The numerical code used in: Modelling the penetration of subsonic rigid projectile probes into granular materials using the cavity expansion theory

Mechanical Properties and Coefficients

Volcanic ash (Change the properties for other target medium here)

In[1]:=

```
ρ = 1620;
ηs = 0.13;
Ee = 3.192 × 10^9;
k = 2 × 10^9;
ν = (3 k - Ee)/(6 k);
Coh = 4.74342 * 10^6;
ϕf = 25.3769 π / 180; (*Y=-(6 (Coh Cos[ϕf]))/(-3+Sin[ϕf]);*)
Y = 2 Coh;
```

In[3]:=

```
cd = Sqrt[(Ee (1 - ν))/((1 + ν) (1 - 2 ν) ρ)]; (*cd=ce:elastic wave speed*)
β = c/Sqrt[Ee/ρ]; ρo = ρ;
W = (12 Coh Cos[ϕf])/(Ee (-3 + Sin[ϕf])); X = (3 - Sin[ϕf])/(3 k + 3 k Sin[ϕf]);
term = .01; (*define the acceptable error |cp2-cp1|<term*)
NR = 1000;
(*define the step-size in the Runge-Kutta method*)
```

Equation 16 a & 16 b

In[6]:=
```
A[c_] := -((4 c^3 Coh (1 + ν) (-1 + 2 ν) Cos[ϕf]) / ((c - cd) Ee
        (3 cd (c + cd) (-1 + 2 ν) + (cd^2 (1 - 2 ν) + 4 c^2 (1 + ν) + c (cd - 2 cd ν)) Sin[ϕf])));
B[c_] := (6 cd Coh (1 + ν) (-1 + 2 ν) Cos[ϕf]) /
    ((c - cd) Ee (3 cd (c + cd) (-1 + 2 ν) + (cd^2 (1 - 2 ν) + 4 c^2 (1 + ν) + c (cd - 2 cd ν)) Sin[ϕf]));
```

Assumption 1 : c_p and S_r (hydrostat plastic region)

Solving Equation 29a & 29b

In[8]:=
```
Cph = Function[{η, V, ϕ, term},
   Clear[out]; error = 1;
   c = V / (η - (2 Coh (-1+η) (1+ν))/Ee)^(1/3);
   If[ϕ == 0, out = c,
    While[Abs[error] > term, out = V / (-B[c] (c - cd) (c + cd) (-1 + η) + η)^(1/3);
     error = out - c;
     c = out]];
   out];
```

Solving Equation 23 & 24 S_0: Equation 30 & 31

In[9]:=

```
Sh = Function[{η, V, ϕ, c, ξ}, ψ = V / c;
  Clear[shy];
  If[ϕ ≠ 0, S0 = (1 + B[c] (-c^2 + cd^2))^2 β^2 η + (3 A[c] (1 + ν) + 2 B[c] (cd^2 (1 - 2 ν) + 3 c^2 ν)) /
      (3 (-1 + ν + 2 ν^2)) + (Coh Cot[ϕf])/Ee - (β^2 ψ^6 (1 + Sin[ϕf]))/(2 (-1 + η)) -
     (2 β^2 ψ^3 (1 + Sin[ϕf])) / ((-1 + η) (-1 + 3 Sin[ϕf])), S0 = (1 + B [c] (-c^2 + cd^2))^2 β^2 η +
     (3 A[c] (1 + ν) + 2 B[c] (cd^2 (1 - 2 ν) + 3 c^2 ν)) / (3 (-1 + ν + 2 ν^2)) - (β^2 ψ^3 (-4 + ψ^3))/(2 (-1 + η))];
  If[ϕ ≠ 0, Shy = S0 ξ^(-(4 Sin[ϕf])/(1+Sin[ϕf])) - (Coh (Cos[ϕf] + Cot[ϕf]))/(Ee (1 + Sin[ϕf])) + (β^2 ψ^6 (1 + Sin[ϕf]))/(2 ξ^4 (-1 + η)) +
     (2 β^2 ψ^3 (1 + Sin[ϕf])) / (ξ (-1 + η) (-1 + 3 Sin[ϕf])),
   Shy = (β^2 ψ^3 (-4 ξ^3 + ψ^3))/(2 ξ^4 (-1 + η)) + S0 - (4 Coh Log[ξ])/Ee];
  Shy];
```

Assumption 1 :
Incompressible Elastic

Equation 32a & 32b

In[10]:=

```
CpIn = Function[{η, V, ϕ, term},
  Clear[out]; error = 1;
  c = V / ((3 Coh - 3 Coh η + Ee η)/Ee)^(1/3);
  ceta = -(2 V^3 ρ Tan[ϕf])/(3 Coh) + (12 × 2^(1/3) V^6 ρ^2 Tan[ϕf]^2) /
     (Coh^2 ((6561 Ee V^3 Sec[ϕf])/Coh - 1/Coh 2187 Ee V^3 Tan[ϕf] - 1/Coh^3 11664 V^9 ρ^3 Tan[ϕf]^3 +
          √(-((136048896 V^18 ρ^6 Tan[ϕf]^6) / Coh^6) + (1/Coh 6561 Ee V^3 Sec[ϕf] -
                1/Coh 2187 Ee V^3 Tan[ϕf] - (11664 V^9 ρ^3 Tan[ϕf]^3) / Coh^3)^2))^(1/3)) +
    1/(27 × 2^(1/3)) ((6561 Ee V^3 Sec[ϕf])/Coh - (2187 Ee V^3 Tan[ϕf])/Coh - (11664 V^9 ρ^3 Tan[ϕf]^3)/Coh^3 +
       √(-((136048896 V^18 ρ^6 Tan[ϕf]^6) / Coh^6) + (1/Coh 6561 Ee V^3 Sec[ϕf] -
             1/Coh 2187 Ee V^3 Tan[ϕf] - (11664 V^9 ρ^3 Tan[ϕf]^3) / Coh^3)^2))^(1/3);
  If[ϕ == 0, out = c, If[η == 0, out = ceta]];
  If[ϕ ≠ 0, If[η ≠ 0, While[Abs[error] > term,
      out = V/ (η + (9 Coh (-1 + η) Cos[ϕf]) / (-3 Ee + (Ee + 18 c^2 ρ) Sin[ϕf]))^(1/3);
      error = out - c;
      c = out]]];
  out];
```

