#3
Ink tree

Reporter: Anastasiia Chervinskaia

Russia, Voronezh

The problem

When a drop of ink is injected inside especially still water, or dropped very close to its surface, it firstly forms a **ring** of ink which then **divides into smaller rings** (see video). The process repeats again and again and forms a **tree-like structure** of ink. What is the maximal number of ring divisions that one can see and how does it depend on the important parameters?

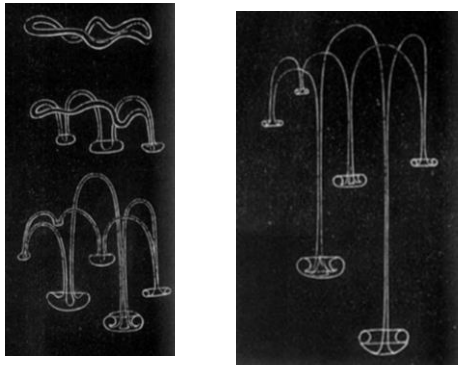
The observation

Literature review

Thomson “On the formation of vortex rings by drops falling into liquids and some allied phenomena”, 1885

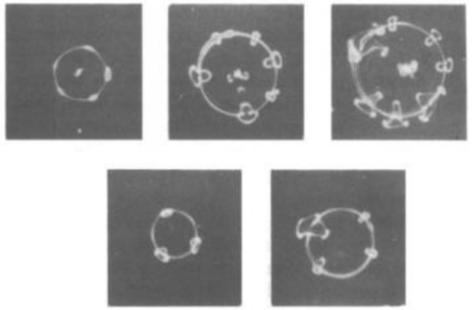
Qualitative explanation

Main physical parameters



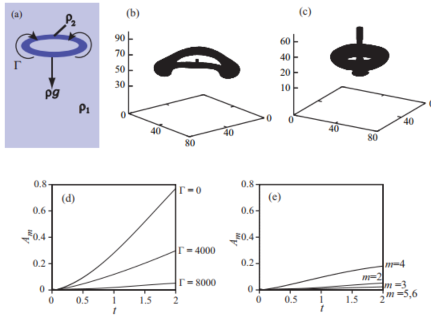
Arecchi, “Fragment Formation in the Break-up of a Drop Falling in a Miscible Liquid”, 1989-2015

Experimental investigation

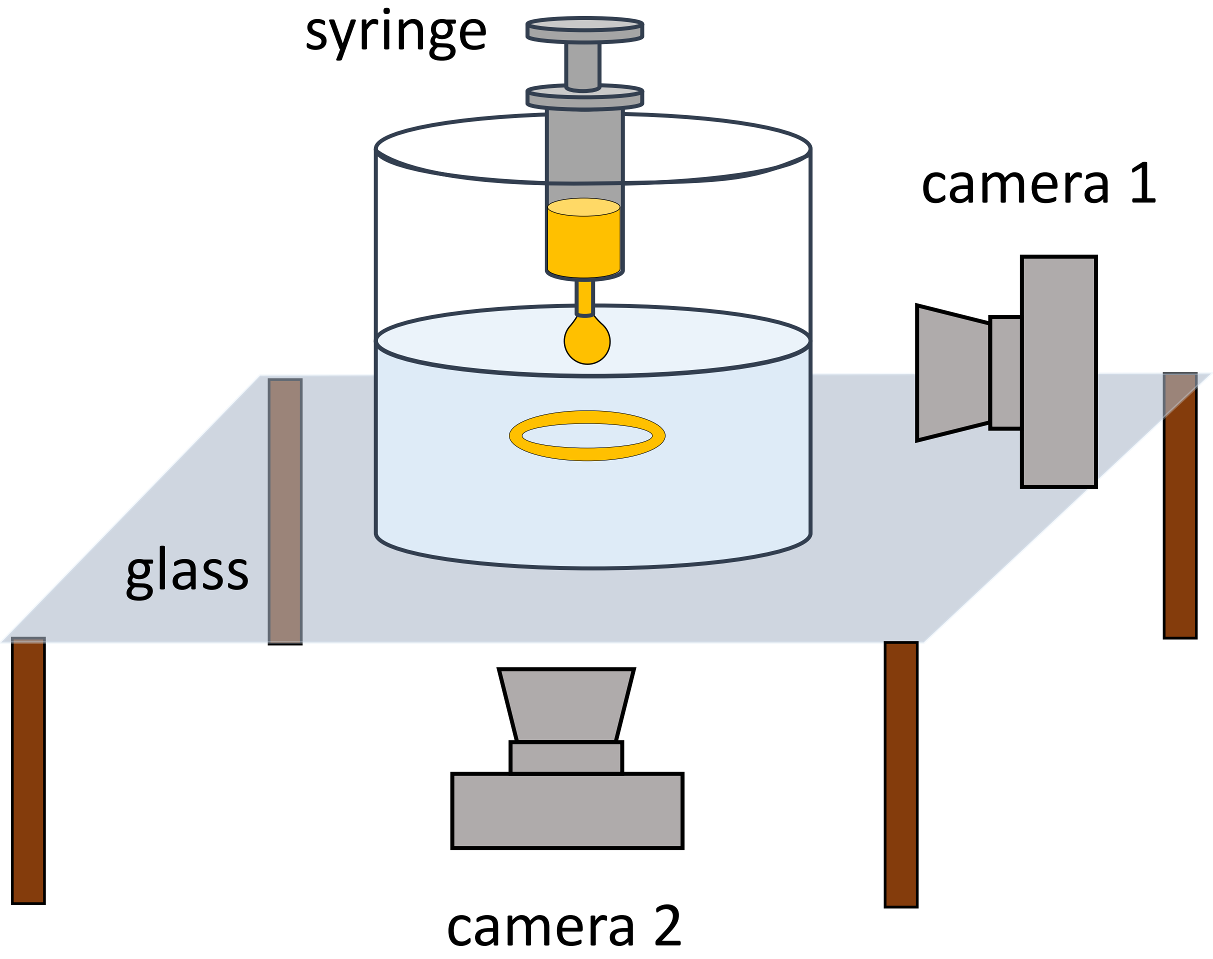


Shimokawa “Breakup and deformation of a droplet falling in a miscible solution”, 2016

Numeric modelling

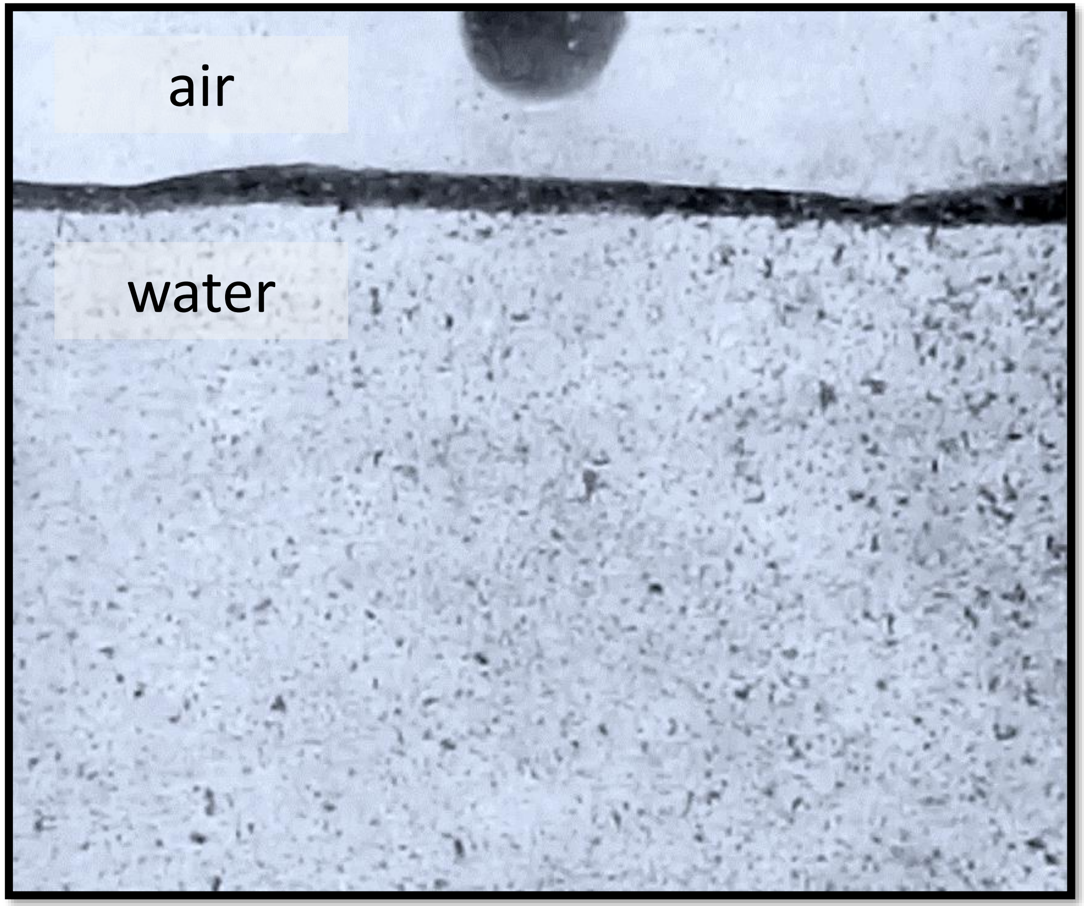


Experimental setup



$$\rho = 1020\ kg/m^3$$
$$\mu = 5 * 10^{-3}\ Pa * s$$
$$\sigma = 4{,}8 * 10^{-2}\ N/m$$

