A strengthened form of the strong Goldbach conjecture

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Abstract. This paper disproves a strengthened form of the strong Goldbach conjecture. Based on this result, it then gives a proof of that statement. The paper thus constitutes an antinomy within ZFC.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Theorem. *Both SSGB and the negation ¬SSGB hold.*

Proof. We define the set S_g := { (pk, mk, qk) | k, m $\in \mathbb{N}$; p, q $\in \mathbb{P}_3$, p < q; m = (p + q) / 2 }.

SSGB is equivalent to saying that every integer x ≥ 4 is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers x ≥ 4 appear as m in a middle component mk of S_g.

There are two possibilities for S_g, exactly one of which must occur: Either there is an n $\in \mathbb{N}_4$ in addition to all the numbers m defined in S_g or there is not. The latter corresponds to SSGB and the former corresponds to the negation ¬SSGB.

The set S_g has the following property: The whole range of $\mathbb{N}_3$ can be expressed by the triple components of S_g, since every integer x ≥ 3 can be written as some pk with k = 1 when x is prime, as some pk with k ≠ 1 when x is composite and not a power of 2, or as (3 + 5)k / 2 when x is a power of 2; p $\in \mathbb{P}_3$, k $\in \mathbb{N}$.

We can split S_g into two complementary subsets: For any y $\in \mathbb{N}_3$, $S_g = S_g+(y) \cup S_g-(y)$, with

$S_g+(y)$ = { (pk', mk', qk') $\in S_g$ | $\exists$ k $\in \mathbb{N}$ pk' = yk $\vee$ mk' = yk $\vee$ qk' = yk } and

$S_g-(y)$ = { (pk', mk', qk') $\in S_g$ | $\forall$ k $\in \mathbb{N}$ pk' ≠ yk $\wedge$ mk' ≠ yk $\wedge$ qk' ≠ yk }.

In the case of ¬SSGB, there is at least one n ≥ 4 additional to all the m that are defined in S_g. The following steps work regardless of the choice of n if there is more than one n.

According to the above three types of expression by S_g triple components, for n we have

(C): $\forall$ k $\in \mathbb{N}$ $\exists$ (pk', mk', qk') $\in S_g$ nk = pk' $\vee$ nk = mk' = 4k'.

Let S_g+ be shorthand for S_g+(n) and let S_g- be shorthand for S_g-(n). Then, because of (C) and because n cannot be the arithmetic mean of a pair of odd primes not used in S_g, there is no other possibility than: ¬SSGB => $S_g = S_g+ \cup S_g-$.

$S_g+ \cup S_g-$ is independent of n, since for every n it equals S_g. On the other hand, $S_g+(y) \cup S_g-(y)$ equals S_g for every y by definition, whether or not we assume ¬SSGB. So, if a set S equals $S_g = S_g+ \cup S_g-$ in the case of ¬SSGB then S equals S_g. Therefore, we obtain

(NG): $\forall$ S (¬SSGB => S_g = S) => S_g = S,

which is equivalent to ¬SSGB, because it is true if ¬SSGB is true, and false if SSGB is true. So, ¬SSGB is proved.

Now, let us assume that ¬SSGB, i.e. (NG), holds. As we have seen above, the validity of (NG) is independent of the n given by ¬SSGB. Since the only difference between SSGB and ¬SSGB is the (non-) existence of n, (NG) still holds if we replace ¬SSGB by SSGB. Therefore, we obtain

(G): $\forall$ S (SSGB => S_g = S) => S_g = S,

which is equivalent to SSGB. So, we have shown ¬SSGB => SSGB, which proves SSGB.

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