Solving Equation 23 & 24; S_0: Equation 33 & 34

In[11]:=

```
SIn = Function[{η, V, ϕ, c, ξ}, ψ = V / c;
   Clear[Sinco];
   If[ϕ ≠ 0, S0 = (Coh Cot[ϕf])/Ee - (2 Coh (2 Ee + 9 c^2 ρ) Cos[ϕf]) / (Ee (-3 Ee +
          (Ee + 18 c^2 ρ) Sin[ϕf])) - (β^2 ψ^3 (1 + Sin[ϕf]) (4 - ψ^3 + 3 ψ^3 Sin[ϕf])) / (2 (-1 + η)
        (-1 + 3 Sin[ϕf])) + η (β + (9 Coh β Cos[ϕf]) / (-3 Ee + (Ee + 18 c^2 ρ) Sin[ϕf]))^2,
    S0 = (β - (3 Coh β)/Ee)^2 η + (2 Coh (2 Ee + 9 c^2 ρ))/(3 Ee^2) - (β^2 ψ^3 (-4 + ψ^3))/(2 (-1 + η))];
   If[ϕ ≠ 0, Sinco = S0 ξ^(-(4 Sin[ϕf])/(1+Sin[ϕf])) - (Coh (Cos[ϕf] + Cot[ϕf]))/(Ee (1 + Sin[ϕf])) + (β^2 ψ^6 (1 + Sin[ϕf]))/(2 ξ^4 (-1 + η)) +
      (2 β^2 ψ^3 (1 + Sin[ϕf])) / (ξ (-1 + η) (-1 + 3 Sin[ϕf])),
    Sinco = (β^2 ψ^3 (-4 ξ^3 + ψ^3))/(2 ξ^4 (-1 + η)) + S0 - (4 Coh Log[ξ])/Ee];
   Sinco];
```

Assumption 2 : Linear Sloution (for linear pressure - volumetric strain aaumption)

Solving Equation 42

In[12]:=
```
CpLinV = Function[{V},
   Clear[out];
   out = FindRoot[(-Ee^2 W (-1 + c^2 X ρ) (-1 + V^2 X ρ) ArcTanh[c √X √ρ]
            (3 cd (c + cd) (-1 + 2 ν) + (cd^2 (1 - 2 ν) + 4 c^2 (1 + ν) + c (cd - 2 cd ν)) Sin[ϕf]) +
          Ee^2 W (-1 + c^2 X ρ) (-1 + V^2 X ρ) ArcTanh[V √X √ρ]
            (3 cd (c + cd) (-1 + 2 ν) + (cd^2 (1 - 2 ν) + 4 c^2 (1 + ν) + c (cd - 2 cd ν)) Sin[ϕf]) +
          √X √ρ (12 c^3 cd (c + cd) Coh (-1 + ν + 2 ν^2) ρ (-1 + V^2 X ρ) Cos[ϕf] +
             Ee (Ee (c - V) W + V (2 V^2 + c Ee (c - V) W X) ρ - 2 c^2 V^3 X ρ^2) (3 cd (c + cd)
                  (-1 + 2 ν) + (cd^2 (1 - 2 ν) + 4 c^2 (1 + ν) + c (cd - 2 cd ν)) Sin[ϕf]))) == 0,
      {c, (V + cd)/2}, AccuracyGoal → 10, PrecisionGoal → 20][[1, 2]];
   out];
```

Equation 17a & 41d & 40b

In[13]:=
```
SrE[c_, ξ_] := (3 A[c] ξ^3 (1 + ν) + 2 B[c] (cd^2 (1 - 2 ν) + 3 c^2 ξ^2 ν)) / (3 ξ^3 (-1 + ν + 2 ν^2));
SrPL0[c_, ξ_] := (3 A[c] (1 + ν) + 2 B[c] (cd^2 (1 - 2 ν) + 3 c^2 ν)) / (3 (-1 + ν + 2 ν^2)) +
    (2 B[c] Ee β^4 - (W + 2 B[c] cd^2 β^2) ρ) / ((-1 + Ee X β^2) ρ) - 1/2 W Log[1/(1 - Ee X β^2)];
SrPL[c_, ξ_] := (W + 2 B[c] (-c^2 + cd^2) β^2)/((-1 + Ee X β^2) ξ) + SrPL0[c, ξ] +
    (W (2 ArcTanh[√(Ee X) β] - 2 ArcTanh[√(Ee X) β ξ] + √(Ee X) β ξ Log[ξ^2/(1 - Ee X β^2 ξ^2)])) /
    (2 √(Ee X) β ξ);
```

Assumption 2 :
Non – Linear Sloution

Solving Equation 43a & 43b with Runge–Kutta method (RK4)