Injection of drop



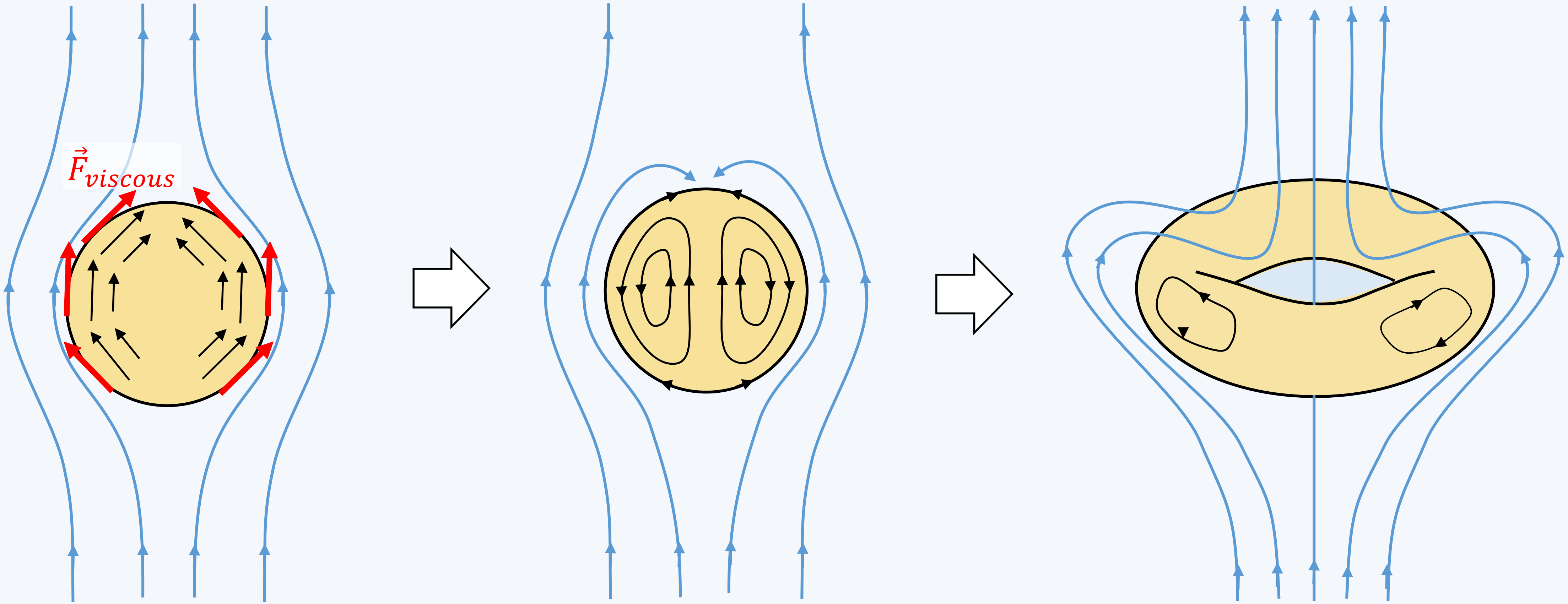
$$E_\sigma \to E_{kinetic}$$

$$\sigma S = \frac{mv^2}{2} \Rightarrow v = \sqrt{\frac{6\sigma}{R\rho}}$$

σ is the coefficient of surface tension
R is radius of the droplet
ρ is the density of ink
v is the velocity

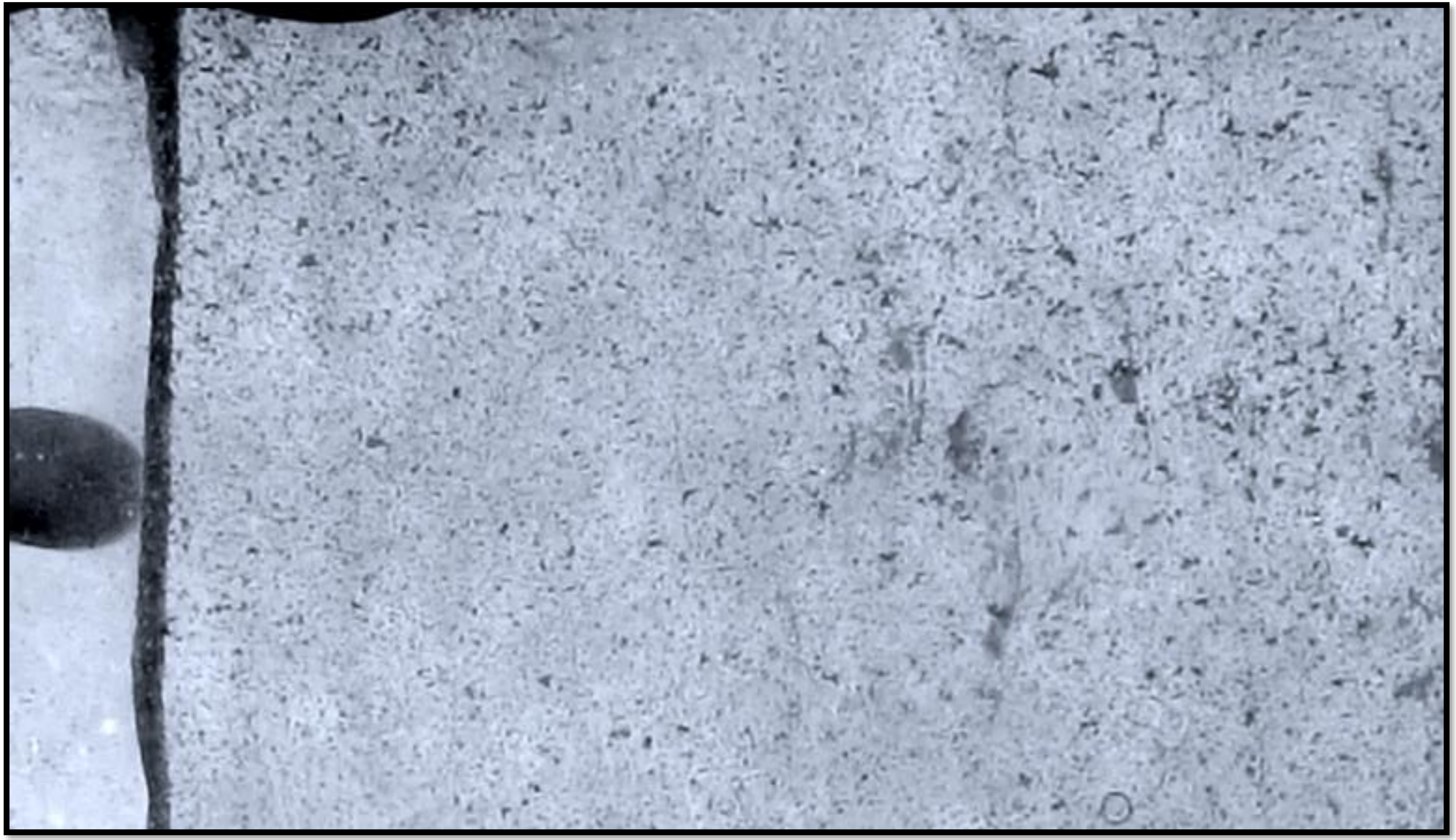
When drop of ink enters the water, its surface energy converts into the kinetic energy

Formation of vortex ring



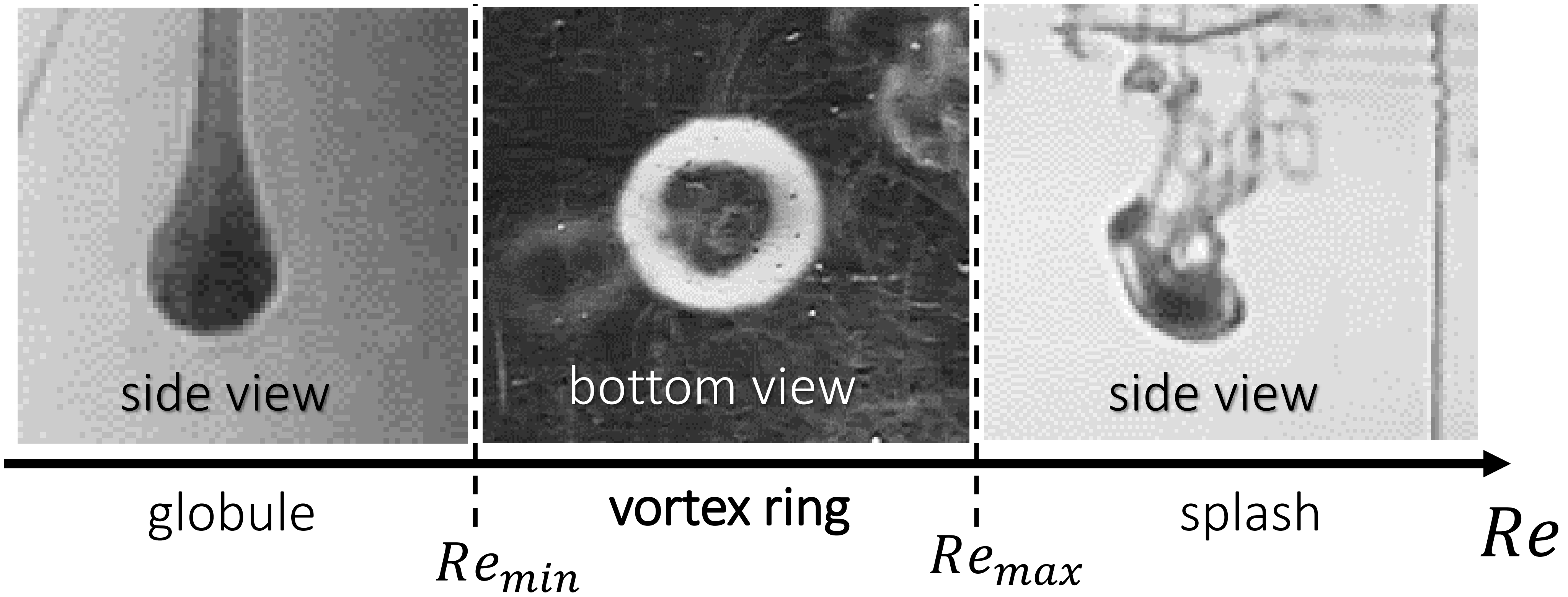
Ink in the drop begins to rotate due to shear stress. Some water begins flowing into the drop, so the vortex ring forms

