

# Styles of reasoning for mathematics education

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## Abstract

Although reasoning is a central concept in mathematics education research, the discipline is still in need of a coherent theory of mathematical reasoning. With respect to epistemological problems in the dominant discourses on proof, mathematical modelling, and post-truth politics in the discipline, and in accordance with trends in the philosophy of mathematics and in mathematics education research in general, it is argued that it is necessary to give a relativist account mathematical reasoning. Hacking's framework of styles of reasoning is introduced as a possible solution. This framework distinguished between at least six different styles of reasoning, many of which are closely connected to mathematics, and argues that these frameworks define what we accept as decidable assertions, as justifications for such assertions, and as possible objects of such assertions. The articles end with a discussion of the implications of the framework for chosen fields of mathematics education research, which may motivate more focussed studies in the future.

## Keywords

Mathematical reasoning, proof, mathematical modelling, post-factual politics, epistemology, styles of reasoning

## 1 Reasoning and mathematics

If we understand 'reasoning' as a discursive practice which justifies assertions, then we will have to admit that mathematics education is closely concerning with reasoning. In this essay, 'mathematical reasoning' is meant to denote of reasoning that involves mathematics – a definition which stays deliberately vague. This includes reasoning *with* mathematics, when we apply 'pure' mathematics to worldly affairs and use mathematics as a means to justify respective assertions, but also reasoning *in* mathematics, which can follow strict deductive lines of inference or less imperative lines of thought.

Jeannotte and Kieran (2017) stated that '[t]he development of students' mathematical reasoning is a goal of several curricula and an essential element of the culture of the mathematics education research community' (p. 1). Every reader might turn to publications from their own cultural background to recognise respective goals. To give a popular German example, Winter (1995) proposed that the educational value of mathematics education lies in its ability to have learners experience, first, that mathematics is useful for modelling problems in reality, second, that mathematics shows 'that rigorous science is possible', and third, that mathematics is a 'school of thought' in the sense of the development of heuristic skills (pp. 39, 42, my translations).<sup>1</sup> Although Winter did not frame his discussion under the term of 'reasoning', it is informative that his three goals could be understood as experiencing, first, reasoning *with* mathematics, second, *deductive* reasoning *in* mathematics, and, third, *more flexible* reasoning *in* mathematics. Read like this, Winter's message would be that mathematics education is all about learning to reason with the help of mathematics.

Not surprisingly, different forms of reasoning play a central part in mathematics education research. A search for the term 'reasoning' in Lerman's (2020) most recent 916-pages long *Encyclopedia of*

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<sup>1</sup> Vohns (2017) provided a more detailed account of Winter's proposal in English.

*Mathematics Education*, which I assume to present contemporary and influential theories in the discipline, yields entries on ‘Deductive reasoning’ (Harel & Weber, 2020), ‘Mathematical proof, argumentation, and reasoning’ (Hanna, 2020) and ‘Quasi-empirical reasoning’ (Sriraman & Mousoulides, 2020), as well as further mentions of more than 30 other forms of reasoning, including ‘algebraic reasoning’, ‘diagrammatic reasoning’, ‘functional reasoning’, ‘geometric reasoning’, ‘inductive reasoning’, ‘mathematical reasoning’, ‘multiplicative reasoning’, ‘scientific reasoning’, and ‘statistical reasoning’.<sup>2</sup>

Although we find ‘mathematical reasoning’ alone mentioned in 28 of the 216 entries of the *Encyclopedia*, it remains unclear what is meant by the term. While some entries lead to the assumption that mathematical reasoning is identical to deductive reasoning, the above list indicates that mathematical reasoning includes many other forms of reasoning. Surprisingly, no entry presents a general answer what reasoning is and what it has to do with mathematics and the above-listed special forms of reasoning.

Neither do I know any study from mathematics education research that would attempt to develop such a general theory. Already Yackel and Hanna (2003) remarked that ‘[w]riting about reasoning in mathematics is complicated by the fact that the term *reasoning*, like understanding, is widely used with the implicit assumption that there is universal agreement on its meaning’, while in fact, the meaning of the term remains widely unclear (p. 228, original emphasis). In a more recent reflection, Jeannotte and Kieran (2017) concluded that ‘the way in which [mathematical reasoning] is described within [curricular] documents tends to be vague, unsystematic, and even contradictory’, that ‘[t]he current state of the field renders difficult any comparison of not only the various approaches to, and characterizations of, [mathematical reasoning] but also the results of related studies’, and that ‘this area is one that could benefit greatly from an attempt at coherent conceptualization’ (pp. 1–2). Jeannotte and Kieran (2017) pioneered by discussing different perspectives from which reasoning has been approached in mathematics education research and by identifying and relating recurrent themes in research on reasoning in the field. However, as their resulting ‘conceptual model’ is mainly a description what was found in the literature, it included the theoretical inconsistencies found in the research field and did not attempt to present a theoretically coherent account of reasoning.