In[16]:=
```
Rungekutta = Function[{V, c, zetaend, zetabeg, NR, y0, y00},
  Clear[ans];
  dydt1[ξ_, u_, s_] := (2 (4 Coh Cos[ϕf] + Ee s (-3 + Sin[ϕf]) + 3 k (1 + Sin[ϕf]))
      (6 k X ξ (Coh Cos[ϕf] + Ee s Sin[ϕf]) + u (2 Coh (2 - 3 k X) Cos[ϕf] +
          3 k (1 + Sin[ϕf]) + Ee s (-3 + Sin[ϕf] - 6 k X Sin[ϕf]))))) /
    (ξ (-Ee^2 s^2 (-3 + Sin[ϕf])^2 - 2 Ee s (-3 + Sin[ϕf]) (4 Coh Cos[ϕf] + 3 k (1 + Sin[ϕf])) +
        (1 + Sin[ϕf]) (-16 Coh^2 + 9 k^2 (-1 + c^2 X ξ^2 ρo) - 24 Coh k Cos[ϕf] + (16 Coh^2 + 9 k^2
              (-1 + c^2 X ξ^2 ρo)) Sin[ϕf] - 9 c^2 k^2 u X (-u + 2 ξ) ρo (1 + Sin[ϕf]))));
  dydt2[ξ_, u_, s_] := -((2 (4 Coh Cos[ϕf] + Ee s (-3 + Sin[ϕf]) + 3 k (1 + Sin[ϕf]))
        (-Ee^2 s^2 (-1 + Cos[2 ϕf] + 6 Sin[ϕf]) + (1 + Sin[ϕf])
            (2 Coh (3 k Cos[ϕf] - 4 Coh (-1 + Sin[ϕf])) + 3 c^2 k u (-u + ξ) ρo (1 + Sin[ϕf])) +
          Ee s (3 k - 6 Coh Cos[ϕf] - 3 k Cos[2 ϕf] + 6 k Sin[ϕf] + 5 Coh Sin[2 ϕf]))) /
      (Ee ξ (1 + Sin[ϕf]) (Ee^2 s^2 (-3 + Sin[ϕf])^2 + 2 Ee s (-3 + Sin[ϕf])
            (4 Coh Cos[ϕf] + 3 k (1 + Sin[ϕf])) +
          (1 + Sin[ϕf]) (16 Coh^2 - 9 k^2 (-1 + c^2 X ξ^2 ρo) + 24 Coh k Cos[ϕf] + (-16 Coh^2 -
                9 k^2 (-1 + c^2 X ξ^2 ρo)) Sin[ϕf] + 9 c^2 k^2 u X (-u + 2 ξ) ρo (1 + Sin[ϕf])))));
  h = (zetabeg - zetaend)/NR;
  ans = {{zetaend, y0, y00}};
  Do[k1 =
    {{dydt1[zet, ans[[1, 2]], ans[[1, 3]]], dydt2[zet, ans[[1, 2]], ans[[1, 3]]]}};
   k2 = {{dydt1[zet + h/2, ans[[1, 2]] + h/2 k1[[1, 1]], ans[[1, 3]] + h/2 k1[[1, 2]]],
      dydt2[zet + h/2, ans[[1, 2]] + h/2 k1[[1, 1]], ans[[1, 3]] + h/2 k1[[1, 2]]]}};
   k3 = {{dydt1[zet + h/2, ans[[1, 2]] + h/2 k2[[1, 1]], ans[[1, 3]] + h/2 k2[[1, 2]]],
      dydt2[zet + h/2, ans[[1, 2]] + h/2 k2[[1, 1]], ans[[1, 3]] + h/2 k2[[1, 2]]]}};
   k4 = {{dydt1[zet + h, ans[[1, 2]] + h k3[[1, 1]], ans[[1, 3]] + h k3[[1, 2]]],
      dydt2[zet + h, ans[[1, 2]] + h k3[[1, 1]], ans[[1, 3]] + h k3[[1, 2]]]}};
   ynp1 = {{ans[[1, 2]], ans[[1, 3]]}} + h/6 (k1 + 2 k2 + 2 k3 + k4);
   ans = Prepend[ans, {zet, ynp1[[1, 1]], ynp1[[1, 2]]}], {zet, zetaend, zetabeg, h}];


  ans];
```

S_r and c_p for differnt assumptions

In[17]:=

```
listCph = {};(*cp: Assumption 1*)
listCpInE = {};(*cp: Assumption 1 incompressible elastic*)
listCpInEP = {};(*cp: Assumption 1 incompressible elastic&plastic*)
listCpLinV = {};(*cp: Assumption 2: Linear*)
listCpFV = {};(*cp: Assumption 2: Non-Linear*)
listSh = {};(*Sr: Assumption 1*)
listSInE = {};(*Sr: Assumption 1 incompressible elastic*)
listSInEP = {};(*Sr: Assumption 1 incompressible elastic&plastic*)
listSLinV = {};(*Sr: Assumption 2: Linear*)
listSFV = {};(*Sr: Assumption 2: Non-Linear*)
```

$$\text{Do}\Big[\text{Vi} = \text{i}\sqrt{\frac{Y}{\rho}};\ \text{c} = \text{CpLinV[Vi]};$$

$$\text{listCpLinV} = \text{Append}\Big[\text{listCpLinV},\ \Big\{\text{Vi}\sqrt{\frac{\rho}{Y}},\ \text{c}\sqrt{\frac{\rho}{Y}}\Big\}\Big];$$

$$\text{listSLinV} = \text{Append}\Big[\text{listSLinV},\ \Big\{\text{Vi}\sqrt{\frac{\rho}{Y}},\ \text{SrPL}\Big[\text{c},\ \frac{\text{Vi}}{\text{c}}\Big]\ \frac{\text{Ee}}{Y}\Big\}\Big];$$