Rotation in the drop



Vortex ring

The dynamics for different Reynolds numbers



Reynolds number

$$Re = \frac{\rho L v}{\mu}$$

L is the characteristic size

ρ is the density of the fluid

v is the velocity of the fluid

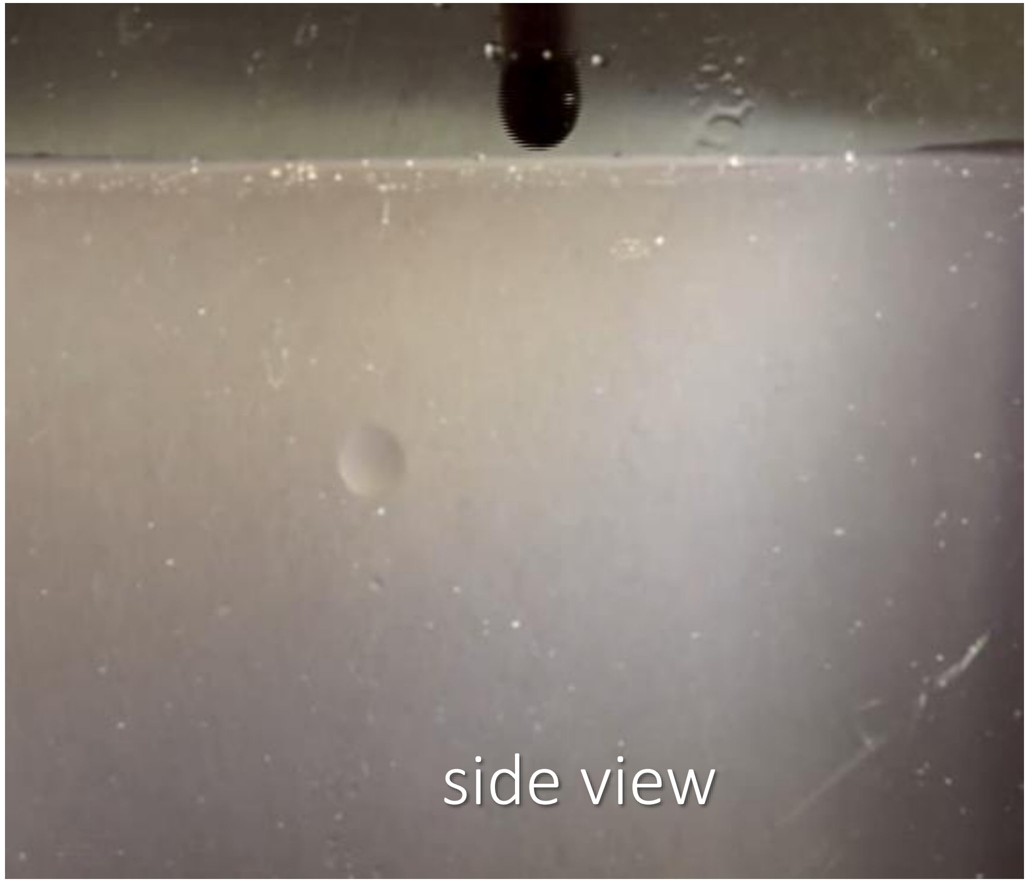
μ is the dynamic viscosity of the fluid

$Re_{min} \simeq 1,5$

$Re_{max} \simeq 500$

The dynamics of the ring

Expansion due to pressure redistribution



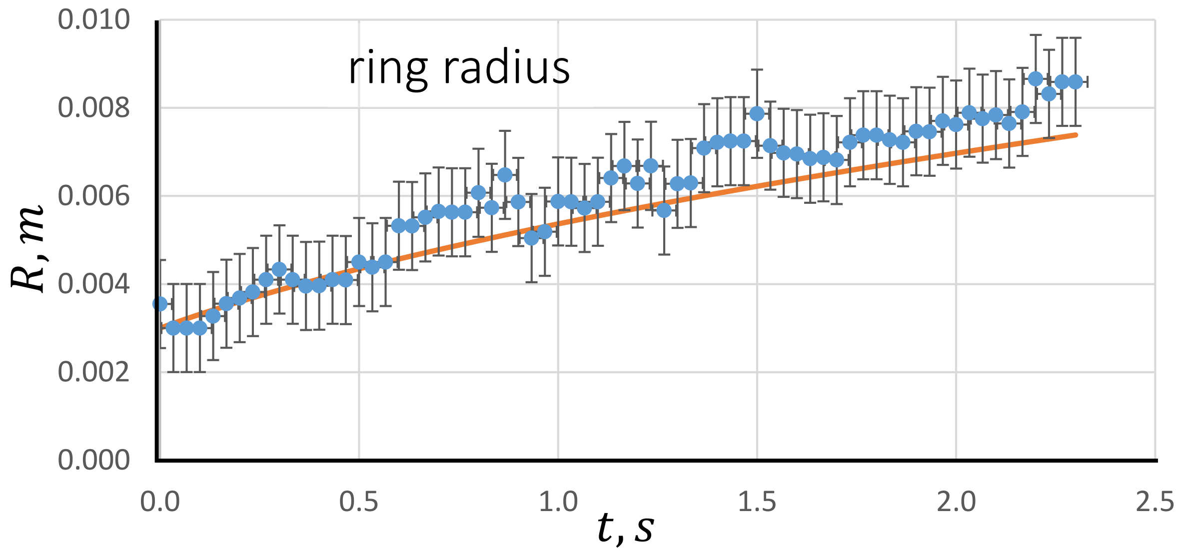
Deceleration due to drag force

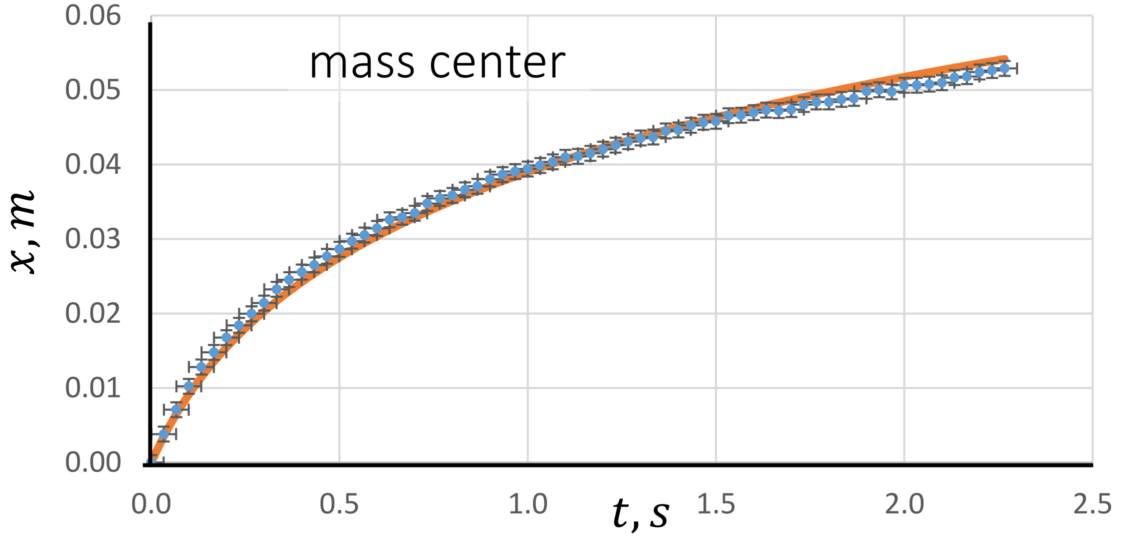
The equations of ring dynamics

Hydrodynamic impulse of the ring: $P = \rho \Gamma R^2$

Newton second law: $\frac{dP}{dt} = B - F_{drag}$

$$\Rightarrow \begin{cases} \Gamma R \frac{dR}{dt} = AVg \\ R\frac{d\Gamma}{dt} = -2\pi C_d a v^2 \end{cases}$$