Reasoning can be discussed from different scholarly perspectives. For example, logical studies focus on the formal structure of arguments and discursive studies ask how language serves to justify knowledge. The perspective of this essay will be epistemological, that is, it will ask how the justification of assertions is possible at all and what it means that an assertion is justified. Any conceptualisation of reasoning from this perspective is complicated by different assumptions concerning the epistemic and ontic functions of reasoning. From the perspective of realism, which assumes that the objects of our discourse or at least their characteristics exist independently of human beings, reasoning allows us to demonstrate what is true or at least reasonable to assume as true. From the perspective of relativism, which assume that objects and their characteristics are products of discourse and therefore mind-dependent, reasoning is productive in shaping our reality. Both perspectives ask for very different functions of reasoning to be described by any conceptualisation, and they invoke very different philosophical problems.

In the next section of this essay, I will address the epistemological positions of realism and relativism along with the problems attached to each position. In the third section, I will describe the development of a relativist understanding of mathematics to motivate the need for a relativist theory of mathematical reasoning. In the fourth section, I will discuss selected fields of mathematics education research to motivate both the need for a general theory of reasoning and the need for a relativist understanding of reasoning. The fifth section is then dedicated to the introduction of the framework of styles of reasoning, which will be developed as a possible solution to the presented problems. In a last section, I will reflect on this solution and propose ways to move forward in future research.

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<sup>2</sup> I refrain from providing the full battery of references here. They are easily found through a digital search in the *Encyclopedia*.

## 2 Realist versus relativist epistemologies

Chakravartty (2017) proposed to call an epistemology ‘realist’ if it follows

- the metaphysical assumption that the objects of reality exist independently of human beings,
- the linguistic assumption that language allows for a true description of reality, and
- the epistemological assumption that knowledge resembles reality.

Especially mathematicians are said to often hold realist positions (Davis & Hersh, 1980). This becomes obvious when we presume that the law of the excluded middle (assuming that a claim is true or false and not beyond or in between that) is an uncircumventable law of thought for any sane person, or that the decimal expansion of  $\pi$ , defined as the ratio of a circle’s circumference to its diameter, stays the same anywhere, at any time, for anybody, also for aliens whom we send this decimal expansion to illustrate the intellectual achievements of us Earthlings.

However, closer analysis reveals some fundamental problems of realism. Although some of these problems were already debated in ancient anti-Eleatic philosophy and re-entered the philosophical stage in the eighteenth century by philosophers such as Berkeley, Hume and Kant (Liston, 2020), they were hardly taken into account in the wider field of science until the emergence of post-structural and constructivist paradigms in the second half of the twentieth century. The following line of argument is an abridged selection from a philosophical discussion which easily fill books (e.g., Audi, 1997/2010; Landesman, 1997):

- Epistemologically, it has been established that we, limited by our sensual perceptions, cannot know in how far our theories accurately resemble an independent existence or if they leave blind spots.
- Metaphysically, this means that neither can we determine if a world independent of human beings exists, nor does it matter epistemologically. The metaphysical assumption is therefore scientifically undecidable and reduced to a matter of belief.
- Linguistically, post-structuralist studies suggest that academic discourses cannot be understood as a true description of reality, but constitute a historically contingent product which depends on values, interests and power relations.<sup>3</sup>

In contrast, Baghramian and Carter (2020) described relativism as ‘the view that truth and falsity, right and wrong, standards of reasoning, and procedures of justification are products of differing conventions and frameworks of assessment and that their authority is confined to the context giving rise to them’ (p. 1). This understanding of relativism would include constructivist and post-structural positions. Even though relativist positions solve the problems identified in the above objections against realism, they provoke new objections (Koertge, 2013; Swoyer, 2015), two of which should be noted here: If knowledge cannot be accessed by reference to a mind-independent reality, then we might find ourselves lacking explanations of why different methods lead to similar insights and why scientific knowledge should be epistemically privileged against other forms of discourse.

Whether new forms of realism and relativism could overcome their above-mentioned problems while keeping their basic assumptions intact is a matter of ongoing philosophical debate. Here, there will be no need to argue that one position makes more sense than the other in general. However, I will deliberately embrace a relativist position in my following discussion of mathematical reasoning, and I will use the next sections for laying out my reasons for doing so. This does not imply, however, that a realist theory of mathematical reasoning is impossible or might not be desirable, albeit for other reasons than the ones I will discuss now.