$$\text{err} = 1;\ \text{If[ArrayDepth[lictcFV]} > 5,\ \text{c} = \text{cp]};$$

$$\text{While}\Big[\text{Abs[err]} > \text{term},\ \text{y0} = \text{B[c]}\ (\text{c} - \text{cd})\ (\text{c} + \text{cd});$$

$$\text{y00} = \big(3\,\text{A[c]}\ (1 + \nu) + 2\,\text{B [c]}\ \big(\text{cd}^2\ (1 - 2\,\nu) + 3\,\text{c}^2\,\nu\big)\big) \big/ \big(3\ \big(-1 + \nu + 2\,\nu^2\big)\big);$$

$$\text{zetaend} = 1;$$

$$\text{zetabeg} = \frac{\text{Vi}}{\text{c}};$$

$$\text{ansFR} = \text{Rungekutta[Vi, c, zetaend, zetabeg, NR, y0, y00]};$$

$$\text{cp} = 0.1\,\frac{\text{Vi}}{\text{ansFR[[1, 2]]}} + 0.9\,\text{c};$$

$$\text{err} = \text{c} - \text{cp};$$

$$\text{c} = \text{cp}\Big];$$

$$\text{listCpFV} = \text{Append}\Big[\text{listCpFV},\ \Big\{\text{Vi}\sqrt{\frac{\rho}{Y}},\ \text{cp}\sqrt{\frac{\rho}{Y}}\Big\}\Big];$$

```
listSFV = Append[listSFV, {Vi Sqrt[ρ/Y], ansFR[[1, 3]] Ee/Y}];
c = Cph[ηs, Vi, ϕf, term];

listCph = Append[listCph, {Vi Sqrt[ρ/Y], c Sqrt[ρ/Y]}];
listSh = Append[listSh, {Vi Sqrt[ρ/Y], Sh[ηs, Vi, ϕf, c, Vi/c] Ee/Y}];
c = CpIn[ηs, Vi, ϕf, term];
listCpInE = Append[listCpInE, {Vi Sqrt[ρ/Y], c Sqrt[ρ/Y]}];
listSInE = Append[listSInE, {Vi Sqrt[ρ/Y], SIn[ηs, Vi, ϕf, c, Vi/c] Ee/Y}];
c = CpIn[0, Vi, ϕf, term];
listCpInEP = Append[listCpInEP, {Vi Sqrt[ρ/Y], c Sqrt[ρ/Y]}];
listSInEP = Append[listSInEP, {Vi Sqrt[ρ/Y], SIn[0, Vi, ϕf, c, Vi/c] Ee/Y}], {i, 0.01, 5.01, .1}]
```

FindRoot: The line search decreased the step size to within tolerance specified by AccuracyGoal and PrecisionGoal but was unable to find a sufficient decrease in the merit function. You may need more than MachinePrecision digits of working precision to meet these tolerances.

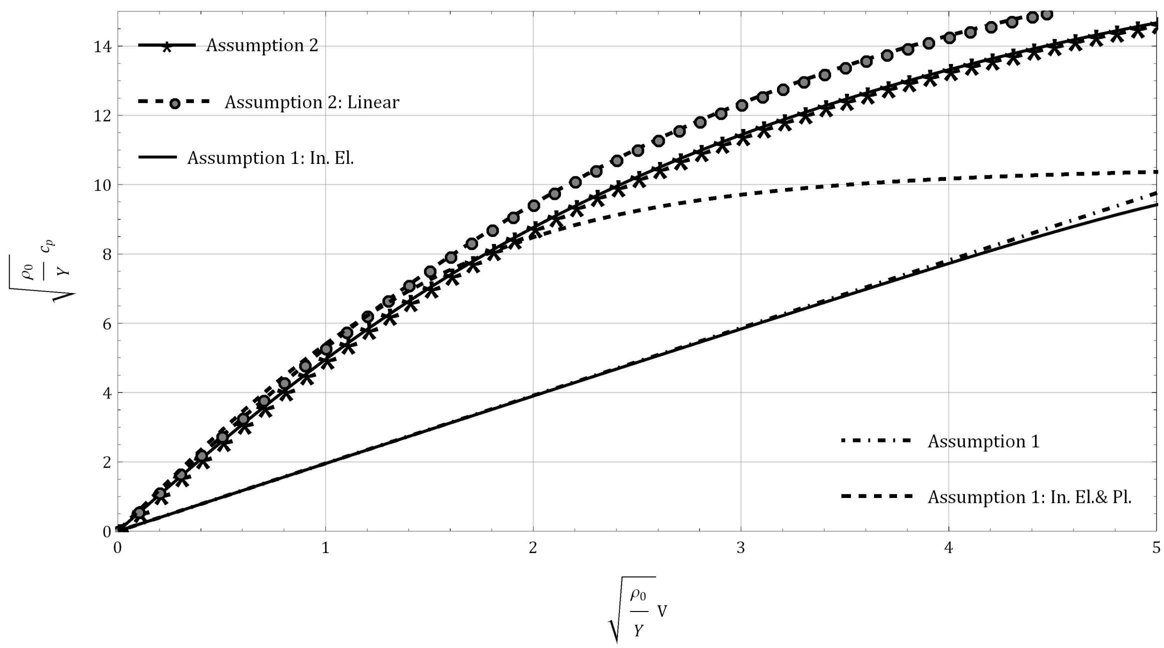
FindRoot: The line search decreased the step size to within tolerance specified by AccuracyGoal and PrecisionGoal but was unable to find a sufficient decrease in the merit function. You may need more than MachinePrecision digits of working precision to meet these tolerances.

FindRoot: The line search decreased the step size to within tolerance specified by AccuracyGoal and PrecisionGoal but was unable to find a sufficient decrease in the merit function. You may need more than MachinePrecision digits of working precision to meet these tolerances.