$\Gamma = \int \vec{v} d\vec{l}$ – circulation of the ring

V – volume of the ring

B – buoyancy force

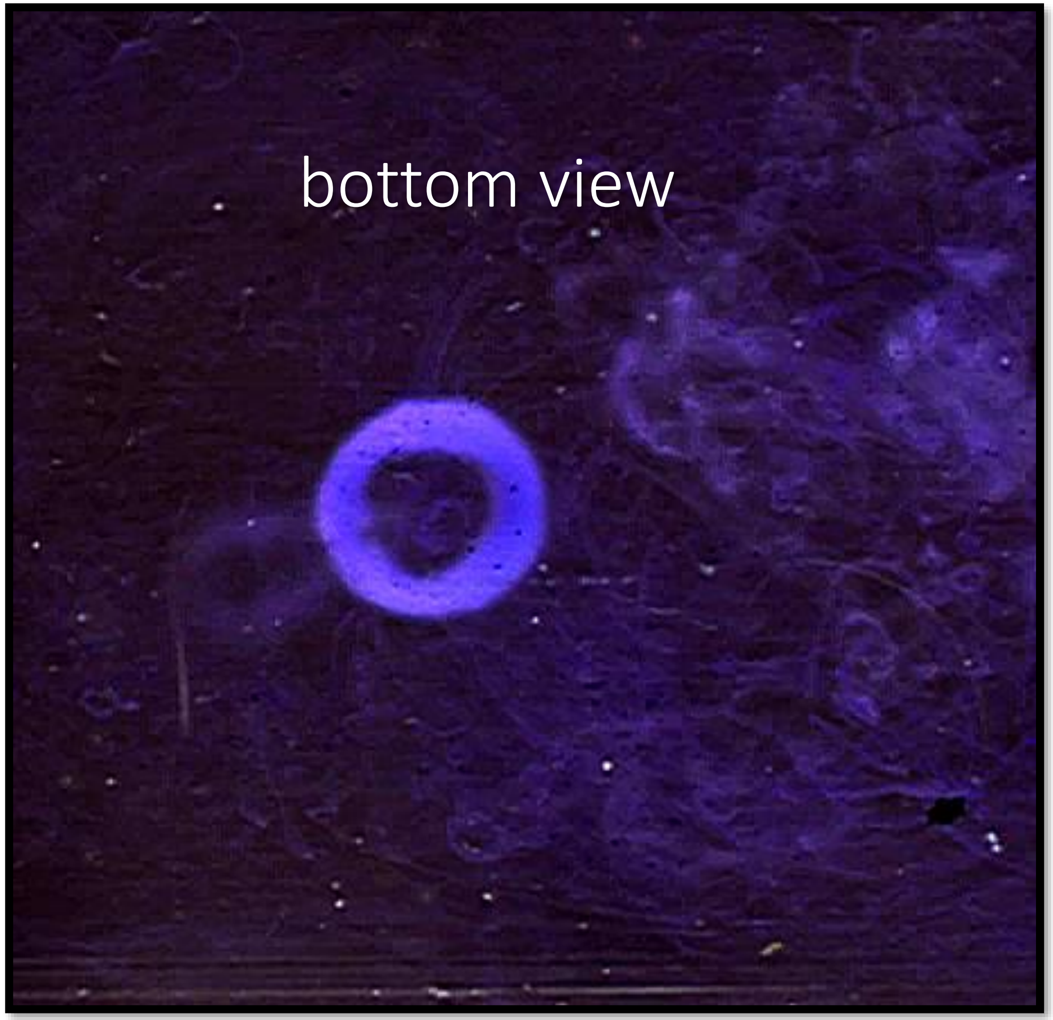
ρ – the density of ink

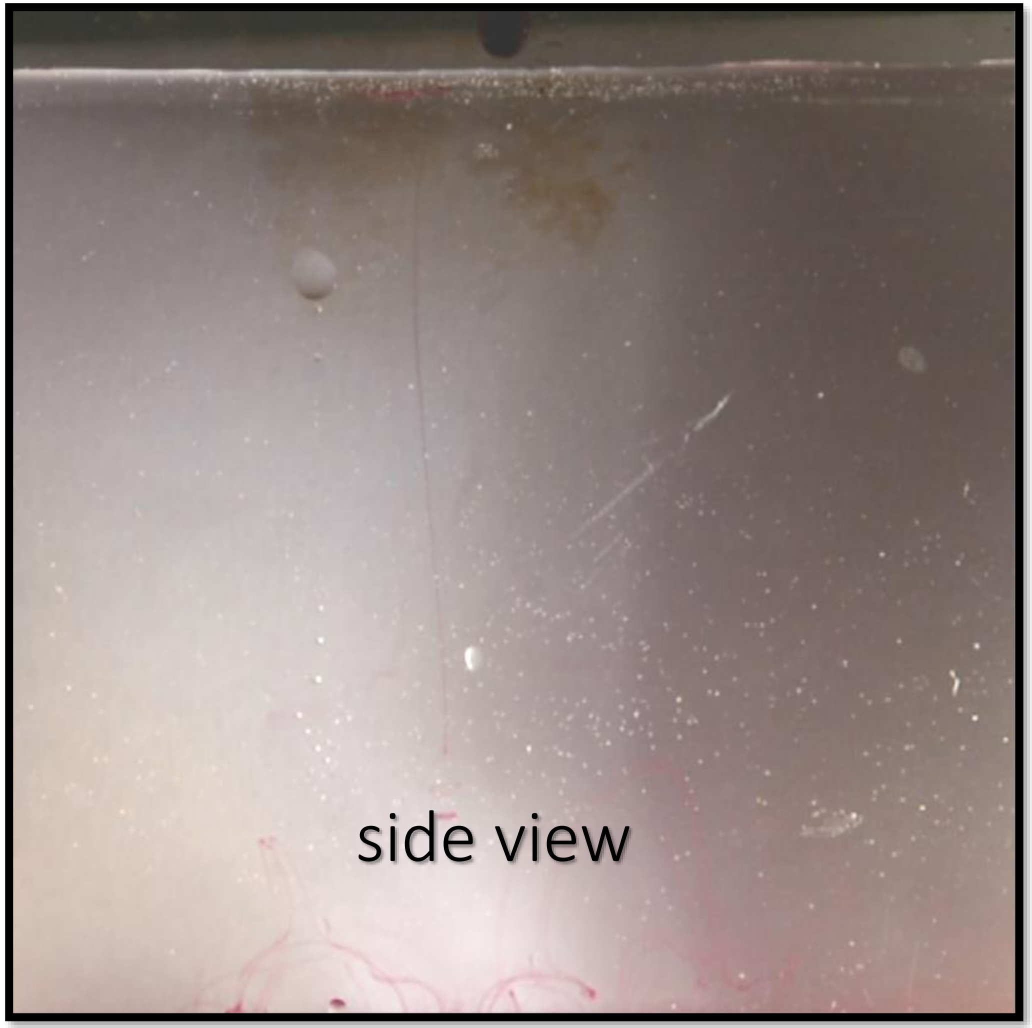
A – Atwood number

$v = \frac{dx}{dt}$ – velocity of the ring center of mass

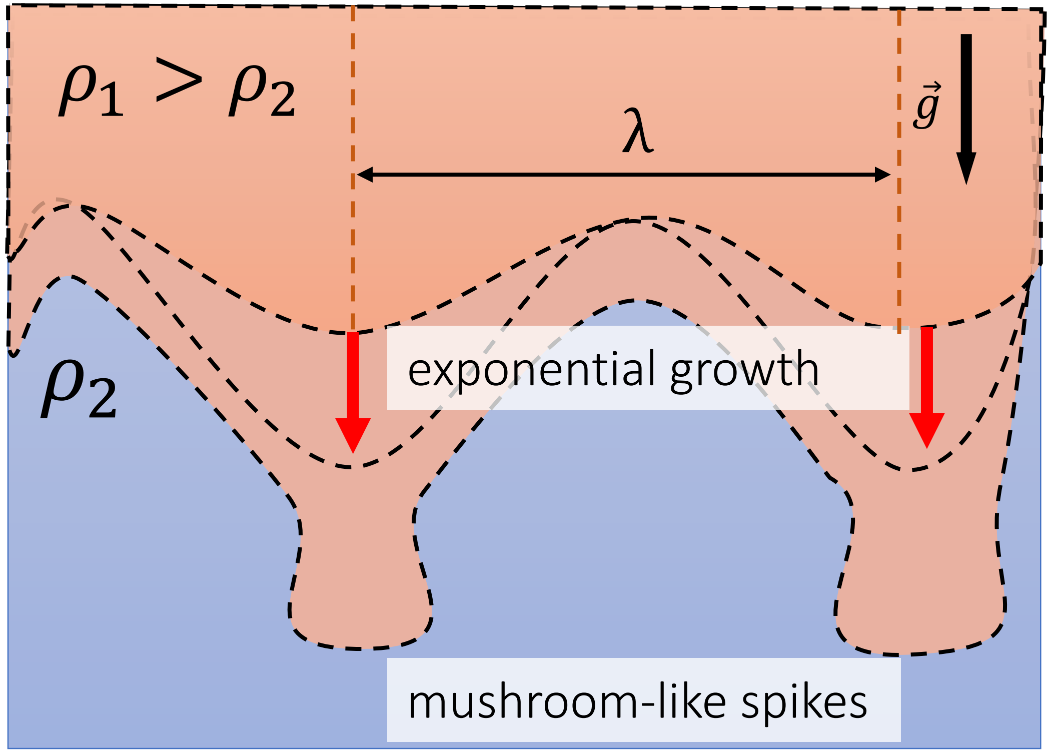
C_d - drag coefficient

Division of the ring.





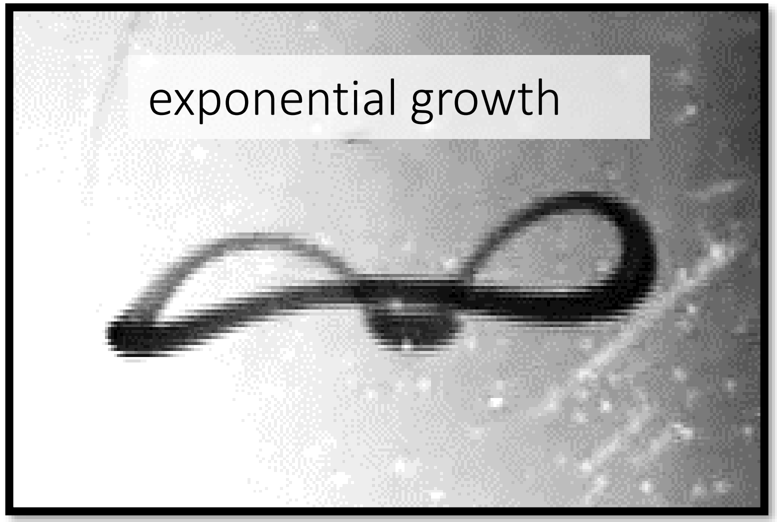
Division of the ring. Rayleigh-Taylor instability.