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<sup>3</sup> Foucault’s (1961) deconstruction of the Modern discourse on insanity is an early example of a study following this paradigm. Elsewhere, I present such a post-structuralist argumentation for logical thinking (Kollosche, 2014b, or, if you can read German, preferably Kollosche, 2014a, Ch. 5). The ineluctable discrepancy between discourses and that which escapes articulation constitutes a central pillar of Lacan’s psychoanalysis (Žižek, 1994).

### 3 Realist versus relativist philosophies of mathematics

Ever since the objects of mathematics and their properties were located in the realm of a mind-independent and eternal truth by the Ancient Greek philosopher Plato, mathematics has served for many as the prime example that knowledge of such truth is possible. With his *Elements*, the Ancient Greek mathematician Euclid has shaped mathematics as a discipline that uses deduction to infer the truth of not-so-obviously true assertions from the most fundamental and obvious truths. The French mathematician and philosopher Descartes (Descartes, 1637/1824) demanded that every discipline which set out to prove the truth of its assertions must follow the example of mathematics. In this vein, the Dutch philosopher Spinoza (1677) presented a theory of ethics ‘in geometrical order’, and the Austrian philosopher Wittgenstein (1922) presented a deductively ordered theory of the logic of language, before he became a critic of the deductive approach in his later career.

In the nineteenth century, developments in mathematics made it impossible to continue to believe in any naïve or Platonist realism (Hersh, 1979): First, while Euclid’s geometry had been considered the true description of the properties of space, the development and application of non-Euclidean geometries showed that mutually contradictory geometries fulfil the requirements of mathematical rigour and can have valuable applications. What sense would it then make to say that one of these geometries was *the* true description of reality? Second, Modern mathematics saw the introduction of mathematical objects whose existence in reality could not easily be shown and which often could not even be imagined, including infinitesimal numbers, space-filling curves, and continuous nowhere-differentiable curves. In which sense would these objects be real if their existence was purely mathematical and opposed to intuition? Third, the Austrian mathematician Gödel could prove that axiomatic theories, which include the arithmetic of natural numbers, do not allow to prove all sentences of that arithmetic, nor can it be guaranteed that they do not lead to contradictions. What reason would be left to assume that any non-trivial mathematical theory could show the true properties of reality?

Shapiro (2000) argued that ‘[a] common response to such dilemmas is to retreat to formalism’ (p. 131), an understanding of mathematics as ‘a game played with linguistic characters’ along the rules of deductive inference (p. 144). The benefit of such a position is that, as Neumann (1931/1983) put it,

even if the statements of classical mathematics should turn out to be false as to content, nevertheless, classical mathematics involves an internally closed procedure which operates according to fixed rules known to all mathematicians and which consists basically in constructing successively certain combinations of primitive symbols which are considered “correct” or “proved.” (pp. 61–62)

But what use has mathematics if it secures statements that are not necessarily true in a realistic sense? Frege (1903) argued that, if we accepted formalism, there would be no reason why mathematical knowledge would be more useful than other purely rule-bound activities such as chess. In this vein, the matured Wittgenstein (1956/1978) stated that ‘[i]t is the use outside mathematics, and so the *meaning* of the signs, that makes the sign-game into mathematics’ (p. 257, original emphasis).

Wittgenstein (1956/1978) was a strong proponent of the relativist assumption that the justification of assertions is ultimately based on *social* grounds, even in mathematics:

Isn’t it like this: so long as one thinks it can’t be otherwise, one draws logical conclusions. This presumably means: so long as *such-and-such is not brought in question at all*. The steps which are not brought in question are logical inferences. But the reason why they are not brought in question is not that they ‘certainly correspond to the truth’ – or something of the sort, – no, it is just this that is called ‘thinking’, ‘speaking’, ‘inferring’, ‘arguing’. There is not any question at all here of some correspondence between what is said and reality; rather is logic *antecedent* to any such correspondence; in the same sense, that is, as that in which the establishment of a method of measurement is *antecedent* to the correctness or incorrectness of a statement of length. (p. 96, original emphases)

Scholars have described from different perspectives how mathematical reasoning does not simply describe reality but actually constitutes it in very specific and socially constructed ways (Bueno & Linnebo, 2009; Davis & Hersh, 1980; Hersh, 1997, 2006; Kerkhove & Bendegem, 2007; Porter, 1996; Restivo, 1992; Restivo, Bendegem, & Fischer, 1993; Tymoczko, 1986). Although these publications and

anthologies have contributed to a better understanding of the relativist epistemology of mathematics, they have not yet led to the development of a relativist theory of mathematical reasoning.

