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HEADWAY IN LARGE-EDDY-SIMULATION WITHIN THE SPH MODELS

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Abstract. In the present paper we show some preliminary results of a novel LES-SPH scheme that extends and generalizes the approach described in [2]. Differently from that work, the proposed scheme is based on the definition of a Quasi-Lagrangian Large-Eddy-Simulation model where a small velocity deviation is added to the actual fluid velocity. When the LES equations are rearranged in the SPH framework, the velocity deviation is modelled through the Particle Shifting Technique (PST), similarly to the δplus-SPH scheme derived in [3]. The use of the PST allows for regular particle distributions, reducing the numerical errors in the evaluation of the spatial differential operators. As a preliminary study of the proposed model, we consider the evolution of freely decaying turbulence in 2D. In particular we show that the present scheme predicts the correct tendencies for the direct and inverse energy cascades.

1 INTRODUCTION

In the last years an increasing number of papers have been dedicated to the extension of the Smoothed Particle Hydrodynamics (SPH) to model problems characterized by turbulent flows [13, 17, 18, 16]. Among the different approaches, the Large-Eddy-Simulation appears as the most suited method to be included in the SPH framework, being based on a filtering of the Navier-Stokes equations that resembles that adopted for the derivation of the smoothed differential operators of the SPH. The present work follows the above-mentioned line of research and extends and generalizes the approach described in [2] where a Lagrangian LES-SPH scheme was proposed. Specifically, the proposed scheme is based on the definition of a Quasi-Lagrangian Large-Eddy-Simulation model where a small velocity deviation is added to the actual fluid velocity. This approach allows us to cast the LES in the framework of the most advanced SPH schemes which rely on a quasi-Lagrangian motion of the fluid particles [19, 20]. In particular, when the LES equations are rearranged in the SPH formalism, the velocity deviation is modelled through the Particle Shifting Technique (PST), similarly to the δplus-SPH scheme derived in [3]. The latter technique proved to be a crucial numerical tool to obtain regular particle distributions, reducing the numerical errors in the evaluation of the spatial differential operators.

The presence of the velocity deviation with respect to the actual fluid velocity leads to the appearance of additional terms in the continuity and momentum equations of the proposed LES-SPH scheme that need proper turbulence closures. Specifically, the term in the momentum equation is represented through a classical LES closure while that in the continuity equation is modelled through the diffusive term of the δ-SPH scheme (see, for example, [1]).

The paper is organized as follows: section §2 briefly introduces the theoretical model and corresponding the numerical scheme and section §3 shows the preliminary results obtained for the evolution of freely decaying turbulence in 2D.

2 QUASI-LAGRANGIAN δLES-SPH

Let us consider the Navier-Stokes equations for a barotropic weakly-compressible Newtonian fluid:

$$\begin{cases} \rho_t \,+\, \nabla\cdot(\rho\,\boldsymbol{u}) \,=\, 0\,, \\ \boldsymbol{u}_t \,+\, (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} \,=\, -\,\dfrac{\nabla p}{\rho} \,+\, \nu\Delta\boldsymbol{u} \,+\, (\lambda' + \nu)\,\nabla\,(\nabla\cdot\boldsymbol{u})\,, \end{cases} \tag{1}$$

where $\boldsymbol{u}$ is the flow velocity, p and ρ denote the pressure and density fields respectively and are related through a state equation, namely $p = F(\rho)$. The hypothesis that the fluid is weakly-compressible corresponds to assume:

$$\frac{dp}{d\rho} \,=\, c^2 \,\gg\, \max\left(\,\|\boldsymbol{u}\|^2\,,\frac{\delta p}{\rho}\,\right)\,, \tag{2}$$

where δp indicates the variation of the pressure field and $c = c(\rho)$ is the sound speed (see *e.g.* [5, 4]). The viscosity coefficients ν, λ' indicate the ratios between the Lamé constants μ, λ and the density ρ. Since the fluid is weakly-compressible, ν and λ' are assumed constant. Now, let us define a generic filter in $\mathbb{R}^3 \times \mathbb{R}$ as follows:

$$\phi = \phi\,(\,\tilde{\boldsymbol{x}}_p(t) - \boldsymbol{y}, t - \tau\,)\,. \tag{3}$$

The above filter is supposed to have a compact support, to depend only on $\|\tilde{\boldsymbol{x}}_p(t) - \boldsymbol{y}\|$ and $|t-\tau|$, and to be an even function with respect to both arguments. Here $\tilde{\boldsymbol{x}}_p(t)$ indicates the position of a *quasi-lagrangian* point that moves in the fluid domain according to the following equation:

$$\boxed{\frac{d\tilde{\boldsymbol{x}}_p}{dt} = \tilde{\boldsymbol{u}}(\,\tilde{\boldsymbol{x}}_p(t), t\,) + \delta\tilde{\boldsymbol{u}}(\,\tilde{\boldsymbol{x}}_p(t), t\,)\,,} \tag{4}$$

where $\delta\tilde{\boldsymbol{u}}$ is a (small) arbitrary velocity deviation (to be specified later) while $\tilde{\boldsymbol{u}}$ is given by the following definition:

$$\boxed{\tilde{\boldsymbol{u}}(\,\tilde{\boldsymbol{x}}_p(t), t\,) = \int_{\mathbb{R}^3}\int_{-\infty}^{+\infty} \phi\,(\tilde{\boldsymbol{x}}_p(t) - \boldsymbol{y}, t-\tau)\,\boldsymbol{u}(\boldsymbol{y},\tau)\,d\tau\,dV_y\,.} \tag{5}$$