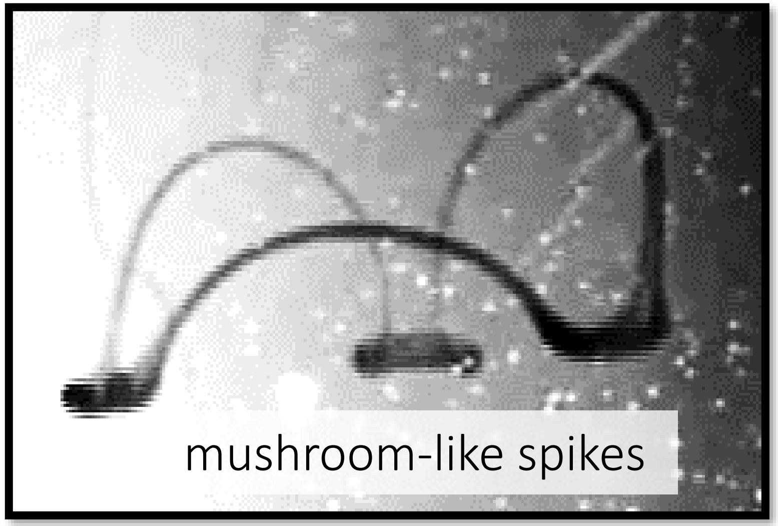


Atwood number

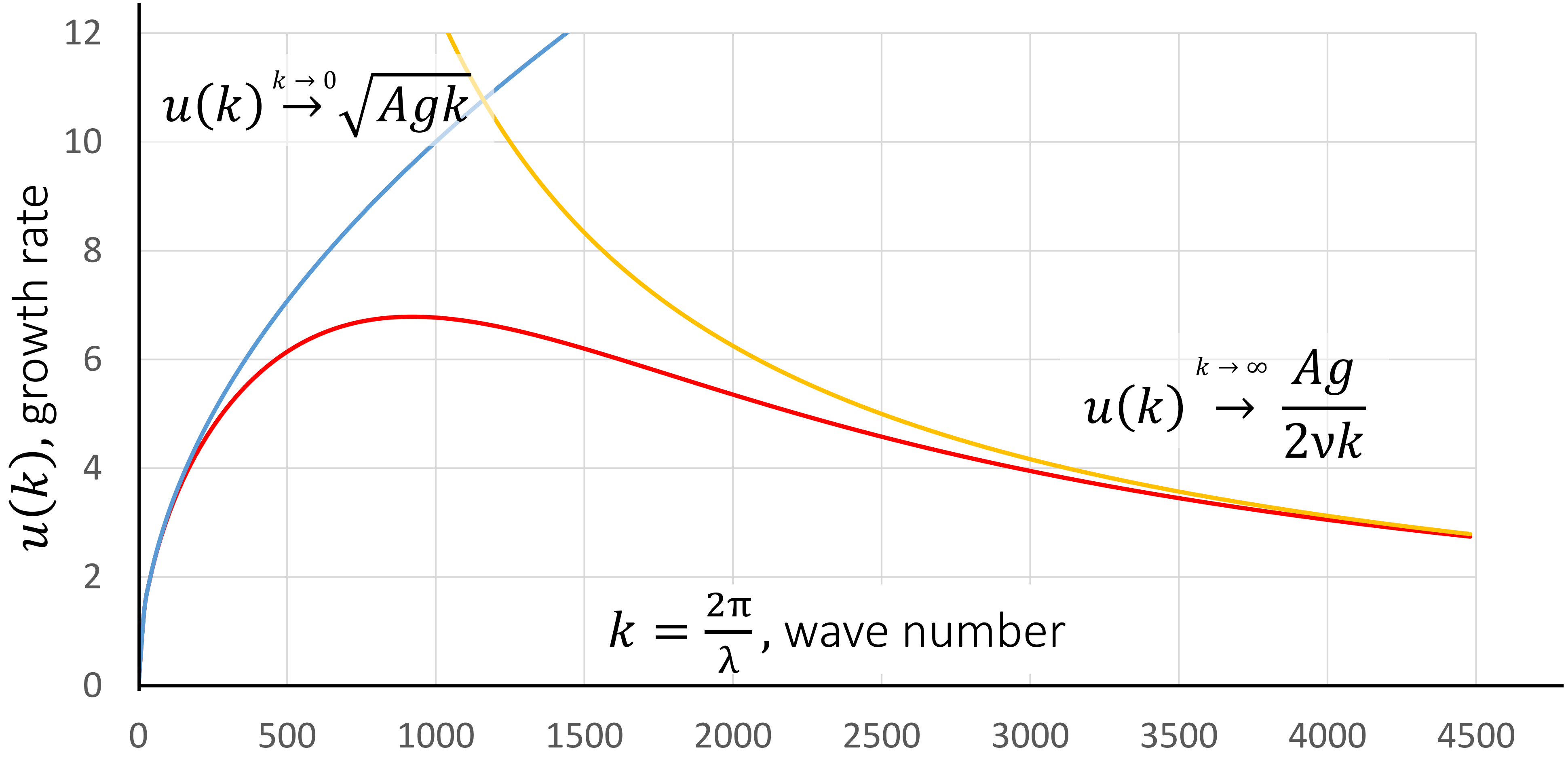
$$A = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}$$

$A \ll 1$ → symmetric fingers





The rate of instability growth

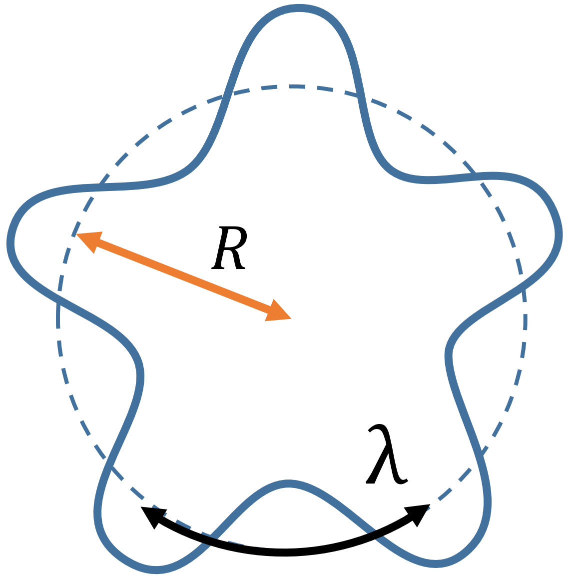


$$u = -\nu k^2 + \sqrt{\nu^2 k^4 + Agk}$$

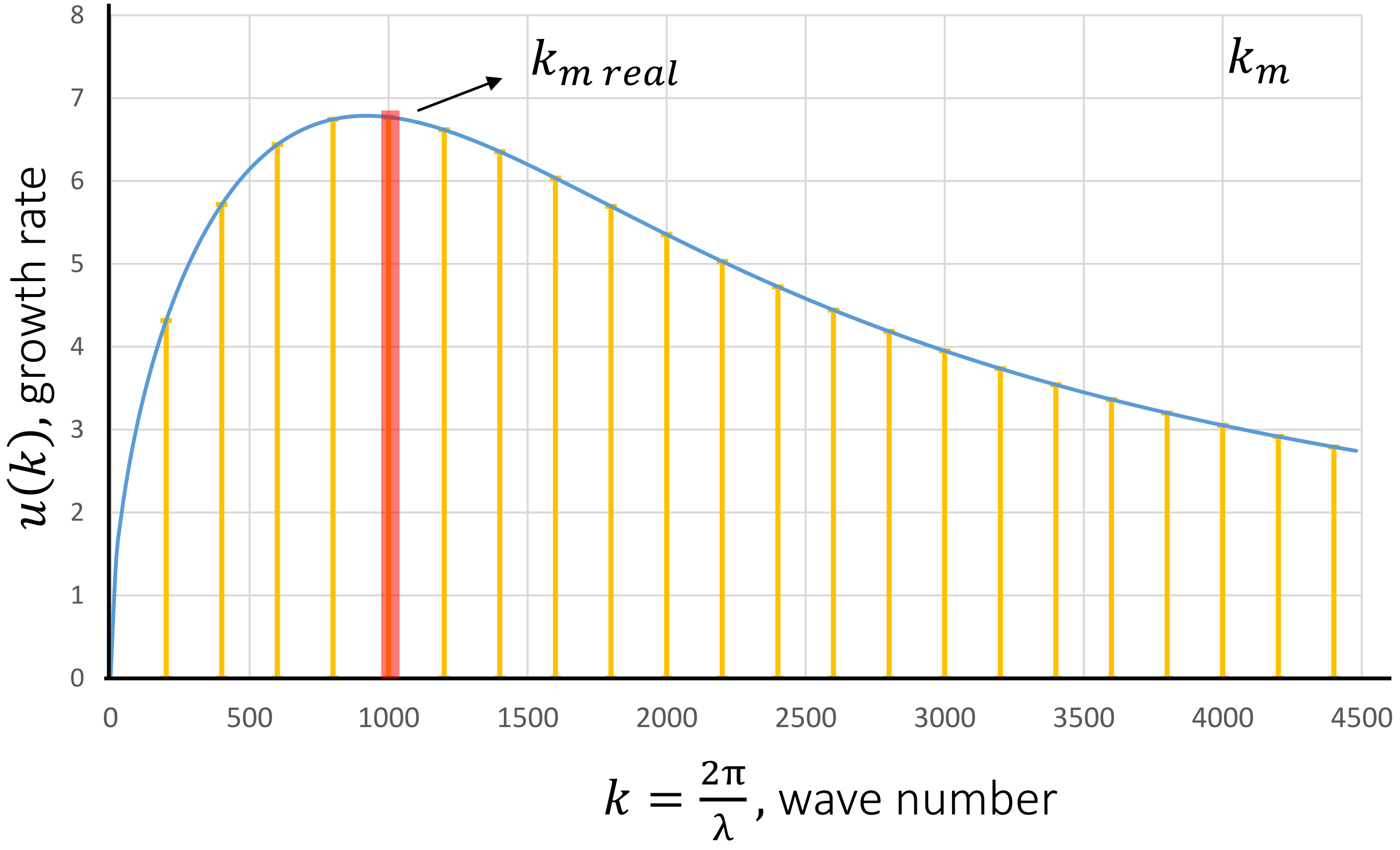
ν – kinematic viscosity
A – Atwood number

Boundary conditions

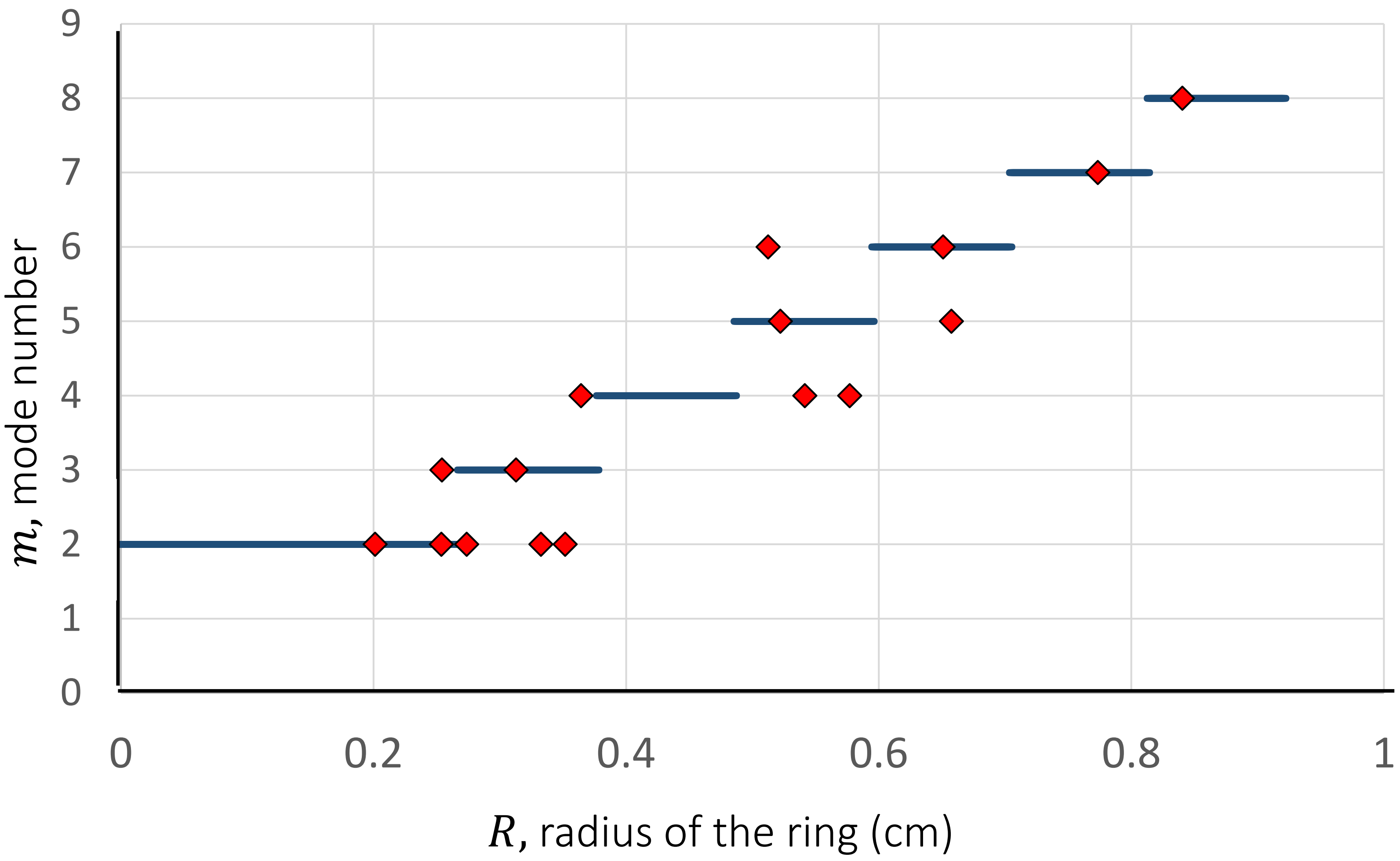
$$\lambda = \frac{2\pi R}{m}$$ m – number of mode