General: Further output of FindRoot::lstol will be suppressed during this calculation.

In[28]:=
```
CPLinV =
  ListPlot[listCpLinV , PlotLegends → Placed[{"Assumption 2: Linear"}, {Left, Top}],
   LabelStyle → (FontFamily → "Cambria"), Joined → True,
   PlotRange → {{0, 5}, {0, 15}}, GridLines → Automatic, AspectRatio → .5,
   Frame → {{True, True}, {True, True}}, PlotStyle → {Black, Dashed, Thick},
   FrameLabel → {{"Sqrt[ρ0/Y] cp", None}, {"Sqrt[ρ0/Y] V", None}}, PlotMarkers → { "●", Medium}];
CPFV = ListPlot[listCpFV , PlotLegends → Placed[{"Assumption 2"}, {Left, Top}],
   LabelStyle → (FontFamily → "Cambria"), Joined → True,
   PlotRange → {{0, 5}, {0, 15}}, GridLines → Automatic, AspectRatio → .5,
   Frame → {{True, True}, {True, True}}, PlotStyle → Black,
   FrameLabel → {{"Sqrt[ρ0/Y] cp", None}, {"Sqrt[ρ0/Y] V", None}}, PlotMarkers → { "*", Large}];
CPH = ListPlot[listCph , PlotLegends → Placed[{"Assumption 1"}, {Right, Bottom}],
   LabelStyle → (FontFamily → "Cambria"), Joined → True,
   PlotRange → {{0, 5}, {0, 15}}, GridLines → Automatic, AspectRatio → .5,
   Frame → {{True, True}, {True, True}}, PlotStyle → {Black, DotDashed, Thick},
   FrameLabel → {{"Sqrt[ρ0/Y] cp", None}, {"Sqrt[ρ0/Y] V", None}}];
CPHINE = ListPlot[listCpInE, PlotLegends → Placed[{"Assumption 1: In. El."},
     {Left, Top}], LabelStyle → (FontFamily → "Cambria"), Joined → True,
   PlotRange → {{0, 5}, {0, 15}}, GridLines → Automatic, AspectRatio → .5,
   Frame → {{True, True}, {True, True}}, PlotStyle → Black,
   FrameLabel → {{"Sqrt[ρ0/Y] cp", None}, {"Sqrt[ρ0/Y] V", None}}];
CPHINEP = ListPlot[listCpInEP, PlotLegends → Placed[{"Assumption 1: In. El.& Pl."},
     {Right, Bottom}], LabelStyle → (FontFamily → "Cambria"), Joined → True,
   PlotRange → {{0, 5}, {0, 15}}, GridLines → Automatic, AspectRatio → .5,
   Frame → {{True, True}, {True, True}}, PlotStyle → {Black, Dashed, Thick},
   FrameLabel → {{"Sqrt[ρ0/Y] cp", None}, {"Sqrt[ρ0/Y] V", None}}];
Show[CPFV, CPLinV, CPH, CPHINE, CPHINEP]
```

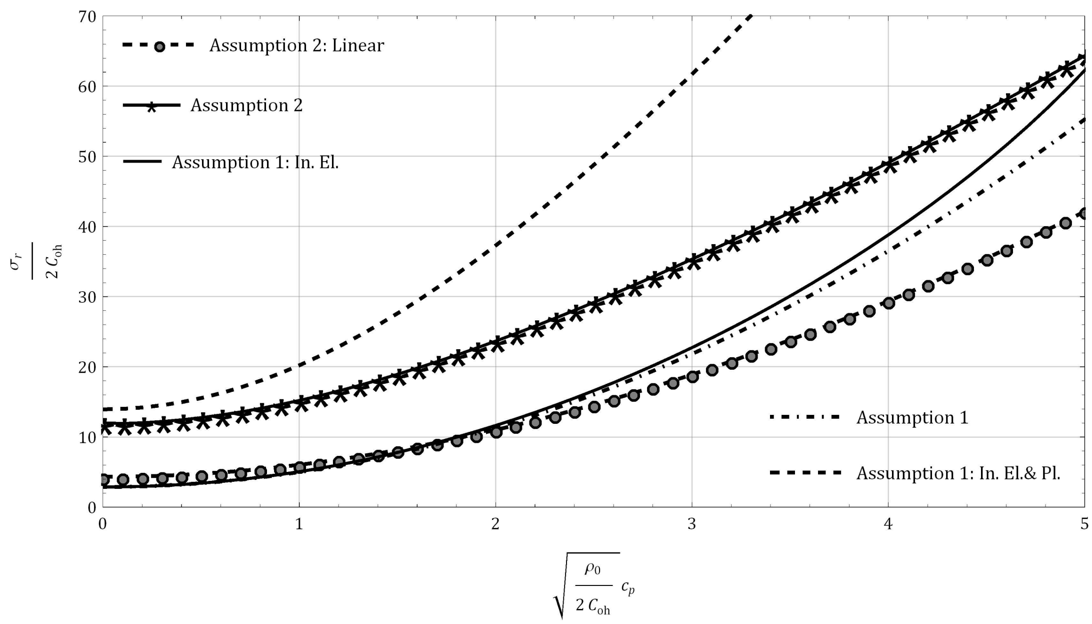
Out[33]=


Assumption 2
Assumption 2: Linear
Assumption 1: In. El.
Assumption 1
Assumption 1: In. El.& Pl.
0
2
4
6
8
10
12
14
0
1
2
3
4
5
$\sqrt{\frac{\rho_0}{Y}}\ c_p$
$\sqrt{\frac{\rho_0}{Y}}\ V$