Hereinafter, we refer to $\tilde{\boldsymbol{x}}_p$ and $\tilde{\boldsymbol{u}}$ as the filtered position and velocity, respectively. Accordingly, the main idea is to rewrite the system (2) in terms of the filtered quantities and obtain a quasi-Lagrangian LES scheme. With respect to this point, we observe that, since the state equation is generally nonlinear, $\widetilde{F(\rho)}$ is different from $F(\tilde{\rho})$ and, consequently, the filtering procedure cannot be applied to both pressure and density. To avoid inconsistency, when we refer to *filtered* pressure we mean $\tilde{p} = F(\tilde{\rho})$. Under this hypothesis, we apply the filter in (3) to the Navier-Stokes equations for weakly-compressible flows and, integrating over $\mathbb{R}^3 \times \mathbb{R}$, we obtain:

$$\begin{cases} \dfrac{d\tilde{\rho}}{dt} = -\,\tilde{\rho}\,\nabla\cdot(\tilde{\boldsymbol{u}} + \delta\tilde{\boldsymbol{u}}) + \nabla\cdot(\,\tilde{\rho}\tilde{\boldsymbol{u}} - \widetilde{\rho\boldsymbol{u}}\,) + \nabla\cdot(\,\tilde{\rho}\,\delta\tilde{\boldsymbol{u}}\,)\,, \\[2ex] \dfrac{d\tilde{\boldsymbol{u}}}{dt} = -\,\dfrac{\nabla\tilde{p}}{\tilde{\rho}} + \nu\Delta\tilde{\boldsymbol{u}} + (\lambda' + \nu)\,\nabla(\nabla\cdot\tilde{\boldsymbol{u}}) - \nabla\left[\widetilde{G(\rho)} - G(\tilde{\rho})\right] + \nabla\cdot\mathbb{T}_\ell \\[2ex] \qquad\quad + \widetilde{\boldsymbol{u}\,\nabla\cdot\boldsymbol{u}} + \nabla\cdot(\,\tilde{\boldsymbol{u}}\otimes\delta\tilde{\boldsymbol{u}}\,) - \tilde{\boldsymbol{u}}\,(\nabla\cdot\delta\tilde{\boldsymbol{u}})\,, \\[2ex] \dfrac{d\tilde{\boldsymbol{x}}_p}{dt} = \tilde{\boldsymbol{u}} + \delta\tilde{\boldsymbol{u}}\,, \qquad \tilde{p} = F(\tilde{\rho})\,, \qquad G(\rho) = \displaystyle\int^{\rho}\frac{1}{s}\frac{dF}{ds}\,ds\,, \end{cases} \tag{6}$$

where the total time derivatives d/dt is done with respect to the velocity $\tilde{\boldsymbol{u}} + \delta\tilde{\boldsymbol{u}}$ and $\mathbb{T}_\ell = \tilde{\boldsymbol{u}}\otimes\tilde{\boldsymbol{u}} - \widetilde{\boldsymbol{u}\otimes\boldsymbol{u}}$. Note that the latter tensor is equivalent to the sub-grid stress tensor. Following the derivation shown in [2], we now rearrange the system (6) in the framework of the Smoothed Particle Hydrodynamics (SPH) scheme. To this purpose, we split the filter ϕ into

$$\phi\,(\,\tilde{\boldsymbol{x}}_p(t) - \boldsymbol{y}, t-\tau\,) = W(\,\tilde{\boldsymbol{x}}_p(t) - \boldsymbol{y}\,)\;\theta(\,t-\tau\,)\,, \tag{7}$$

where W indicates the SPH kernel and denote the spatial and time filtering as below:

$$\langle f\rangle(\,\tilde{\boldsymbol{x}}_p(t), t\,) = \int_{\mathbb{R}^3} W(\,\tilde{\boldsymbol{x}}_p(t) - \boldsymbol{y}\,)\; f(\boldsymbol{y}, t)\,dV_y \qquad \overline{f}\,(\boldsymbol{y}, t) = \int_{\mathbb{R}} \theta(\,t-\tau\,)\; f(\boldsymbol{y},\tau)\,d\tau\,.$$

Using the above definitions it is easy to prove that $\tilde{f} = \langle\,\overline{f}\,\rangle$ and that, generally, the time and space filters do not commute, i.e. $\langle\,\overline{f}\,\rangle \neq \overline{\langle f\rangle}$ (see, for example, [2]). Since the time filter is the inner one, the overall LES-SPH scheme may be regarded as a spatial Lagrangian filter applied to a set of time-averaged variables. In this sense, the time filter may be thought as an implicit filter whose presence is accounted for through the modeling of the additional terms.

Now, suppose that we want to model a high Reynolds number flow, for which LES filtering is required. Then, we need the filtered variables $\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\rho}$ for each fluid particle at positions $\tilde{\boldsymbol{x}}_p$; at the same time, we want to approximate the operators in equation (6) in the SPH fashion. For example, using the above definitions, we can rearrange the gradient of $\tilde{f}$ as follows:

$$\nabla \tilde{f} = \langle \nabla \cdot \overline{f} \rangle = \langle \nabla(\overline{f} + \tilde{f} - \tilde{f}) \rangle = \langle \nabla \tilde{f} \rangle + \langle \nabla(\overline{f} - \tilde{f}) \rangle \,, \tag{8}$$