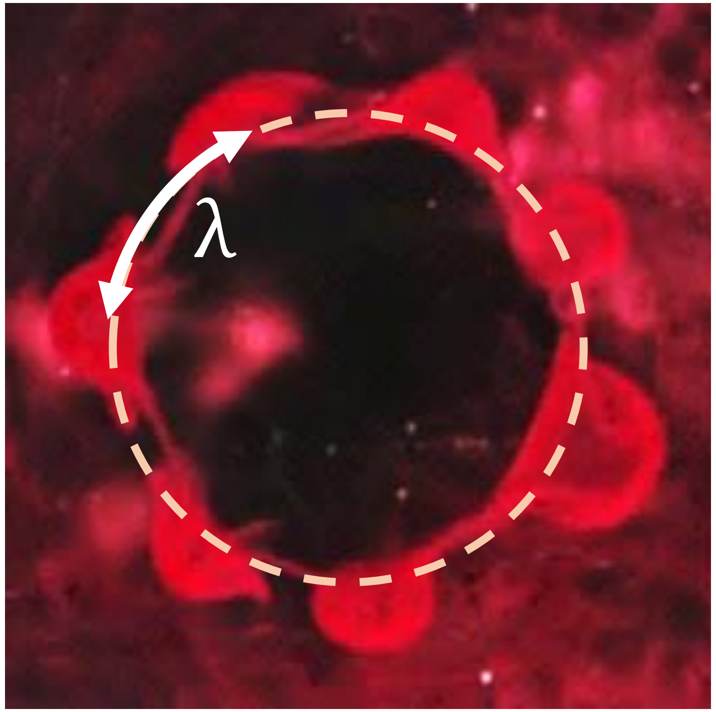


wave number: $k_m = \dfrac{m}{R}$

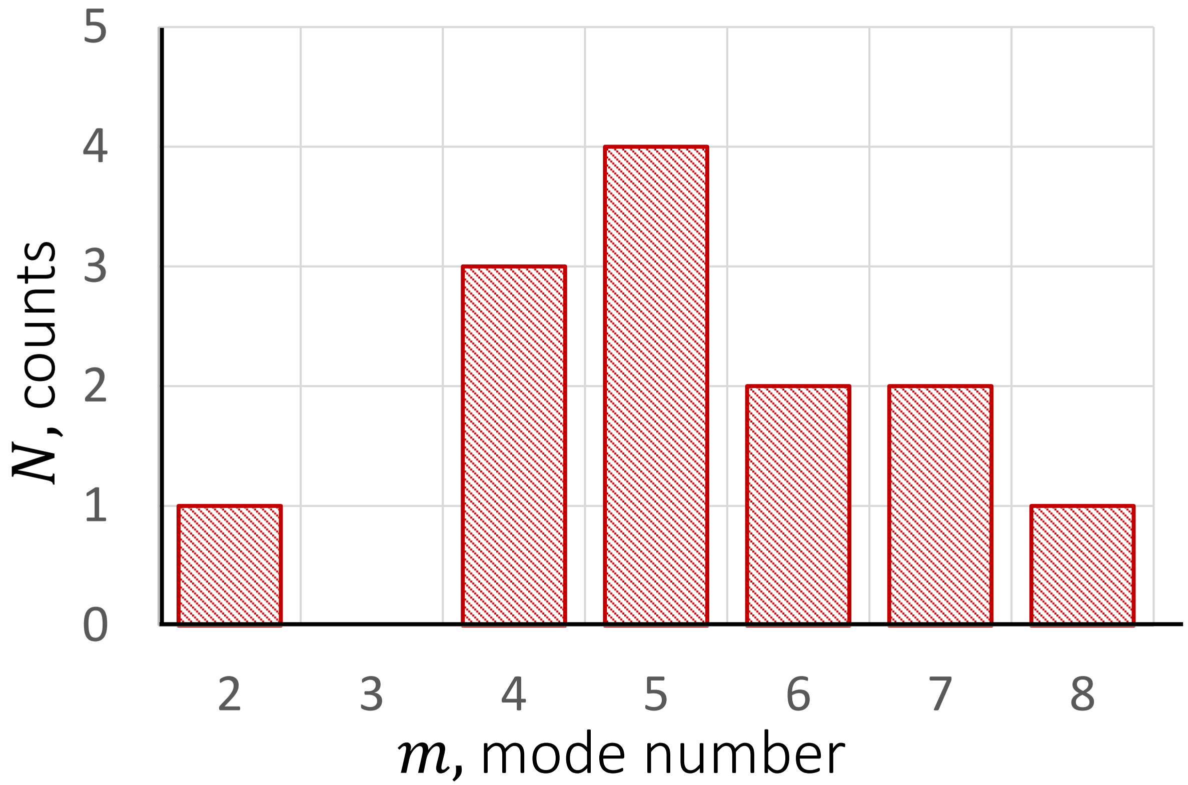


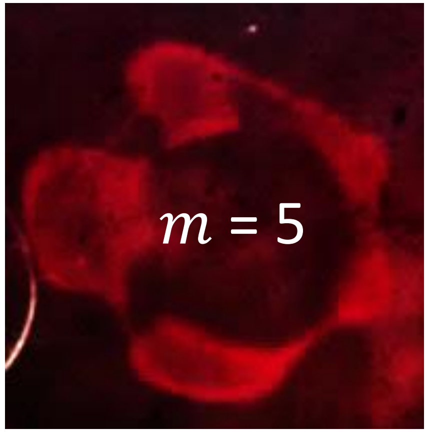
Number of mode vs ring radius

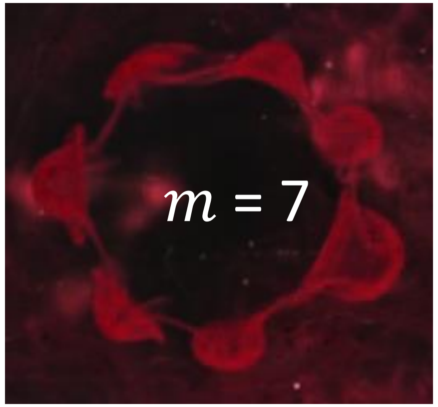


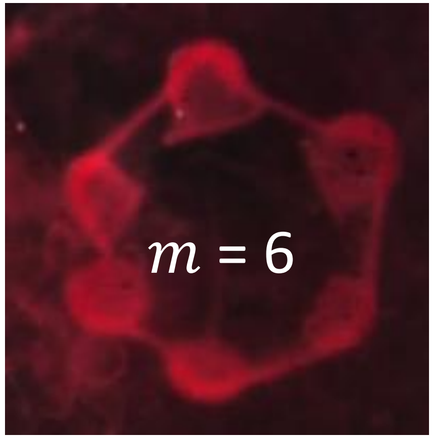


Distribution of mode numbers for fixed droplet radii





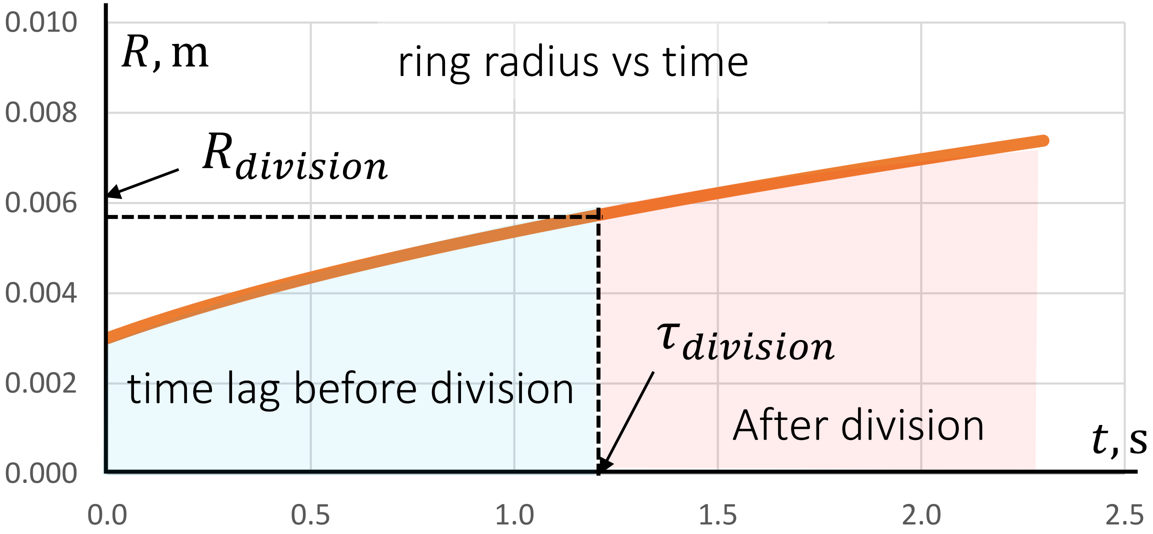




Big variance, because any asymmetry leads to the growth of instability

Method of calculation

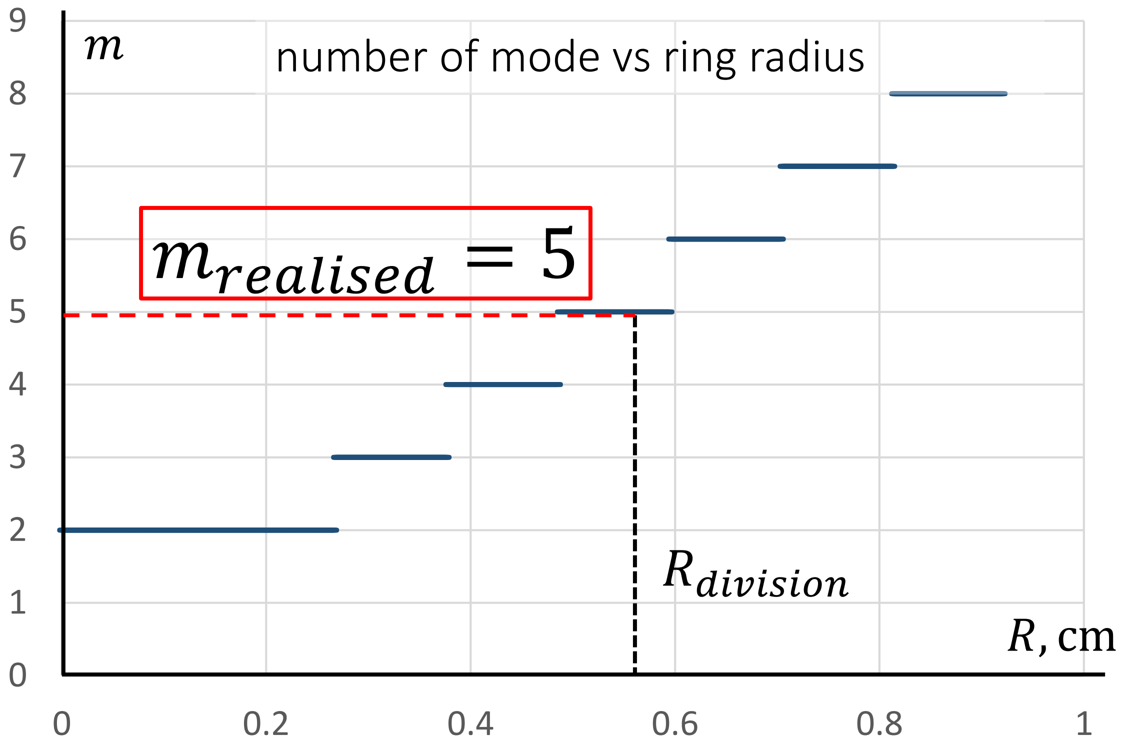
Parameters of the system are substituted in the numeric model of ring dynamics



$$\tau_{division} = \frac{1}{u_{max}(k)} = \nu^{\frac{1}{3}} \left(\frac{2A}{g}\right)^{\frac{2}{3}}$$