In[34]:=

```
SLinV = ListPlot[listSLinV , PlotLegends → Placed[{"Assumption 2: Linear"}, {Left, Top}],
   LabelStyle → (FontFamily → "Cambria") , Joined → True,
   PlotRange → {{0, 5}, {0, 70}}, GridLines → Automatic, AspectRatio → .5,
   Frame → {{True, True}, {True, True}}, PlotStyle → {Black, Dashed, Thick},
   FrameLabel → {{"σr/(2 Coh)", None}, {"Sqrt[ρθ/(2 Coh)] cp", None}}, PlotMarkers → { "●", Medium}];
SFV = ListPlot[listSFV , PlotLegends → Placed[{"Assumption 2"}, {Left, Top}],
   LabelStyle → (FontFamily → "Cambria") , Joined → True,
   PlotRange → {{0, 5}, {0, 70}}, GridLines → Automatic, AspectRatio → .5,
   Frame → {{True, True}, {True, True}}, PlotStyle → Black,
   FrameLabel → {{"σr/(2 Coh)", None}, {"Sqrt[ρθ/(2 Coh)] cp", None}}, PlotMarkers → { "*", Large}];
SH = ListPlot[listSh , PlotLegends → Placed[{"Assumption 1"}, {Right, Bottom}],
   LabelStyle → (FontFamily → "Cambria") , Joined → True,
   PlotRange → {{0, 5}, {0, 70}}, GridLines → Automatic, AspectRatio → .5,
   Frame → {{True, True}, {True, True}}, PlotStyle → {Black, DotDashed, Thick},
   FrameLabel → {{"σr/(2 Coh)", None}, {"Sqrt[ρθ/(2 Coh)] cp", None}}];
SHINE = ListPlot[listSInE, PlotLegends → Placed[{"Assumption 1: In. El."}, {Left, Top}],
   LabelStyle → (FontFamily → "Cambria") , Joined → True, PlotRange → {{0, 5}, {0, 70}},
   GridLines → Automatic, AspectRatio → .5, Frame → {{True, True}, {True, True}},
   PlotStyle → Black, FrameLabel → {{"σr/(2 Coh)", None}, {"Sqrt[ρθ/(2 Coh)] cp", None}}];
SHINEP = ListPlot[listSInEP, PlotLegends → Placed[{"Assumption 1: In. El.& Pl."},
      {Right, Bottom}], LabelStyle → (FontFamily → "Cambria") , Joined → True,
   PlotRange → {{0, 5}, {0, 70}}, GridLines → Automatic, AspectRatio → .5,
   Frame → {{True, True}, {True, True}}, PlotStyle → {Black, Dashed, Thick},
   FrameLabel → {{"σr/(2 Coh)", None}, {"Sqrt[ρθ/(2 Coh)] cp", None}}];
Show[SLinV, SFV, SH, SHINE, SHINEP]
```

Out[39]=



Penetration

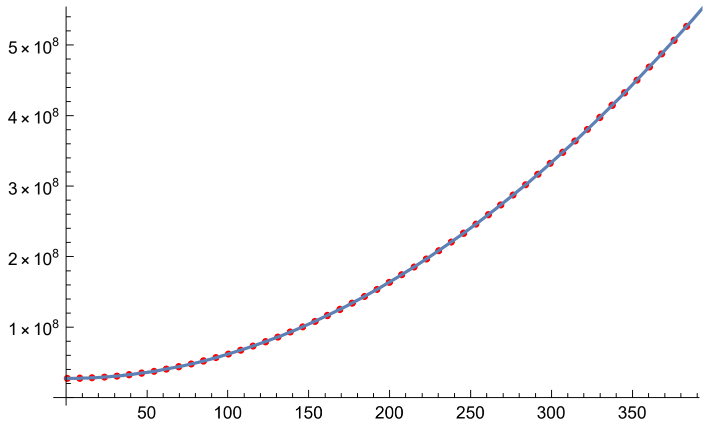
Geometry of the probe

In[40]:=

```
a = 0.156; M = 162 + 55.3;
Vp = 520; μ = 0;
CHR = 6;
s = 2 a CHR; θo = ArcSin[(s - a)/s];
```

Curve fitting for Assumption 1 (Hydrostat model)

In[44]:=
```
listSh2 = listSh;
listSh2[[All, 1]] = listSh2[[All, 1]] / Sqrt[ρ/Y];
listSh2[[All, 2]] = listSh2[[All, 2]] Y;
fith = Fit[listSh2, {1, x, x^2, x^3, x^4}, x];
g[x_] := fith
Show[ListPlot[listSh2, PlotStyle → Red], Plot[g[x], {x, 0, 500}]]
SIGMAFh[vp_, ϕ_] := fith /. x → vp Cos[ϕ];(*V=vp Cos[ϕ]*)
```

Out[49]=


Equation 49

In[51]:=
```
dah = (-SIGMAFh[vp, ϕ] / M) (μ Sin[ϕ] + Cos[ϕ]) (Sin[ϕ] - (s - a)/s) 2 π s^2 // FullSimplify
```

Out[51]= $0.000010533 \operatorname{Cos}[\phi] \left(-2.62668 \times 10^{11} + \text{vp} \operatorname{Cos}[\phi] \left(-5.56098 \times 10^{6} + \text{vp} \operatorname{Cos}[\phi] \left(-3.29592 \times 10^{7} + \text{vp} \operatorname{Cos}[\phi] \left(431.272 + 1.\, \text{vp} \operatorname{Cos}[\phi]\right)\right)\right)\right) (-0.916667 + \operatorname{Sin}[\phi])$