where the first term in the right-hand side is the SPH operator while the latter term accounts for small scale "fluctuations" in space, hereinafter denoted through $f' = \overline{f} - \tilde{f}$. For confined flows, the non-commutability of filtering and differentiation must be taken into account for a rigorous extension of the filtering close to the boundaries (i.e. in those points whose distance from the boundaries is smaller than the kernel radius). The above procedure can be applied to all the remaining operators. By doing so, in SPH formalism the system (6) reads:

$$\begin{cases} \dfrac{d\tilde{\rho}}{dt} = -\,\tilde{\rho}\,\langle \nabla \cdot (\tilde{\boldsymbol{u}} + \delta\tilde{\boldsymbol{u}}) \rangle \,+\, \langle \nabla \cdot (\tilde{\rho}\,\delta\tilde{\boldsymbol{u}}) \rangle \,+\, \mathcal{C}_1 \,+\, \mathcal{C}_2 \,+\, \mathcal{C}_3 \,, \\[2ex] \dfrac{d\tilde{\boldsymbol{u}}}{dt} = -\,\dfrac{\langle \nabla \tilde{p} \rangle}{\tilde{\rho}} \,+\, \nu \langle \Delta \tilde{\boldsymbol{u}} \rangle \,+\, (\lambda' + \nu)\,\langle \nabla\,(\nabla \cdot \tilde{\boldsymbol{u}}) \rangle \\[2ex] \qquad\quad + \langle \nabla \cdot (\tilde{\boldsymbol{u}} \otimes \delta\tilde{\boldsymbol{u}}) \rangle \,-\, \tilde{\boldsymbol{u}}\,\langle \nabla \cdot \delta\tilde{\boldsymbol{u}} \rangle \,+\, \mathcal{M}_1 \,+\, \mathcal{M}_2 \,+\, \mathcal{M}_3 \,, \\[2ex] \dfrac{d\tilde{\boldsymbol{x}}_p}{dt} = \tilde{\boldsymbol{u}} \,+\, \delta\tilde{\boldsymbol{u}} \,, \qquad \tilde{p} \,=\, F\,(\tilde{\rho}) \,, \end{cases} \tag{9}$$

where:

$$\mathcal{C}_1 \;=\; -\,\tilde{\rho}\,\langle \nabla \!\cdot\! \boldsymbol{u}' \rangle \,, \qquad \mathcal{C}_2 \,=\, \nabla \cdot (\,\tilde{\rho}\tilde{\boldsymbol{u}} - \widetilde{\rho\boldsymbol{u}}\,) \,, \tag{10}$$

$$\mathcal{C}_3 \;=\; -\,\tilde{\rho}\;\nabla \!\cdot (\,\delta\tilde{\boldsymbol{u}} - \langle \delta\tilde{\boldsymbol{u}} \rangle) \,+\, \nabla \cdot (\,\tilde{\rho}\,\delta\tilde{\boldsymbol{u}} - \langle \tilde{\rho}\,\delta\tilde{\boldsymbol{u}} \rangle\,) \,, \tag{11}$$

$$\mathcal{M}_1 \;=\; -\,\frac{\langle\, \nabla p' \,\rangle}{\tilde{\rho}} \,+\, \nu\,\langle\, \Delta \boldsymbol{u}' \,\rangle \,+\, (\lambda' + \nu)\,\langle\, \nabla\,(\nabla \!\cdot\! \boldsymbol{u}')\,\rangle \,, \tag{12}$$

$$\mathcal{M}_2 \;=\; -\,\nabla \left[\, \widetilde{G(\rho)} \,-\, G(\tilde{\rho}) \,\right] \,+\, \widetilde{\boldsymbol{u}\,\nabla \!\cdot\! \boldsymbol{u}} \,+\, \nabla \cdot \mathbb{T}_\ell \tag{13}$$

$$\mathcal{M}_3 \;=\; \nabla \cdot (\,\tilde{\boldsymbol{u}} \otimes \delta\tilde{\boldsymbol{u}} - \langle \tilde{\boldsymbol{u}} \otimes \delta\tilde{\boldsymbol{u}} \rangle\,) \,-\, \tilde{\boldsymbol{u}}\,\nabla \cdot (\,\delta\tilde{\boldsymbol{u}} - \langle \delta\tilde{\boldsymbol{u}} \rangle\,) \,. \tag{14}$$

Here $\mathcal{C}_1$ and $\mathcal{M}_1$ come from the SPH approximation procedure and require a SPH closure, whereas $\mathcal{C}_2$ and $\mathcal{M}_2$ include all terms from the Lagrangian LES and require a LES closure.

Finally, $\mathcal{C}_3$ and $\mathcal{M}_3$ come from the use of the generic deviation velocity $\delta\tilde{\boldsymbol{u}}$. Using a Taylor expansion and the hypothesis that the fluid is weakly-compressible, it is possible to show that both $\mathcal{C}_3$ and $\mathcal{M}_3$ are negligible while $\mathcal{C}_1$ and $\mathcal{M}_1$ play a minor role with respect to the terms $\mathcal{C}_2$ and $\mathcal{M}_2$. As a consequence, only $\mathcal{C}_2$ and $\mathcal{M}_2$ are retained and modelled following a LES closure approach. In particular, $\mathcal{M}_2$ is modelled as done in [2]:

$$\mathcal{M}_2 \simeq \nabla \cdot \left[-\frac{q^2}{3}\,\mathbb{1} - \frac{2}{3}\,\nu_T\,\mathrm{Tr}(\widetilde{\mathbb{D}})\,\mathbb{1} + 2\,\nu_T\,\widetilde{\mathbb{D}} \right] , \tag{15}$$