ν – kinematic viscosity A – Atwood number

$u_{max}(k)$ – maximum growth rate

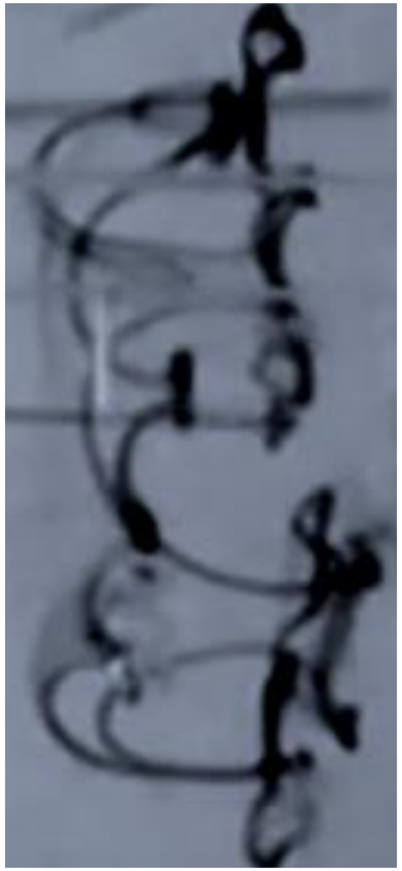


Iterations

Initial ring

spikes

secondary rings

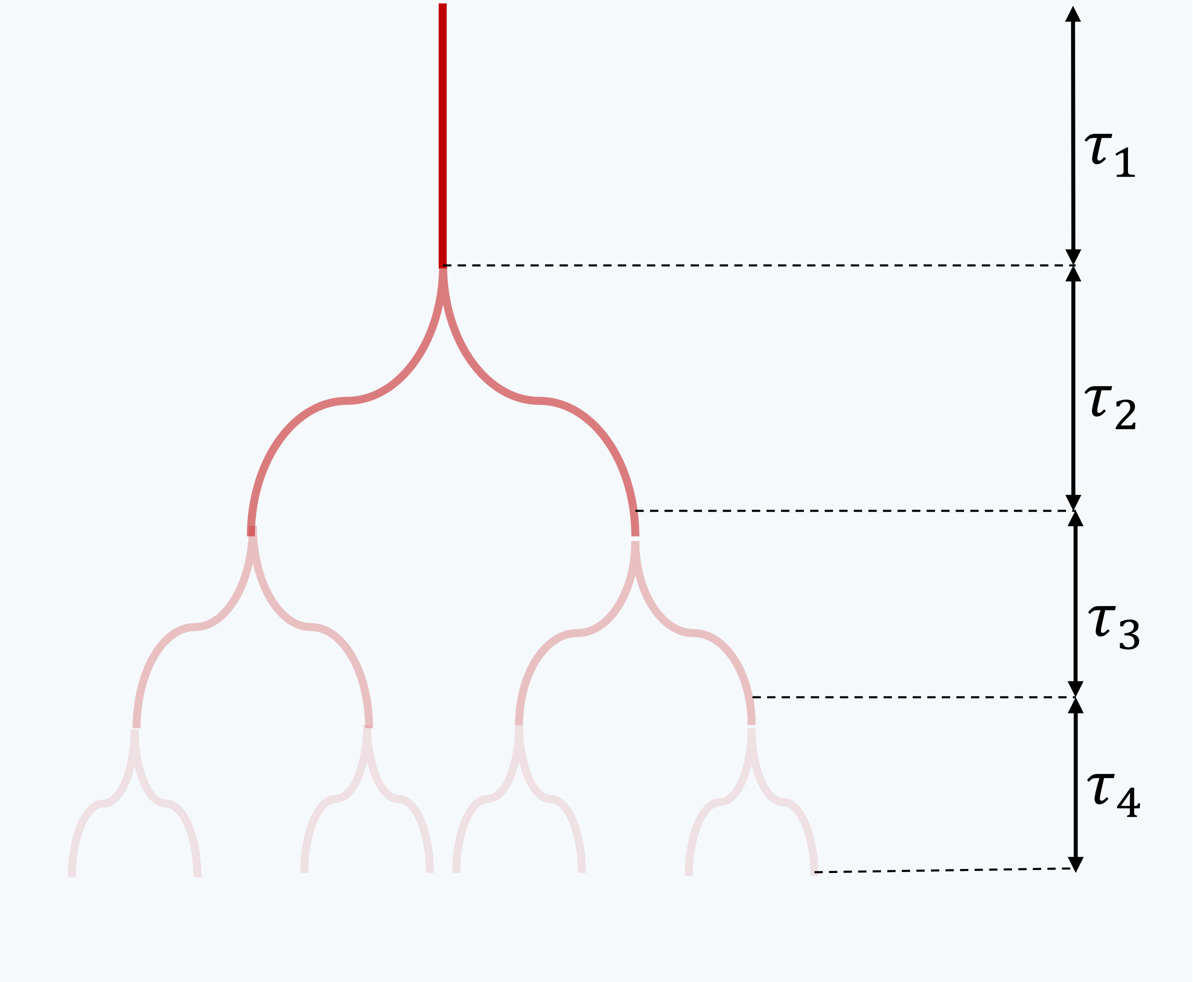


rings divisions

tree-like structure

The process is limited by diffusion

The technique to estimate the number of divisions



τ_i : time-lag between division and formation of each ring

$$\tau_{dif} \simeq \frac{a^2}{4D}$$: characteristic time of diffusion

a is the core diameter

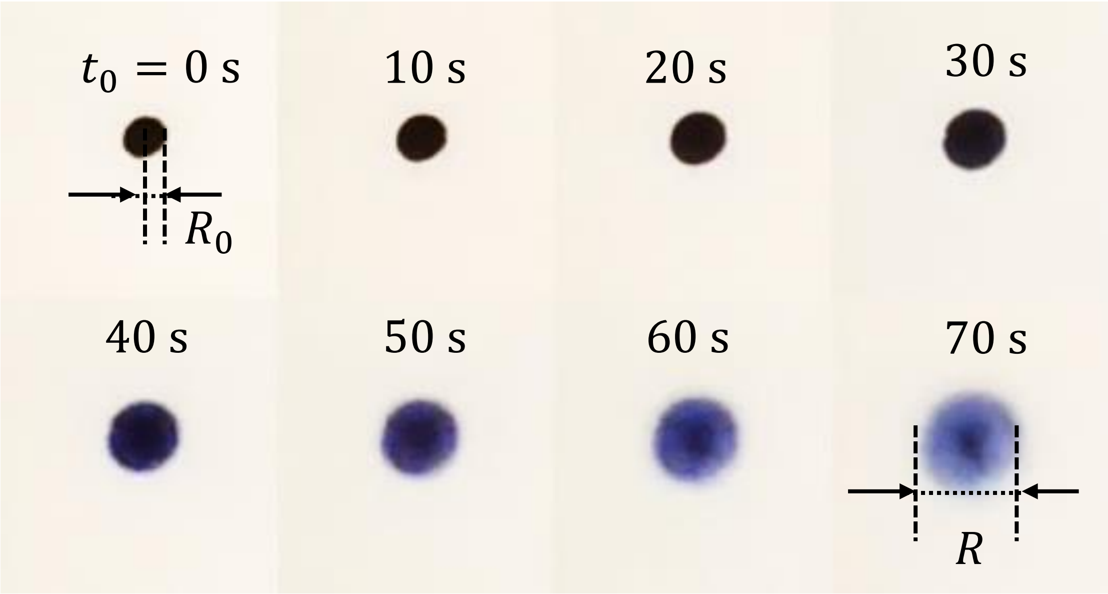
criteria of division for i -th stage

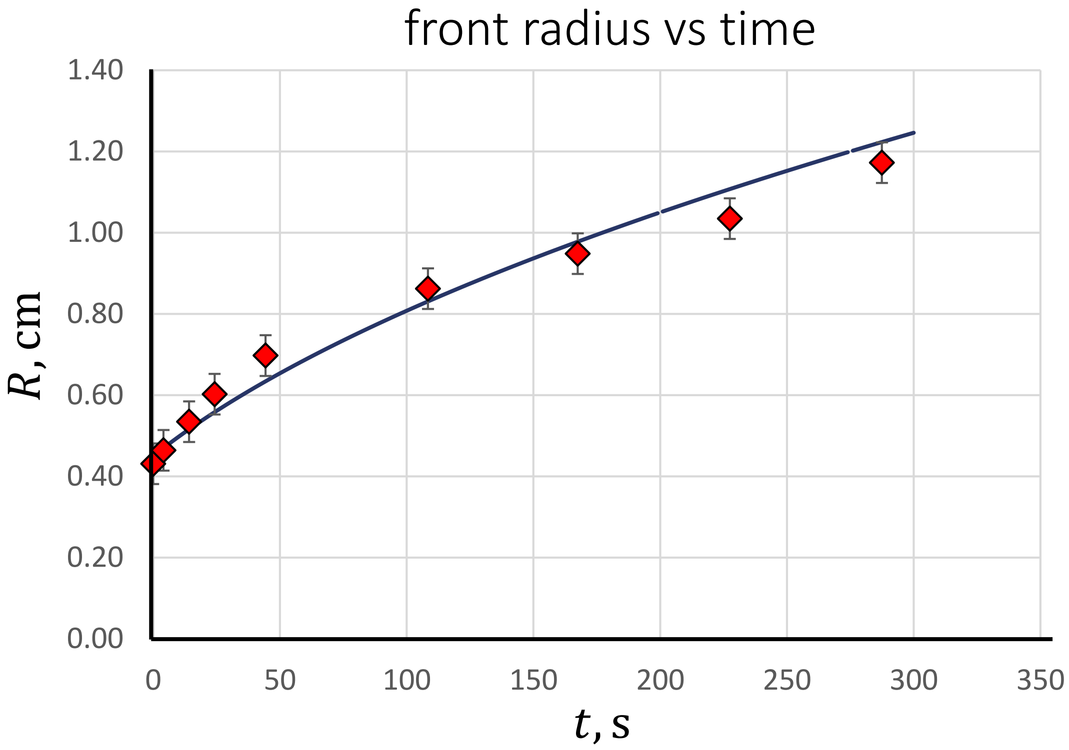
$$\tau_i < \tau_{dif}$$

The technique to measure the diffusion coefficient.

$$D\left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r}\right) = \frac{\partial C}{\partial t} \qquad \left.\frac{\partial C}{\partial r}\right|_{r=R} = max$$

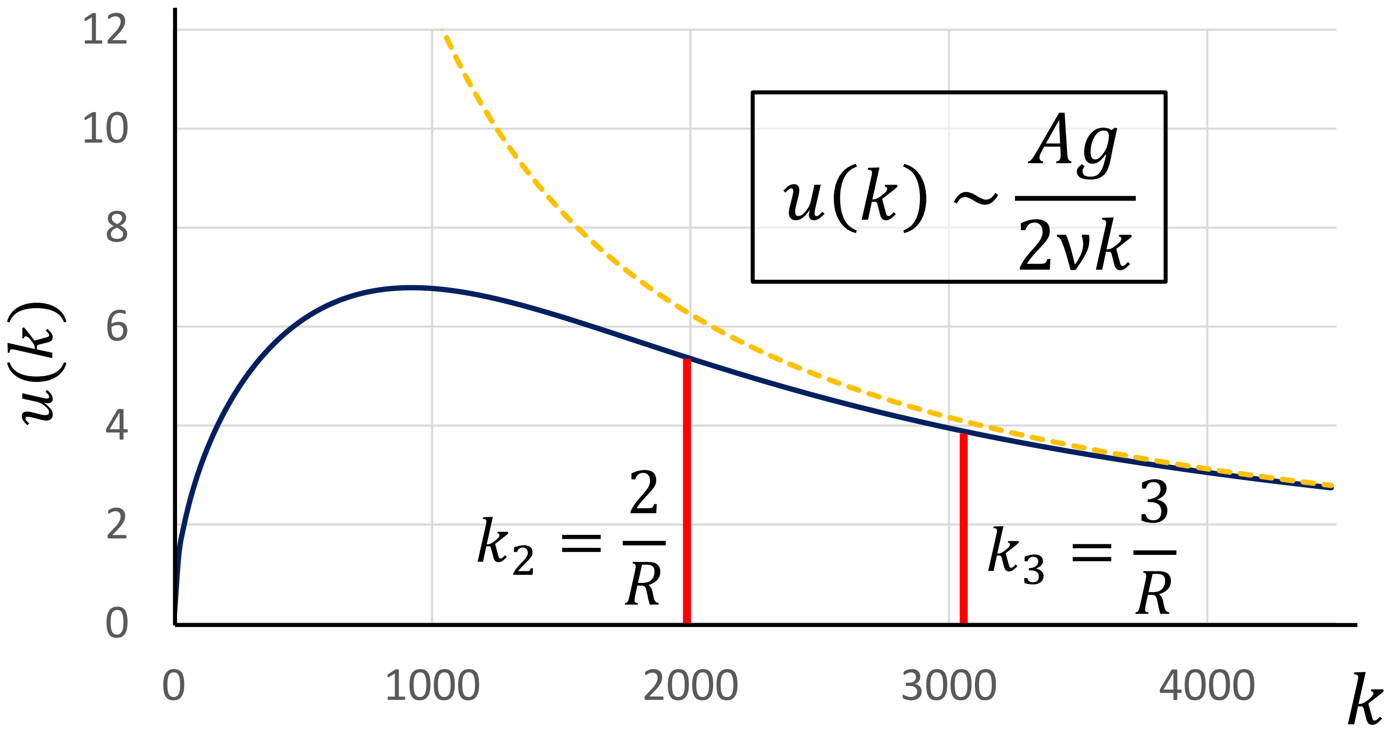
D – coefficient of diffusion
R – radius of front of diffusion
C – concentration





$$D = 1.1 * 10^{-9} м^2/c$$

The estimation of time-lag between division and formation for secondary rings