In[52]:=
```
azh = Integrate[dah, {ϕ, ϕ0, π/2}] /. ϕ0 → θo // Simplify
```

Out[52]= $-9606.55 - 0.043885\, \text{vp} - 0.0655723\, \text{vp}^2 + 2.36844 \times 10^{-7}\, \text{vp}^3 + 1.60971 \times 10^{-10}\, \text{vp}^4$

In[53]:=
```
azhy[v_] := azh /. vp → v
```

Equation 50

In[54]:=
```
dzh = vp / azhy[vp] // Simplify
```

Out[54]= $\text{vp} \big/ \left(-9606.55 - 0.043885\, \text{vp} - 0.0655723\, \text{vp}^2 + 2.36844 \times 10^{-7}\, \text{vp}^3 + 1.60971 \times 10^{-10}\, \text{vp}^4\right)$

In[55]:=
```
Zhvp = Integrate[dzh, {vp, Vp0, Vph}, Assumptions → 1000 > Vp0 > Vph > 0]
```

Out[55]=
$$-0.0448554\,\text{ArcTan}[0.00156442 + 0.0026131\,\text{Vp0}] + 0.0448554\,\text{ArcTan}[0.00156442 + 0.0026131\,\text{Vph}] - 7.89624\,\text{Log}[19465. - 1.\,\text{Vp0}] - 7.34299\,\text{Log}[20935.2 + 1.\,\text{Vp0}] + 7.61961\,\text{Log}\left[146450. + 1.19736\,\text{Vp0} + 1.\,\text{Vp0}^2\right] + 7.89624\,\text{Log}[19465. - 1.\,\text{Vph}] + 7.34299\,\text{Log}[20935.2 + 1.\,\text{Vph}] - 7.61961\,\text{Log}\left[146450. + 1.19736\,\text{Vph} + 1.\,\text{Vph}^2\right]$$

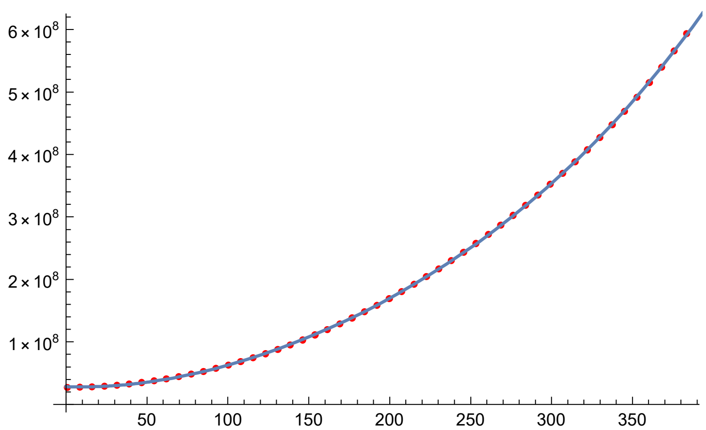
In[56]:=
```
Zh[v_, v0_] := Zhvp /. Vph → v /. Vp0 → v0
```

In[57]:=
```
Zh[0, Vp]
```

Out[57]= 7.97351

Curve fitting for Assumption 1 (incompressible elastic)

In[58]:=
```
listSInE2 = listSInE;
listSInE2[[All, 1]] = listSInE2[[All, 1]] / Sqrt[ρ/Y];
listSInE2[[All, 2]] = listSInE2[[All, 2]] Y;
fitInE = Fit[listSInE2, {1, x, x^2, x^3, x^4}, x];
gInE[x_] := fitInE
Show[ListPlot[listSInE2, PlotStyle → Red], Plot[ gInE[x] , {x, 0, 500}]]
SIGMAFInE[vp_, ϕ_] := fitInE /. x → vp Cos[ϕ];(*V=vp Cos[ϕ]*)
```

Out[63]=


Equation 49

In[65]:=
$$\text{daInE} = \frac{-\text{SIGMAFInE}[\text{vp}, \phi]}{\text{M}} (\mu \text{Sin}[\phi] + \text{Cos}[\phi]) \left(\text{Sin}[\phi] - \frac{\text{s} - \text{a}}{\text{s}}\right) 2 \pi \text{s}^2 \text{ // FullSimplify}$$

Out[65]=
$$-0.00137261 \text{Cos}[\phi] \left(2.09649 \times 10^9 + \text{vp Cos}[\phi] \left(-6.58686 \times 10^6 + \text{vp Cos}[\phi] (364451. + \text{vp Cos}[\phi] (-552.028 + 1. \text{ vp Cos}[\phi]))\right)\right) (-0.916667 + \text{Sin}[\phi])$$

In[66]:=
$$\text{azInE} = \text{Integrate}\left[\text{daInE}, \left\{\phi, \phi0, \frac{\pi}{2}\right\}\right] \text{ /. } \phi0 \rightarrow \theta o \text{ // Simplify}$$

Out[66]=
$$-9991.84 + 6.77387 \text{ vp} - 0.0944881 \text{ vp}^2 + 0.0000395063 \text{ vp}^3 - 2.09769 \times 10^{-8} \text{ vp}^4$$

In[67]:=
```
azIncE[v_] := azInE /. vp → v
```

Equation 50

In[68]:=
$$\text{dzIncE} = \frac{\text{vp}}{\text{azIncE}[\text{vp}]} \text{ // Simplify}$$

Out[68]=
$$\text{vp} \Big/ \left(-9991.84 + 6.77387 \text{ vp} - 0.0944881 \text{ vp}^2 + 0.0000395063 \text{ vp}^3 - 2.09769 \times 10^{-8} \text{ vp}^4\right)$$