where q^2 represents the turbulent kinetic energy, ν_T is the turbulent kinetic viscosity and $\widetilde{\mathbb{D}}$ is the strain-rate tensor, that is $\widetilde{\mathbb{D}} = (\nabla\tilde{\boldsymbol{u}} + \nabla\tilde{\boldsymbol{u}}^T)/2$. Similarly to [12], we assume:

$$q^2 = 2\,C_Y\,\ell^2\,\|\,\widetilde{\mathbb{D}}\,\|^2 , \qquad \nu_T = (C_S\,\ell)^2\,\|\,\widetilde{\mathbb{D}}\,\| , \tag{16}$$

where $\|\widetilde{\mathbb{D}}\|$ is a rescaled Frobenius norm, namely $\|\,\widetilde{\mathbb{D}}\,\| = \sqrt{2\,\widetilde{\mathbb{D}}:\widetilde{\mathbb{D}}}$ and ℓ is the radius of the SPH spatial kernel. The dimensionless parameters C_Y and C_S are respectively called the Yoshizawa and Smagorinsky constants.

For what concerns $\mathcal{C}_2$, the closure proposed in [2] corresponds to $\mathcal{C}_2 = \nabla\cdot(\,\nu_\delta\,\nabla\tilde{\rho}\,)$ where ν_δ is assumed to be a function of $\widetilde{\mathbb{D}}$. In agreement with the usual approaches adopted in the LES framework, this is equivalent to model $\mathcal{C}_2$ as a diffusive term. In the present work we adopt a finer closure and write:

$$\mathcal{C}_2 = \ell^2\,\nabla\cdot(\,\nu_\delta\,\nabla\,\Delta\tilde{\rho}\,) . \tag{17}$$

In the SPH framework a simple way to model $\mathcal{C}_2$ is obtained by using the diffusive term proposed in [10], since this contains fourth-order spatial derivatives of the density field.

2.1 Numerical Scheme

To write the system (6) in the discrete formalism, we rely on the work of Sun et al. [3] where the additional $\delta\tilde{\boldsymbol{u}}$-terms are included in the SPH framework in a consistent way. In particular we obtain:

$$\begin{cases} \dfrac{d\tilde{\rho}_i}{dt} = -\,\tilde{\rho}_i \displaystyle\sum_j \left[(\tilde{\boldsymbol{u}}_j + \delta\tilde{\boldsymbol{u}}_j) - (\tilde{\boldsymbol{u}}_i + \delta\tilde{\boldsymbol{u}}_i)\right]\cdot\nabla_i W_{ij}\,V_j \,+ \\[2ex] \qquad \displaystyle\sum_j (\tilde{\rho}_j\delta\tilde{\boldsymbol{u}}_j + \tilde{\rho}_i\delta\tilde{\boldsymbol{u}}_i)\cdot\nabla_i W_{ij}\,V_j + \sum_j \delta_{ij}\,\boldsymbol{\psi}_{ji}\cdot\nabla_i W_{ij}\,V_j \\[2ex] \dfrac{d\tilde{\boldsymbol{u}}_i}{dt} = -\dfrac{1}{\tilde{\rho}_i}\displaystyle\sum_j (\tilde{p}_j + \tilde{p}_i)\nabla_i W_{ij}\,V_j + \dfrac{\rho_0}{\tilde{\rho}_i}\,K \sum_j \alpha_{ij}\,\pi_{ij}\nabla_i W_{ij}\,V_j \,+ \\[2ex] \qquad \dfrac{\rho_0}{\tilde{\rho}_i}\displaystyle\sum_j (\tilde{\boldsymbol{u}}_j\otimes\delta\tilde{\boldsymbol{u}}_j + \tilde{\boldsymbol{u}}_i\otimes\delta\tilde{\boldsymbol{u}}_i)\cdot\nabla_i W_{ij}\,V_j \\[2ex] \dfrac{d\tilde{\boldsymbol{x}}_i}{dt} = \tilde{\boldsymbol{u}}_i + \delta\tilde{\boldsymbol{u}}_i , \qquad \tilde{p}_i = F(\tilde{\rho}_i) , \qquad V_i = \dfrac{m_i}{\tilde{\rho}_i} , \end{cases} \tag{18}$$

where m_i is the i-th particle mass (assumed to be constant) and V_i is its volume while $W_{ij} = W(\tilde{\boldsymbol{x}}_i - \tilde{\boldsymbol{x}}_j)$. The coefficient of the viscous term is $K = 2(n+2)$ where n is the number of spatial dimensions while its arguments are:

$$\pi_{ij} \;=\; \frac{(\tilde{\boldsymbol{u}}_j - \tilde{\boldsymbol{u}}_i)\cdot(\tilde{\boldsymbol{x}}_j - \tilde{\boldsymbol{x}}_i)}{\|\tilde{\boldsymbol{x}}_j - \tilde{\boldsymbol{x}}_i\|^2} \qquad \alpha_{ij} \;=\; \frac{\mu}{\rho_0} + 2\,\frac{\nu_i^T\,\nu_j^T}{\nu_i^T + \nu_j^T}\,,$$

where $\nu_i^T = (C_s\,\ell)^2\,\|\widetilde{\mathbb{D}}_i\|$ and C_s is the Smagorinsky constant (set equal to 0.12). The first contribution in the expression of α_{ij} represents the actual fluid viscosity while the latter one is the LES closure for turbulence. Note that all contributions related to the fluid compressibility have been neglected, since they are negligible in comparison to the leading order stress tensor. The symbol ψ_{ij} is the argument of the diffusive term of [10], namely:

$$\boldsymbol{\psi}_{ij} \;=\; \left[(\rho_j - \rho_i) - \frac{1}{2}\left(\langle\nabla\rho\rangle_i^L + \langle\nabla\rho\rangle_j^L\right)\cdot(\tilde{\boldsymbol{x}}_j - \tilde{\boldsymbol{x}}_i)\right]\frac{(\tilde{\boldsymbol{x}}_j - \tilde{\boldsymbol{x}}_i)}{\|\tilde{\boldsymbol{x}}_j - \tilde{\boldsymbol{x}}_i\|^2}\,. \tag{19}$$

