the rational numbers, and in particular the ordinate pairs, are not denumerable if it appears from the diagonal listing that all the elements of the table can be numbered? A first response was given above, demonstrating that the succession defined by the diagonal procedure is not linear but divergent. A second reply comes from the mathematical definition of infinite. Until the 19th century, the definition of infinite was based, as it is based in this paper, on the principle of induction, i.e. on the definition of a natural number. Subsequently, in order to be able to justify the study of "actual" infinite numbers, a new definition was introduced, identifying the infinite set through the property of being able to be placed in bijective correspondence with a subset of itself. But it is clear that such a definition must be rejected since it is self-contradictory: in fact to say that two sets are in bijection correspondence means that it is possible to put in relation each and every element of one set with each and every element of the other. If the second set is a proper subset of the first, only those elements belonging to the subset (or equivalent) can be "mapped" because of the principle of reflexivity of the relation of equivalence (generated by the bijective correspondence) and from the principle of identity. I.e there must be some elements of the superset not in bijection with the subset (for definition of subset itself). This statement can be shown formally in the following manner: we consider the currently accepted definitions of proper subset, equivalent sets and infinite sets [3, p.6].

Definition 5.1. Inclusion and proper inclusion.

Let A and B be two sets. If all the elements constituting A also belong to B (not excluding the case $A = B$), then A is said to be a subset of B and is indicated by $A \subseteq B$. If $A \neq B$ and $A \neq \emptyset$ it is said to be a proper subset and is indicated by $A \subset B$.

Definition 5.2. Equivalence of sets.

Two sets M and N are said to be equivalent (notation $M \sim N$) if a *one-to-one* correspondence can be established between their elements.

Definition 5.3. Equipotence of sets.

If two finite sets are equivalent, they are composed of the same number of elements. If instead two equivalent sets M and N are arbitrary, it is said that M and N have equal potency.

Definition 5.4. One-to-one correspondence.

The elements of two sets are said to be in *one-to-one* correspondence ($\rightleftarrows$) when each element in one set corresponds to one, and only one, element of the other and vice versa. It is evident that a *one-to-one* correspondence can be established between two finite sets if, and only if, they have the same number of elements.

Definition 5.5. Infinite set.

Each infinite set is equivalent to a proper subset of itself.

This property can be accepted as defining an infinite set.

From Definition 5.1, if $A \subset B$ then every element of A can be put in correspondence with one and only one element of B, $A \rightarrow B$, but not the opposite $A \not\leftarrow B$. Thus, rewriting the definition of an infinite set using mathematical formalism, gives:

1) *A, B infinites;* $A \subset B$*; by hypothesis*
2) $A \sim B \Rightarrow A \rightleftarrows B$*; by definition of infinite set*
3) $A \subset B \Rightarrow A \rightarrow B,\ A \not\leftarrow B$*; by definition of proper subset*
4) *So we have:* $A \rightleftarrows B$ *and* $A \not\leftarrow B$ *from 1), 2) and 3).*

We have, therefore, a contradiction in the currently accepted definition of an infinite set. The contradiction arises from the illusion that we can define an actual infinite and that we are able to manipulate it. A particular bijective correspondence is given by the identity function, I, which makes a given element of a set correspond to the element itself. The infinite is always in potency and never in act, at least in science. In the end, an answer of heuristic type can be given by one of Escher's drawings. Eyesight, being one of man's senses, is easily deceived and therefore, as the rationalists but also Aristotle affirmed, for problems of logic and mathematics we must scrupulously adhere to the strictest formal rigour. What emerges from the diagonal procedure is that, in effect, considering a single row or column of the doubly infinite table, a form of extraction is performed to obtain a denumerable succession from a much richer whole, and that the only real originality is that in this form we have managed to conceal its true nature.

6. CONCLUSIONS

The nineteenth century has seen a huge enhancement of the research in all the fields of science, led by the the industrial revolution and its technological needs. Their great discoveries gave to the humankind a sense of power never felt before. In that mood, scientists threw themselves enthusiastically in the research of the very origin of everything.

Cosmology on one side, atomic and, later, nuclear physics on the other side, received a great boost in the physical realm. In the mathematical realm, among the others, the research of an axiomatisation and formalisation of the mathematics were two of the main fields. In that context the studies of Sets led to the (ancient) question of the Infinite Sets and their definitions and properties.

It led researchers to believe that the human beings could reach, by means of science, the complete knowledge and domination of the Universe. Laically speaking this is a huge logical mistake, because the content never can contain its container. We humans are part of the Universe so we cannot *comprehend* it all.

But the previous statement has not to be taken as diminutive of the potentials of humankind, on the contrary! We experience every day the dichotomy of our nature: physical and mental. Our mind is capable of stunning abstractions and discoveries. They resemble the somewhat mysterious "emerging properties". Maybe every time we make a discovery we enrich the Universe expanding it and at every expansion there are more and more things to discover. In a sense we are (maybe) participating to the creation of our Universe. I want to stress that I'm talking here from a laic, a philosophical point of view, theology is beyond the scope of these reasonings.

What is the Universe, where does it come from, why it exists, ontologically speaking, are questions that own to philosophy and theology. Science can only try to understand *how* it works. In particular philosophy should take back its prerogative to be, also, the bridge between science and theology.

7. References

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