## 4 Realist versus relativist positions in mathematics education research

Relativist positions have a long history and broad variety and have become influential in mathematics education research in very different forms. Some prominent examples are radical constructivism (Glaserfeld, 1991), post-structuralism (Walshaw, 2007), and Lacanian psychoanalysis (Brown, 2008). Indeed, relativist theories have allowed for substantially new insights, for example concerning the learning of mathematics (Lerman, 2000) or the socio-political role of mathematics and mathematics education (Valero, 2004). A striking sign of the impact of this development on the discipline is that Lerman's (2020) *Encyclopedia* includes no entry on any realist epistemological position, but entries on the relativist positions of 'constructivism in mathematics education', 'quasi-empirical reasoning', and 'poststructuralist and psychoanalytic approaches in mathematics education'. It would seem that relativism is the epistemological position of choice for many who engage in theory building in mathematics education research today.

However, not all fields of mathematics education research have relativist theories to offer, especially not when it comes to mathematical reasoning. In fact, the theories in use can often be described as employing a naïve realism, whereby they also inherit the most fundamental problems of this epistemological position. I can only briefly address proof and modelling as two prominent examples.

### 4.1 Proof and reasoning in mathematics

Proof is often considered to be a part or even the essence of the most vividly discussed form of reasoning in mathematics. Looking at Lerman's (2020) *Encyclopedia*, the importance of proof for mathematics education research shows up in the sheer mass of entries that touch the topic or parts thereof. These *Encyclopedia* entries share a formalist position in the philosophy of mathematics, where mathematical objects are conceived of as sign games and proofs do not produce 'truth' (in a realist sense) but merely 'deducibility'. For example, Harel and Weber (2020) reported that 'contemporary mathematicians view axiom systems as being a *freestanding system* of relationships between *undefined terms*' (p. 185, my emphases), and Umland and Sriraman (2020) stated that, in mathematics, 'the validity of an argument in its final form is judged *solely* on whether it is *logically* consistent' (p. 61, my emphases).

Apart from the above-mentioned Fregian concern that such a formalist view on proof is ill-fit to explain the applicability of mathematics, it pretends as if no reasons beyond deductive inference would impact our decision which mathematical assertions to consider justified knowledge. Especially, this perspective cannot explain why we would choose our set of axioms in one way or another. In contrast, Hilbert (1930) stated that the fact that the interior angles of a triangle in Euclidean geometry add up to  $180^\circ$  was an empirical truth, and that any axiom system for Euclidean geometry that did not allow to deduce this theorem would be discarded. Roughly put, a certain type of application often defines the truths that an axiomatic theory should produce.

As a result of this 'retreat' to formalism, it remains obscure why mathematical theories are shaped in specific ways, in which sense specific theorems make sense beyond their mere deducibility, what it means epistemically that a mathematical statement is 'true', and how the employed form of reasoning is effective in justifying knowledge at all. In this vein, Jahnke (2007) was concerned 'that the usual teaching of mathematics is not successful in explaining the *epistemological meaning* of proof' (p. 80, original emphasis). More recently, Jahnke and Krömer (2020) called for a focus on the 'justification' of axioms and definitions in mathematics education, and argued that the activity of testing the consequences of different hypotheses will be more authentic and insightful for the understanding of mathematical reasoning than learning proof alone.

In reflection, we do not only need an explanation in how far hypothetico-deductive reasoning – thus Jahnke's (2007, p. 83) wording to highlight the interplay of assuming and deducing – produces justified

knowledge about our world. This explanation would also have to explain the influences of other justifications on the body of knowledge built with hypothetico-deductive reasoning.

#### 4.2 Modelling and reasoning with mathematics

The second example is mathematical modelling, which Kaiser (2020) understood as the activity ‘to solve real-world problems using mathematics’ (p. 554) in her *Encyclopedia* entry on the topic. She explained that ‘[i]n nearly all approaches, the idealized process of mathematical modelling is described as a cyclic process’ (p. 556). The cyclic process assumes a separation between reality and mathematics and is supposed to start with a problem in a real situation, which leads to a real model in a step not involving mathematics, before the model is translated into a mathematical model and solved within mathematics. In a last step, the mathematical solution is interpreted and validated against the original problem in the real world. Theories of mathematical modelling which rely on some version of the modelling cycle have not only informed the wide majority of studies on mathematical modelling, they have also been used to explain and structure mathematical modelling for pupils (e.g., Maroska, Olpp, Bachmann, Banach, & Birkendorf, 2007, p. 148) and to ground the understanding of the relation of reality and mathematics in the PISA studies (e.g., OECD, 2009).