Here the superscript L indicates that the gradient is evaluated through the renormalized gradient formula [9] as follows:

$$\begin{cases} \langle\nabla\rho\rangle_i^L \;=\; \displaystyle\sum_j (\rho_j - \rho_i)\boldsymbol{L}_i\nabla_i W_{ij} V_j\,, \\[2ex] \boldsymbol{L}_i \;=\; \left[\displaystyle\sum_j (\tilde{\boldsymbol{x}}_j - \tilde{\boldsymbol{x}}_i)\otimes\nabla_i W_{ij} V_j\right]^{-1} \end{cases} \tag{20}$$

As done in [2] and in [11], the diffusive term is not multiplied by an external parameter but the latter is included directly inside the summation and modelled as a viscous-like coefficient following a standard LES approach. In particular we choose:

$$\delta_{ij} \;=\; 2\,\frac{\nu_i^\delta\,\nu_j^\delta}{\nu_i^\delta + \nu_j^\delta}\,, \qquad \text{where} \quad \nu_i^\delta \;=\; (C_\delta\,\ell)^2\,\|\widetilde{\mathbb{D}}_i\|\,. \tag{21}$$

in which C_δ is a dimensionless constant set equal to 1.5 . The value of this constant has been calibrated in [2] by simulating free decay turbulence in periodic domains in both 2D and 3D frameworks.

As often done for weakly-compressible fluids, the state equation is linearized around the reference density ρ_0, leading to $\tilde{p}_i = c_0^2\,(\tilde{\rho}_i - \rho_0)$ where $c_0 = c(\rho_0)$ is a numerical sound speed that satisfies the following requirement:

$$M_a \;=\; \frac{U_{ref}}{c_0} \;\le\; 0.1 \qquad \text{with} \quad U_{ref} \;=\; \max\left(U_{max}\,,\sqrt{\frac{\delta\tilde{p}_{max}}{\rho_0}}\right)\,. \tag{22}$$

Here U_{max} and $\delta\tilde{p}_{max}$ indicate the maximum fluid velocity and the maximum pressure variation expected during the simulation. The above inequality allows the density variations to be below 1% (see [5]).

Finally, following [3], the velocity deviation is defined as:

$$\delta\tilde{\boldsymbol{u}}_i = \min\left(\,\|\delta\hat{\boldsymbol{u}}_i\|\,,\,\frac{U_{\max}}{2}\right)\frac{\delta\hat{\boldsymbol{u}}_i}{\|\delta\hat{\boldsymbol{u}}_i\|} \tag{23}$$

where:

$$\delta\hat{\boldsymbol{u}}_i \;=\; -\,M_a\,\ell\,c_0\,\sum_j\left[\,1+R\left(\frac{W_{ij}}{W(\Delta x)}\right)^n\,\right]\nabla_i W_{ij} V_j\,. \tag{24}$$

and Δx is the initial mean particle distance. Here the constants R and n are set equal to 0.2 and 4 respectively. The expression in (23) has to be further corrected for particles close to the domain boundaries to avoid a non-physical particle motion. For further details we address the interested reader to [3].

3 APPLICATIONS

In the next section we consider the evolution of freely decaying turbulence in a two-dimensional bi-periodic squared domain. The Taylor-Green vortex solution (see [14]) is used to initialize the computations. In particular a path of 8×8 vortex cells is considered. The Reynolds number is defined as $Re = UL/\nu$ where L is the side of the squared domain (that is, $L = 8\,\ell$ where ℓ is the length of a vortex cell), U is the reference initial velocity and ν is the kinematic viscosity. The SPH particles are placed by using the Particle Packing Algorithm described in [15].

3.1 Freely decaying turbulence in 2D

Figure 1 shows the comparison between the energy spectra at $Re = 10,000$ obtained by using three different SPH models for turbulence, namely the LES model of [13], the δ-LES SPH scheme defined in [2] and the present scheme. The spectra have been computed by using a Moving Least Square interpolation with a Gaussian kernel with the same radius of that adopted in the SPH simulations. Four different spatial resolutions have been considered, that is $L/\Delta x = 150, 300, 600, 1200$ where Δx is the initial mean particle distance. In the figure the direct and inverse cascade trends have been plotted by green and orange dashed lines respectively and the wave number associated with the kernel radius has been indicated by the symbol k_R. Generally, all the SPH models correctly predict these trends but show a non-physical increase of the energy spectrum at wave numbers comparable with k_R as the resolution increases. This behaviour is common to all the SPH schemes considered here and seems to be an intrinsic characteristic of the SPH models themselves. In any case, the proposed model, thanks to the action of the Particle Shifting Technique, predicts a reduction of the energy at large wave numbers in comparison to the LES model of [13] and the δ-LES SPH scheme of [2]. This overall trend is confirmed in figure 2 where the same results are displayed for $Re = 1,000,000$.

