A strengthened form of the strong Goldbach conjecture

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Abstract. This paper disproves a strengthened form of the strong Goldbach conjecture. Based on this result, it then gives a proof of that statement. The paper thus constitutes an antinomy within ZFC.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Theorem. *Both SSGB and the negation ¬SSGB hold.*

Proof. We define the set $S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \}$.

SSGB is equivalent to saying that every integer $x \geq 4$ is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers $x \geq 4$ appear as m in a middle component mk of S_g.

There are two possibilities for S_g, exactly one of which must occur:

(E) Either there is an $n \in \mathbb{N}_4$ in addition to all the numbers m defined in S_g or there is not. The latter corresponds to SSGB and the former corresponds to the negation ¬SSGB.

The set S_g has the following property: The whole range of $\mathbb{N}_3$ can be expressed by the triple components of S_g, since every integer $x \geq 3$ can be written as some pk with k = 1 when x is prime, as some pk with $k \neq 1$ when x is composite and not a power of 2, or as (3 + 5)k / 2 when x is a power of 2; $p \in \mathbb{P}_3$, $k \in \mathbb{N}$.

We can split S_g into two complementary subsets: For any $y \in \mathbb{N}_3$, $S_g = S_g+(y) \cup S_g-(y)$, with

$S_g+(y) = \{ (pk', mk', qk') \in S_g \mid \exists\, k \in \mathbb{N} \quad pk' = yk \ \vee\ mk' = yk \ \vee\ qk' = yk \}$ and

$S_g-(y) = \{ (pk', mk', qk') \in S_g \mid \forall\, k \in \mathbb{N} \quad pk' \neq yk \ \wedge\ mk' \neq yk \ \wedge\ qk' \neq yk \}$.

In the case of ¬SSGB, there is at least one $n \geq 4$ additional to all the m that are defined in S_g. The following steps work regardless of the choice of n if there is more than one n.

According to the above three types of expression by S_g triple components, for n we have

(C) $\forall\, k \in \mathbb{N} \quad \exists\, (pk', mk', qk') \in S_g \qquad nk = pk' \ \vee\ nk = mk' = 4k'$.

Let S_g+ be shorthand for S_g+(n) and let S_g- be shorthand for S_g-(n). Then, because of (C) and because n cannot be the arithmetic mean of a pair of odd primes not used in S_g, only the following is possible: $\neg$SSGB => $S_g = S_g+ \cup S_g-$.

By definition, $S_g+(y) \cup S_g-(y)$ equals S_g for every y, whether or not we assume $\neg$SSGB. Therefore, we obtain

(NG) $\forall$ S ($\neg$SSGB => S_g = S) => S_g = S,

which is equivalent to $\neg$SSGB, because it is true if $\neg$SSGB is true, and false if SSGB is true. So, $\neg$SSGB is proved.

Since under $\neg$SSGB there is an n different from all the m defined in S_g, according to (E) there is at least one set S that does not fulfill (NG). Therefore, we obtain

$\exists$ S ($\neg$SSGB => S_g = S) and $S_g \neq S$,

which is equivalent to SSGB. So, SSGB is proved.

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