Voigt (2011) highlighted that it would be naïve to conceive of reality as separate from mathematics, only to be mathematised in a later step. Instead, our perception of reality is always-already influenced by our knowledge of mathematics, just as mathematics is not independent of reality. Schürmann (2016) argued that the separation between reality and mathematics in the modelling cycle can be understood as the paradoxical attempt to rescue the formalist assumption of the independence of mathematics from reality while simultaneously postulating a compatibility of mathematics and reality. The validation step in the modelling cycle, which tests the suitability of mathematical solutions for the initial ‘real-world problem’, acts as if the real-world problem was not being redefined by mathematics but existing independently of mathematics. However, Schürmann (2021) demonstrated that many modelling activities translate between mathematics and other elaborated theories rather than between mathematics and an empirical reality. For example, in a comment on a modelling task where it has to be decided if it is worth driving to a different country to buy cheaper petrol there, Schürmann (2016) argued that it would be impractical to refer to a potential environmental damage of that trip. This reveals that the decision to approach this problem from the perspective of mathematics has already formatted our perception of it in economic terms. In this vein, Jablonka and Gellert (2007) argued that ‘[m]athematics is a means for the generation of new realities not only by providing descriptions of “real world situations”, but also by colonising, permeating and transforming reality’, adding that ‘[m]odels become the reality, which they set out to model’ (p. 6).

If we want to understand how our perception of reality is influenced by mathematics, we need a theory of mathematical modelling as a form of reasoning with mathematics, which explains how mathematical modelling contributes to justifying assertions about the ‘real world’, what it means for an assertion about the ‘real world’ to be justified mathematically in specific, and thus in which way and with which effects our reality is constituted mathematically.

#### 4.3 Mathematics education and post-truth politics

An issue which is rather new on the agenda of mathematics education research is that of its relation to post-truth politics, which can be understood as a form of political discourse and practice which no longer relies on justified knowledge. Thereby, ‘politics’ should be understood in a broad sense, not only as that which is negotiated between politicians, but as decision making in any social sense, including a discussion on how to deal with anthropogenic climate change with friends and acquaintances in a local pub. Those who think that mathematics is immune to post-truth attitudes might be convinced otherwise by Maddox’ (2017) short film and alternative-truth parody *Alternative Math* or by the Twitter and blog discussions unleashed and documented by Lindsay (2020), both

unfolding around controversy whether  $2+2=4$ .<sup>4</sup> In any case, the unjustified use of numbers and scepticism towards the meaning of numbers have already entered the political stage, as documented by Hauge, Kacerja and Lilland (2019).

Irrespective of whether post-truth politics enters the mathematics classroom or not, mathematics education will have to prepare adolescents to understand the nature, the potential and the limitations of forms of reasoning which are associated with mathematics. Accordingly, Hauge (2019) called for explicit reflections on ‘fake news’, ‘source criticism’, ‘mathematical models’, ‘mathematics based argumentation’ and ‘argumentation structures’ more generally in the mathematics classroom (pp. 492–493). However, as I pointed out elsewhere, her attempts, and that of other colleagues publishing on the issue, are misleading as long as it restricts itself to these reflections (Kollosche, 2021): As MacMullen (2020) pointed out, post-truth attitudes are often nurtured by the assumptions that all knowledge is necessarily constructed and interest-driven and that the validity of specific assertions cannot be established or checked by the individual anyway. Obviously, such attitudes are created and strengthened by interventions that focus on source criticism and on questioning mathematical models.

In his very detailed study, McIntyre (2018) could show that post-truth politics were inspired by relativist epistemologies and build on their problems to explain why scientific knowledge should be epistemically privileged against other forms of discourse. This underlines that implications from relativist epistemologies have already entered public discourse and therefore cannot be ignored. Consequently, Zhao (2020) proposed ‘to rethink math and science education not only for global competition and monetary gain, but as part of the *basic epistemic education* where students learn to seek and process knowledge as part of their way of existing in the world’ (p. 400, emphasis added). As the discussion of post-truth politics shows, this goal of mathematics education should not be ignorant of relativist positions in epistemology, but explain how justified knowledge is possible if it cannot refer to any mind-independent reality.

## 5 Styles of reasoning and mathematics

The previous sections have documented the need for a theory of mathematical reasoning which follows a relativist epistemology, which is suited to explain how assertions are justified with the help of mathematics, and which provides some explanation why some forms of reasoning are labelled ‘scientific’ while others are not. Such a theory would be especially interesting for mathematics education research if it could structure and explain forms of mathematical reasoning, including answers to specialised questions such as the ones we ended up with in the discussion of proof and modelling.