To understand if the LES closures described in the equations (15) and (17) work properly, it is useful to analyse the evolution of the heat generated by $\mathcal{M}_2$ and $\mathcal{C}_2$ in comparison with the heat produced by the viscous fluid term $\nu\langle\Delta\tilde{\boldsymbol{u}}\rangle$. Hereinafter we

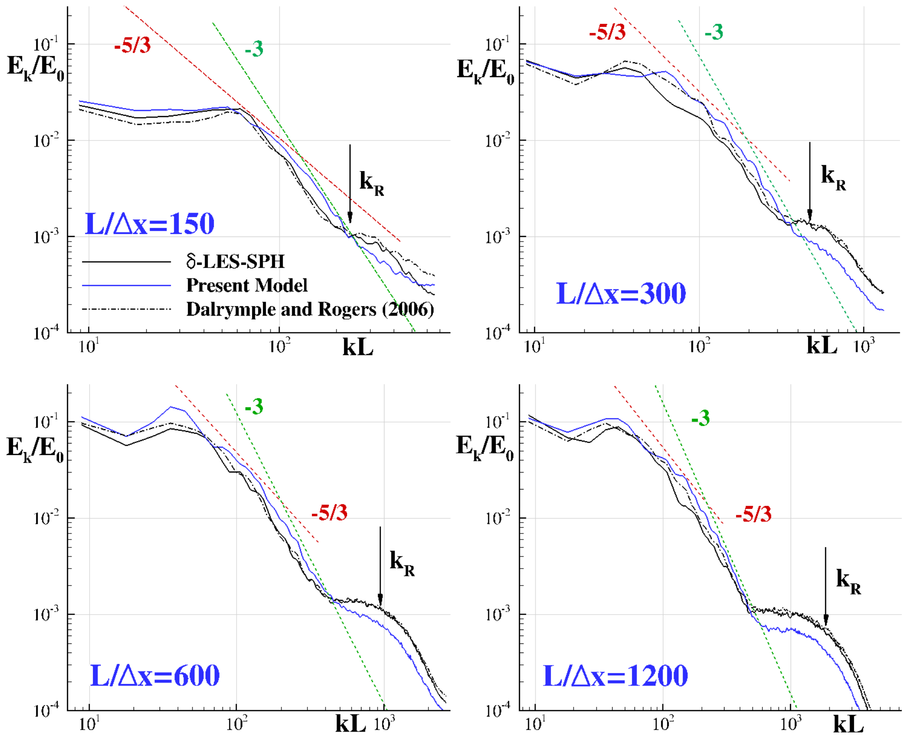

Figure 1: Freely decaying turbulence at $Re = 10,000$ for different spatial resolutions. The symbol k_R indicates the wave number associated to the kernel radius while Δx is the initial mean particle distance.

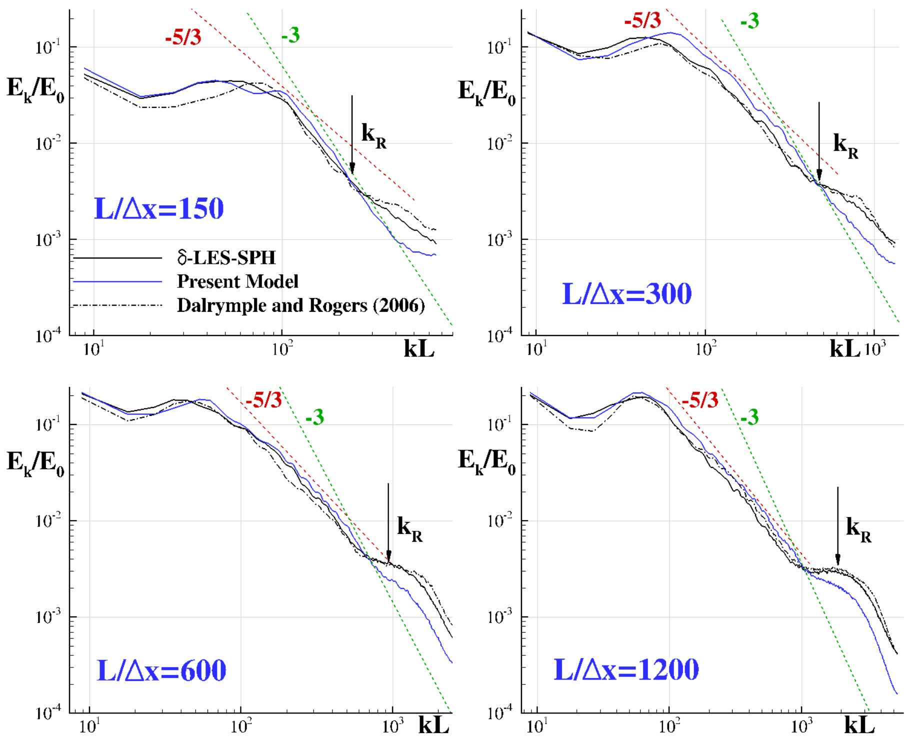

Figure 2: Freely decaying turbulence at $Re = 1,000,000$ for different spatial resolutions. The symbol k_R indicates the wave number associated to the kernel radius while Δx is the initial mean particle distance.

