A strengthened form of the strong Goldbach conjecture

Ralf Wüsthofen

Abstract. This paper disproves a strengthened form of the strong Goldbach conjecture. Based on this result, it then gives a proof of that statement. The paper thus constitutes an antinomy within ZFC.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Theorem. *Both SSGB and the negation ¬SSGB hold.*

Proof. We define the set S_g := { (pk, mk, qk) | k, m $\in \mathbb{N}$; p, q $\in \mathbb{P}_3$, p < q; m = (p + q) / 2 }.

SSGB is equivalent to saying that every integer x ≥ 4 is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers x ≥ 4 appear as m in a middle component mk of S_g.

There are two possibilities for S_g, exactly one of which must occur: Either there is an n $\in \mathbb{N}_4$ in addition to all the numbers m defined in S_g or there is not. The latter corresponds to SSGB and the former corresponds to the negation ¬SSGB. This implies

(A): Assuming the existence of n, i.e. assuming ¬SSGB, means that the numbers m defined in S_g do not take all integer values x ≥ 4. Not assuming the existence of n, i.e. not assuming ¬SSGB, means that the numbers m defined in S_g take all integer values x ≥ 4.

The set S_g has the following property: The whole range of $\mathbb{N}_3$ can be expressed by the triple components of S_g, since every integer x ≥ 3 can be written as some pk with k = 1 when x is prime, as some pk with k ≠ 1 when x is composite and not a power of 2, or as (3 + 5)k / 2 when x is a power of 2; p $\in \mathbb{P}_3$, k $\in \mathbb{N}$.

We can split S_g into two complementary subsets: For any y $\in \mathbb{N}_3$, $S_g = S_g+(y) \cup S_g-(y)$, with

$S_g+(y)$ = { (pk', mk', qk') $\in S_g$ | ∃ k $\in \mathbb{N}$ pk' = yk ∨ mk' = yk ∨ qk' = yk } and

$S_g-(y)$ = { (pk', mk', qk') $\in S_g$ | ∀ k $\in \mathbb{N}$ pk' ≠ yk ∧ mk' ≠ yk ∧ qk' ≠ yk }.

In the case of ¬SSGB, there is at least one n ≥ 4 additional to all the m that are defined in S_g. The following steps work regardless of the choice of n if there is more than one n.

Let S_g+ be shorthand for $S_g+(n)$ and let S_g- be shorthand for $S_g-(n)$. Then, ¬SSGB => $S_g = S_g+ \cup S_g-$.

According to the above three types of expression by S_g triple components, for n we have

(C): $\forall k \in \mathbb{N} \quad \exists (pk', mk', qk') \in S_g \quad nk = pk' \lor nk = mk' = 4k'$.

Because of (C) and because n cannot be the arithmetic mean of a pair of odd primes not used in S_g, the only remaining possibility is as follows.

By definition, $S_g+(y) \cup S_g-(y)$ equals S_g for every y, whether or not we assume ¬SSGB. Therefore, we obtain

(NG): $\forall S \quad (\neg SSGB \Rightarrow S_g = S) \Rightarrow S_g = S$,

which is equivalent to ¬SSGB, because it is true if ¬SSGB is true, and false if SSGB is true. So, ¬SSGB is proved.

On the other hand, under ¬SSGB there is an n different from all the m defined in S_g, and so (A) implies that there is at least one set S that does not fulfill (NG). Therefore, we obtain

$\exists S \quad (\neg SSGB \Rightarrow S_g = S)$ and $S_g \neq S$,

which is equivalent to SSGB. So, SSGB is proved.

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