Hacking’s framework of *styles of reasoning* has the potential to be such a theory. A journal article does not provide the space to fully develop the framework, nor does it allow to deeply elaborate its implications for any field of mathematics education research, but it may serve to outline the framework’s characteristics and potential and to motivate specialised studies on its implications for mathematics education research.

### 5.1 Introducing styles of reasoning

So, how can we explain how we come to justified knowledge if we cannot assume that such justification can be evaluated by comparison with an existence independent of humans? Hacking’s (1982) framework of *styles of reasoning* reacted to this question and ‘has become widespread in the literature of the history and philosophy of science’ (Sciortino, 2017, p. 243). Hacking (1992a) explained that a style of reasoning comprises

- the identification of a class of propositions as candidates for truth-or-falsehood,
- methods for the justification of such propositions,

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<sup>4</sup> I discuss both episodes in a conference paper in the Topic Study Group on ‘Philosophy of Mathematics and Mathematics Education’ at the 2021 International Congress on Mathematical Education (ICME).

- the creation of objects of scientific study for which such propositions hold,
- the formulation of scientific laws of a particular kind, and
- the facilitation of new possibilities for understanding.

Hacking's idea of styles of reasoning is an epistemological reinterpretation of the six styles that were described in enormous detail on more than 2400 pages by science historian Crombie (1994). But, while Crombie was only interested in describing how these styles came into being and developed, Hacking turned to the question how these styles allowed for reasoning. I provide a quick characterisation of the styles based on my reading of Crombie (1994) and Hacking (1982, 1990, 1992a, 2002, 2012, 2014):

- (a) The *postulation style of reasoning* creates deductively structured theories which are based on presupposed objects with presupposed properties, and in which the justification of an assertion must be deduced. Not only does it proceed from as uncontroversial assumptions as possible, it also adjusts its assumptions in light of the acceptability of the inferences that can be obtained on their basis. Its objects are defined by their suitability for a deductive theory. Mathematical theories are often referred to as ideal examples of this style, but there are also other theories building on postulation.
- (b) The *experimental style of reasoning* defines its objects by their observable and measurable characteristics. Through induction, such observation and measurement then allow for justifying assertions about these objects. Although experimentation is more commonly associated with the natural sciences, there are experimental approaches to mathematics, and mathematics plays a crucial part in measurement.
- (c) The *modelling style of reasoning* defines its objects through analogies to objects of other theories. In consequence, legitimate assertions about the model object are taken to be legitimate assertions about the modelled object. In mathematical modelling, this style relies on models taken from mathematics, but mathematics itself works with models for mathematical objects, for example with visual models such as Venn diagrams.
- (d) The *taxonomic style of reasoning* defines its objects by their positions in classifications. Jointly classified objects are considered to share properties, and general assertions about classes are taken to apply for the classified objects as well. Typical examples are Ancient Greek medicine and the Modern taxonomy of living organisms.
- (e) The *statistical style of reasoning* defines its objects by the regularities in their randomly appearing properties. It allows assertions about chances and justifies them through the processing of experimental data and methods of calculation. Thus, statistics is often considered a part of mathematics. Hacking (1990) presented a detailed study conceptualising statistics as a style of reasoning.
- (f) The *genetic style of reasoning* defines its objects as developmental. Here, legitimate assertions about the past properties of such developments are extended to legitimate assertions about the present or future properties of such developments. Mathematics seems to have little to do with this style, probably because mathematics idealises its objects as time-independent.

Imagine how we might justify the assertion that a specific virus is dangerous for humans. In style (a), we might use the characteristics of the virus to prove that it has the pre-defined property of being dangerous. In style (b), observation and measurement might provide data to decide on the dangerousness of the virus. In style (c), we might find an analogy for how the virus works, for example by comparing it to a locust infestation, to establish its dangerousness. In style (d), we might give reasons why our virus falls into the same category as other dangerous viruses. In style (e), we might study a population of infected people and judge that it is very likely that the virus is dangerous. In style (f), an earlier spread of the virus in a certain area might be taken as paradigmatic for the spread of the virus elsewhere. Each of these justifications employs a different style of reasoning, uses a different understanding of what it means for a virus to be dangerous, includes different methods to determine whether or not a virus is dangerous, and thus broadens our understanding of the dangerousness of viruses.