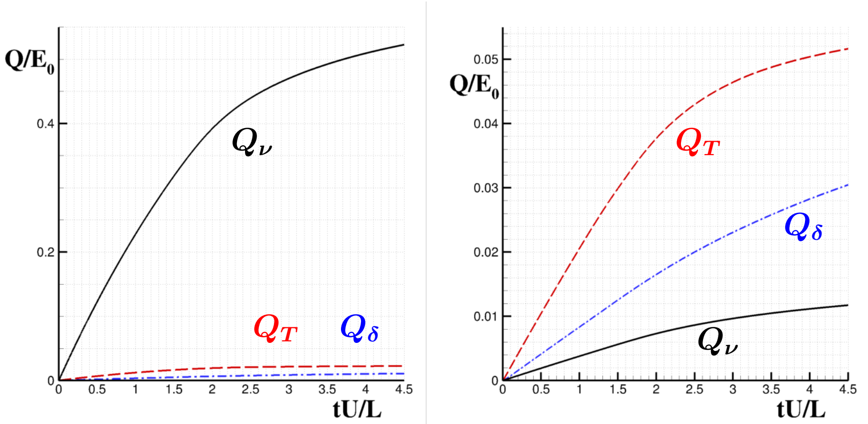

Figure 3: Freely decaying turbulence: the heat generated in the present model by the fluid viscous term (namely, Q_ν), by the LES model in the momentum equation (that is Q_T) and by the LES model in the continuity equation (i.e. Q_δ) for $Re = 10,000$ (left panel) and $Re = 1,000,000$ (right panel) for $L/\Delta x = 1,200$.

indicate these contributions through Q_T, Q_δ and Q_ν respectively. Figure (3) shows these terms for the finest spatial resolutions, namely $L/\Delta x = 1,200$, and for $Re = 10,000$ (left panel) and $Re = 1,000,000$ (right panel). In the former case, the simulation is very close to a DNS simulation and, consistently, the heat produced by the terms related to the LES closures (namely Q_T and Q_δ) is small in comparison to the dissipation caused by the actual viscosity of the fluid (i.e. Q_ν). On the contrary, for $Re = 1,000,000$, the spatial resolution is too coarse for a DNS simulation and, consequently, the LES terms give the main contribution to the heat production and make Q_T and Q_δ predominant over Q_ν.

Finally, in figure (4) we compare some snapshots of the pressure field for the three models under investigation, namely the LES model of [13] (top panels), the δ-LES SPH scheme defined in [2] (middle panels) and the present scheme (bottom panels) for the finest spatial resolution (that is $L/\Delta x = 1,200$). The left column displays the outputs obtained for $Re = 10,000$ while the right column contains the results for $Re = 1,000,000$. In both the cases, the pressure field predicted by the model described in [13] is much more noisy than those obtained though the LES model of [2] and by the present one. In particular, the latter are comparable even though the present model tends to preserve the larger eddies more efficiently, consistently with the energy spectra of figures (1) and (2).

4 CONCLUSIONS

In the present work we propose an extension of the LES model described in [2] using a quasi-Lagrangian LES formulation. This model is based on the assumption that the fluid particles move following a velocity that is made by the actual fluid velocity plus a small deviation. The latter is modelled through the Particle Shifting Technique, as described in [3].

The proposed model is preliminary tested by simulating the problem of freely decaying turbulence and comparing the results with the LES models described in [13] and [2]. The