In[69]:=
```
ZIncEvp = Integrate[dzIncE, {vp, Vp0, Vph}, Assumptions → 1000 > Vp0 > Vph > 0]
```

Out[69]=
$$1.13391 \text{ ArcTan}[0.0415442 - 0.00302241 \text{ Vp0}] - 5.66949 \text{ ArcTan}[0.497224 - 0.000535849 \text{ Vp0}] - 1.13391 \text{ ArcTan}[0.0415442 - 0.00302241 \text{ Vph}] + 5.66949 \text{ ArcTan}[0.497224 - 0.000535849 \text{ Vph}] - 5.58166 \text{ Log}\left[4.34372 \times 10^6 - 1855.84 \text{ Vp0} + 1. \text{ Vp0}^2\right] + 5.58166 \text{ Log}\left[109659. - 27.4908 \text{ Vp0} + 1. \text{ Vp0}^2\right] + 5.58166 \text{ Log}\left[4.34372 \times 10^6 - 1855.84 \text{ Vph} + 1. \text{ Vph}^2\right] - 5.58166 \text{ Log}\left[109659. - 27.4908 \text{ Vph} + 1. \text{ Vph}^2\right]$$

In[70]:=
```
ZIncE[v_, v0_] := ZIncEvp /. Vph → v /. Vp0 → v0
```

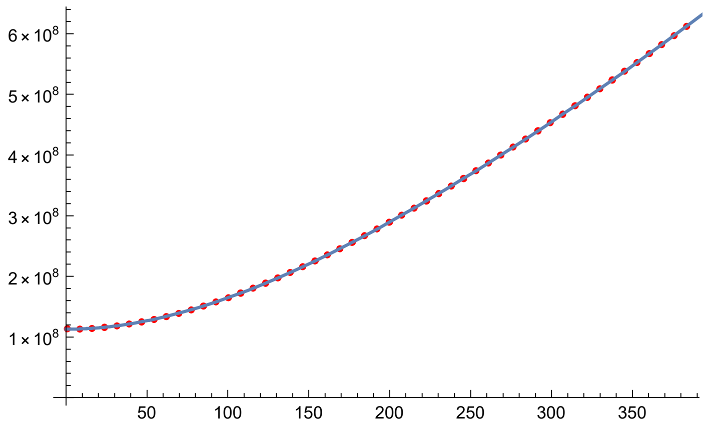
In[71]:=
```
ZIncE[0, Vp]
```

Out[71]= 7.92051

Curve fitting for Assumption 2

In[72]:=
```
listSFV2 = listSFV;
listSFV2[[All, 1]] = listSFV2[[All, 1]] / Sqrt[ρ/Y];
listSFV2[[All, 2]] = listSFV2[[All, 2]] Y;
fitFV = Fit[listSFV2, {1, x, x^2, x^3, x^4}, x];
gFV[x_] := fitFV
Show[ListPlot[listSFV2, PlotStyle → Red], Plot[gFV[x] , {x, 0, 500}]]
SIGMAFFV[vp_, ϕ_] := fitFV /. x → vp Cos[ϕ];(*V=vp Cos[ϕ]*)
```

Out[77]=



Equation 49

In[79]:=
```
daFV = -SIGMAFFV[vp, ϕ]/M (μ Sin[ϕ] + Cos[ϕ]) (Sin[ϕ] - (s - a)/s) 2 π s^2 // FullSimplify
```

Out[79]= $-0.000803805 \text{Cos}[\phi] \left(1.42417 \times 10^{10} + \text{vp Cos}[\phi] \left(-1.35328 \times 10^{6} + \text{vp Cos}[\phi] (784214. + \text{vp Cos}[\phi] (-1302.85 + 1. \text{ vp Cos}[\phi]))\right)\right) (-0.916667 + \text{Sin}[\phi])$

In[80]:=
```
azFV = Integrate[daFV, {ϕ, ϕ0, π/2}] /. ϕ0 → θo // Simplify
```

Out[80]= $-39748.3 + 0.814987 \text{ vp} - 0.119063 \text{ vp}^2 + 0.0000546012 \text{ vp}^3 - 1.22842 \times 10^{-8} \text{ vp}^4$

In[81]:=
```
azFVN[v_] := azFV /. vp → v
```

Equation 50

In[82]:=
```
dzFV = vp/azFVN[vp] // Simplify
```

Out[82]= $\text{vp} / \left(-39748.3 + 0.814987 \text{ vp} - 0.119063 \text{ vp}^2 + 0.0000546012 \text{ vp}^3 - 1.22842 \times 10^{-8} \text{ vp}^4\right)$

In[83]:=
```
ZFV = Integrate[dzFV, {vp, Vp0, Vph}, Assumptions → 1000 > Vp0 > Vph > 0]
```

Out[83]=
```
-8.70795 ArcTan[1.05112 - 0.000458392 Vp0] - 2.96523 ArcTan[0.125245 + 0.00177324 Vp0] +
 8.70795 ArcTan[1.05112 - 0.000458392 Vph] + 2.96523 ArcTan[0.125245 + 0.00177324 Vph] -
 3.66473 Log[1.00172 × 10^7 - 4586.11 Vp0 + 1. Vp0^2] + 3.66473 Log[323017. + 141.262 Vp0 + 1. Vp0^2] +
 3.66473 Log[1.00172 × 10^7 - 4586.11 Vph + 1. Vph^2] - 3.66473 Log[323017. + 141.262 Vph + 1. Vph^2]
```

In[84]:=
```
ZFVN[v_, v0_] := ZFV /. Vph → v /. Vp0 → v0
```

In[85]:=
```
ZFVN[0, Vp]
```

Out[85]=
```
2.61069
```