Crombie's (1994) list might not be perfect. At least, it is utterly Eurocentric (and admittedly so, as the title of Crombie's three-volume book is *Styles of scientific thinking in the European tradition*). As Kusch (2010) pointed out in his critique of both Crombie's and Hacking's approaches, it is at least strange that Hacking uncritically adopted Crombie's controversially discussed list of scientific styles of thinking. My guess is that Hacking was less concerned with the composition of the list than with the question of how styles allow for reasoning. At least, Hacking (1992a) admitted that it might be reasonable to add, differentiate, connect, merge or question styles. There are styles that came and disappeared such as the episteme of resemblance, which was studied by Foucault (1966). Hacking (1992b) described how styles (b) and (c) enter into a synthesis which he called the laboratory style. Hacking (1992a) also proposed to add an 'algorismic style', which he characterised as 'little interested in postulation but dedicated to finding algorithms' (p. 8).

The framework of styles of reasoning allows us to tackle some of the fundamental problems outlined earlier: It qualifies as a relativist epistemology as it assumes that the truth-aptness of a statement, the logic of argumentation, and the ontic nature of the objects that we are reasoning about is *relative* to a style of reasoning. Note that this does not imply that each and every assertion could be justified by an appropriate style. In our virus example, the final decision whether or not we call our virus dangerous depends not only on the applied style of reasoning but on aspects external to the styles. The question why certain styles should be accepted as scientific but other imaginable styles should not, finds a first answer in the sense that they have proven successful in a historical perspective. Whether this success is a mere coincidence or well-founded is another question, which is still open for debate.

Even for the question why different styles lead to similar insights, explanations have been offered: Ruphy (2011) talked about 'ontological enrichment' (p. 1212) to describe the process in which one concept is reconceptualised differently within different styles of reasoning (just like the dangerousness of a virus in the above example). In a relativist paradigm it does not make sense to argue that these conceptualisations refer to a pre-existing 'same'. Instead, we use them as if they were the same. Thereby, we gain what Ruphy (2011) called a 'foliated pluralism' of understanding. For the sake of this pluralism, we might actively choose reconceptualisations which reproduce chosen insights from other styles of reasoning and discard reconceptualisations whose insights are not sufficiently compatible to the insights from other styles of reasoning – at least as long they do not open new ways of understanding that one does not want to give up.

## 5.2 Styles of reasoning in mathematics education research

Styles of reasoning offer a framework to conceptualise mathematical reasoning from a relativist perspective. This could allow to locate and define the different forms of reasoning discussed in mathematics education research within a holistic theory. For example, 'diagrammatic reasoning' could be described as employing the modelling style of reasoning so that mathematics is modelled by diagrams, while 'multiplicative reasoning' would appear to be a special case of what Hacking (1992a) called the algorismic style of reasoning. A thorough discussion of more problematic cases, for example concerning how to accommodate 'functional reasoning' in the framework, may be informative both for the discussion on the specific form of mathematical reasoning and on the potential and limits of the framework itself.

From the perspective of the framework of styles of reasoning, probability and statistics education is an interesting field of mathematics education research (Sriraman & Chernoff, 2020): This field is characterised by an outstanding awareness of the particularities of the forms of reasoning it studies. The identified problems of learners and even of experts to use these forms of reasoning document impressively that (at least some) styles of reasoning do not come naturally but are a cultural product whose use has to be learnt and negotiated. This perspective led to a psychologically inspired interest in how people make sense of probability and statistics in reasoning activities. Ben-Zvi's (2020) postulation that '[t]he goal of teaching statistics is to produce statistically educated students who develop statistical literacy and the ability to reason statistically', where the latter 'is the way people reason with the "big statistical ideas" and make sense of statistical information during a data-based activity' (p. 178) or Gal's (2002) proposal that '[a]dults should develop a positive view of themselves

as capable of statistical and probabilistic reasoning as well as a willingness and interest to “think statistically” in relevant situations’ (p. 19) show that ‘learning to reason’ is already a central part of the self-understanding of this research field. However, as Sriraman & Chernoff (2020) acknowledged in their *Encyclopedia* entry on ‘Probabilistic and Statistical Thinking’, the perspective of the field is predominantly psychological and not epistemological. Thus, the question how probabilistic and statistical reasoning relates to other forms of reasoning, how it justifies assertions, and how it contributes to our understanding of the world, are not yet part of the academic reflections of the field.