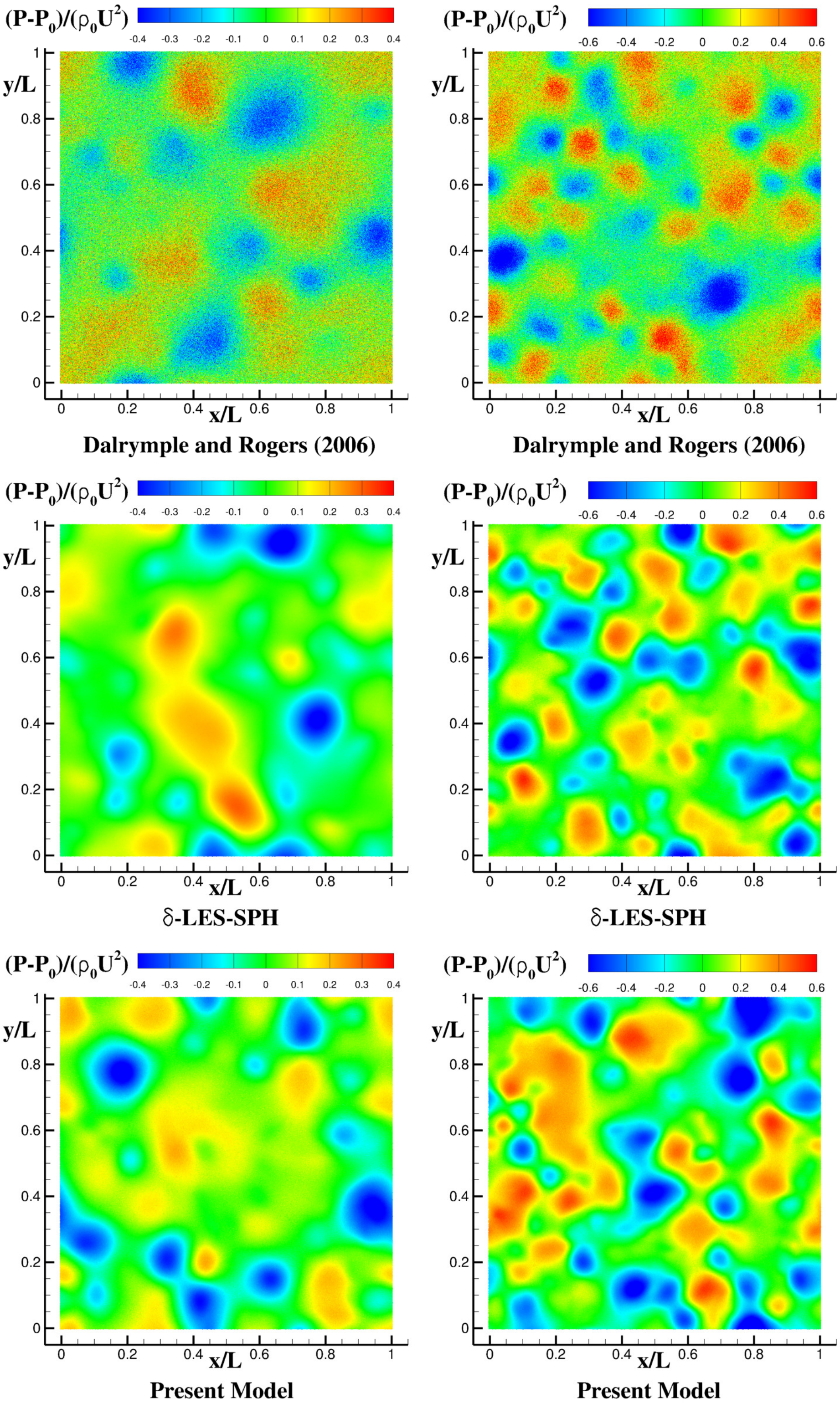

Figure 4: Freely decaying turbulence: pressure field predicted by the different SPH models at $Re = 10,000$ (left column) and $Re = 1,000,000$ (right column) for $L/\Delta x = 1,200$.

results show that the present model reduces the amount of energy at large wave numbers in comparison with the above models. Further, similarly to the scheme of [2], it predicts a pressure field that is free from high-frequency spurious noise.

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REFERENCES

[1] Antuono M., Colagrossi A., and Marrone S., Numerical diffusive terms in weakly-compressible SPH schemes, *Computer Physics Communications*, (2012) **183**(12):2570–2580.

[2] Di Mascio A., Antuono M., Colagrossi A., and Marrone S., Smoothed particle hydrodynamics method from a large eddy simulation perspective. *Physics of Fluids*, (2017) **29**(035102).

[3] Sun P. N., Colagrossi A., Marrone S., Antuono M., and Zhang A. M., A consistent approach to particle shifting in the δ-Plus-SPH model. *Computer Methods in Applied Mechanics and Engineering*, (2019) **348**:912–934.

[4] Marrone S., Colagrossi A., Di Mascio A., and Le Touzé D., Prediction of energy losses in water impacts using incompressible and weakly compressible models, *Journal of Fluids and Structures*, (2015) **54**:802–822.

[5] Monaghan J., Simulating Free Surface Flows with SPH, *J. Comp. Phys.*, (1994) **110**(2):39–406

[6] Colagrossi A., Antuono M., and Le Touzé D., Theoretical considerations on the free-surface role in the Smoothed-particle-hydrodynamics model, *Physical Review E*, (2009) **79**(5):056701.

[7] Colagrossi A., Antuono M., Souto-Iglesias A., and Le Touzé D., Theoretical analysis and numerical verification of the consistency of viscous smoothed-particle-hydrodynamics formulations in simulating free-surface flows, *Physical Review E*, (2011) **84**:026705.

[8] Moin P., Squires K., Cabot W., and Lee S., A dynamic subgrid-scale model for compressible turbulence and scalar transport, *Physics of Fluids A*, (1991) **3**(11):2746–2757.

[9] Antuono M., Colagrossi A., Marrone S., Molteni D., Free-surface flows solved by means of SPH schemes with numerical diffusive terms, *Comp. Phys. Comm.*, (2010) **181**:532–549.

[10] Antuono M., Colagrossi A., Marrone S., Numerical diffusive terms in weakly-compressible SPH schemes, *Comp. Phys. Comm.*, (2012) **183**:2570–2580.

[11] Meringolo D.D., Marrone, S. Colagrossi A., Liu Y., A dynamic δ-SPH model: How to get rid of diffusive parameter tuning, *Computers & Fluids* (2019) **179**:334–355.

[12] Yoshizawa A., Statistical theory for compressible turbulent shear flows, with the application to subgrid modeling, *Physics of Fluids*, (1986) **29**:2152.

[13] Dalrymple R. and Rogers B., Numerical modeling of water waves with the SPH method, *Coastal Eng.*,(2006) **53**(2-3): 141–147.

[14] Taylor, G. I. and Green, A. E., Mechanism of the Production of Small Eddies from Large Ones, *Proc. R. Soc. Lond. A*, (1937) **158**: 499–521.

[15] Colagrossi A., Bouscasse B., Antuono M., and Marrone S., Particle packing algorithm for SPH schemes, *Comput. Phys. Commun.* (2012) **183**: 1641–1653.

[16] Mayrhofer A., Laurence D., Rogers B., and Violeau D., DNS and LES of 3-D wall-bounded turbulence using smoothed particle hydrodynamics, *Comput. Fluids* (2015) **115**: 86–97.

[17] Shao S., Ji C., SPH computation of plunging waves using a 2-D sub-particle scale (SPS) turbulence model. *International Journal for numerical methods in fluids*, (2006) **51**(8): 913–936.

[18] Price D.J., Resolving high Reynolds numbers in smoothed particle hydrodynamics simulations of subsonic turbulence. *Monthly Notices of the Royal Astronomical Society: Letters*, (2012) **420**(1): L33–L37.

[19] Xu, R., Stansby P., Laurence D., Accuracy and stability in incompressible SPH (ISPH) based on the projection method and a new approach. *Journal of computational Physics*, (2009) **228**(18): 6703–6725.

[20] Nestor R.M., Basa M., Lastiwka M., Quinlan N.J., Extension of the finite volume particle method to viscous flow. *Journal of Computational Physics*, (2009) **228**(5): 1733–1749.