For the case of mathematical proof, the postulation style of reasoning unfolds an understanding which appears to be compatible to the arguments brought forward by Jahnke (2007). In particular, neither Crombie nor Hacking included logic, especially deduction, as a style of reasoning. Hacking (1982) reminded us that logic ‘merely preserves’ truth, allows ‘jumping from truth to truth’, it does not ‘create the possibility for truth and falsehood’ and therefore does not qualify as a style of reasoning (p. 57). Just as described by Jahnke (2007), the postulation style of reasoning requires meaning beyond itself to decide which assertions it should be able to justify and which assumptions to begin with. This meaning might enter postulational reasoning by ontological enrichment as in the case described by Hilbert (1930), where diagrammatic insights were expected to be reproduced by an axiomatic theory of geometry. An open question then is which objects from other theories were used for ontological enrichment through mathematics. This perspective can help to explain in which sense mathematics has meaning, which might be especially important for approaching mathematics in teaching and learning activities. This perspective also allows to explain how axiomatic theories have meaning in some respect and, at the same time, can then be approached in a game-like fashion.

For the case of mathematical modelling, the modelling style of reasoning allows us to understand mathematical theories as models in reasoning processes. Thereby, mathematical models add to the foliated pluralism of our understanding by an ontological enrichment, which reinterprets the phenomena of study as mathematical objects. This perspective explains in how far mathematical modelling operates within mathematics and is still engaged in our construction of reality. As models only offer analogies for a *selection* of the modelled objects of a theory, mathematical models necessarily format our understanding in a different way than the original theory and impose a mathematical structure on the specific subject matter. Although there are some studies which address the way in which reasoning with mathematics formats our understanding (e.g., Porter, 1996), the question for the potential and limitations of the use of specific mathematical theories for modelling is only seldomly discussed in mathematics education.

The styles-of-reasoning framework is also helpful to explain how justified knowledge is possible in a relativist epistemology: Scientifically justified knowledge is gained through specific forms of reasoning. These forms have evolved throughout history, have gained wide acceptance in science, and have added profoundly to human understanding. No style offers an ultimate judgement about the validity of any assertion and the choice of styles might depend on one’s presumptions and interests, but that does not imply that scientific reasoning can be twisted to justify any claim. From the perspective of mathematics education, it is especially interesting that many styles of reasoning are closely related to mathematics. I will have to leave it open for discussion from the perspective of educational theory in how far this suggests that the mathematics classroom might be well suited or even a necessary place to reflect on specific forms of scientific reasoning.

## 6 Conclusion

This essay began by noticing the central role of the concept of reasoning in mathematics education research and the absence of a coherent theory of mathematical reasoning. On this basis, it argued for the need of a coherent theory to describe reasoning, including forms of reasoning in and with mathematics. In relation to specific trends in the philosophy of mathematics and in mathematics education research and answering to epistemological needs in special fields of mathematics education research, it was argued that the required theory should assume a relativist position in epistemology.

Hacking's framework of styles of reasoning was then presented as a possible solution to the outlined requirements, and selected implications of the framework for mathematics education research were outlined.

Although more questions may have been raised than answered, the introduction of the framework of styles of reasoning has already been useful to get a relativist description of the discursive practices with which assertions gain legitimacy, especially in and with mathematics. Styles of reasoning allowed to rearticulate concerns in various fields of mathematics education research within a consistent theoretical framework, to legitimise these concerns on a general epistemological basis, and to show up ways of understanding that may allow us to address these concerns in accordance with relativist theories of understanding. Thereby, it would seem especially promising that future research could resort to the rich repertoire of insights and episodes from the history and philosophy of science and mathematics in order to better understand how mathematics is involved in human reasoning, how reasoning in and with mathematics can be understood within mathematics education research, and how it can be addressed in the mathematics classroom.

Considering the styles listed by Crombie (1994) and Hacking (1982), it is striking how many forms of reasoning are closely linked to mathematics. This raises the question in how far mathematics education research has paid appropriate attention to the forms of reasoning which are described in the respective styles. Just consider the large dominance of reflections on deductive inference, which could be argued to play a marginal role in many mathematics classrooms, in contrast to the low number of studies on experimental approaches in mathematics education and in the discipline itself (Jinyuan-Li, 2003; Philipp, 2013). I content that the framework introduced here has the potential to inspire and guide future research on reasoning in and with mathematics.

Vice versa, mathematics education research might also have something to offer for the philosophy of science and mathematics and for epistemology in general. In their reflections on research on probabilistic and statistical thinking, Sriraman & Chernoff (2020) proposed that 'a field interested in the teaching and learning of probability and statistics and probabilistic and statistical thinking, such as the fields of probability and statistics education, is uniquely positioned to continue to investigate probabilistic and statistical thinking' (p. 676). I would agree and argue that the same can be expected from mathematics education research in general for insights on a broader range of styles of reasoning.

## 7 References

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