time-lag between division and formation of each ring

$$\tau \sim \frac{1}{u(k)} \Rightarrow \tau \sim \frac{\rho \nu}{\Delta \rho g R}$$

ν – kinematic viscosity
A – Atwood number
R – radius of the ring

Fragmentation number. The criteria of ring division

The dimensionless ratio:

$$F = \frac{\tau_{diffusion}}{\tau_{growth}} = \frac{R_i a_i^2 \Delta\rho_i g}{D\rho\nu}$$

$F > F_{crit}$

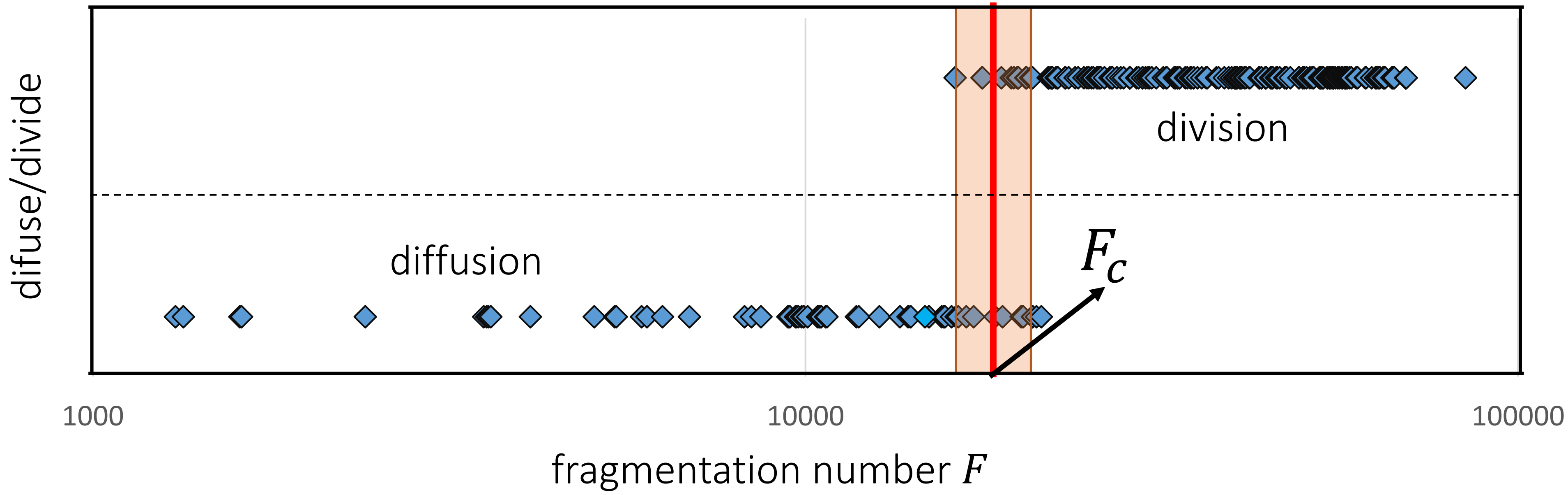
the ring will divide

$F < F_{crit}$

the ring will dissolve

Experimental estimation of critical fragmentation number

We used **250** experimental situations:



Critical fragmentation number: $F_c \approx 2 * 10^4$

Estimation of number of divisions

Volume of drop for N-th iteration

$$V_{i+1} \approx V_i / m_i \qquad V_N = \frac{V_0}{m_0 * 2^N}$$

The condition of division

$$F_N = \frac{V_0}{m_0 * 2^N} \frac{\Delta \rho g}{D\mu} < F_{Crit}$$

Number of division

$$N \approx log_2 \left(\frac{V_0}{m_0 * F_{crit}} \frac{\Delta \rho g}{D\mu} \right) + 1$$

V_0 - initial volume of drop

m_0 – number of divisions for the first iteration

Experimental test of the model

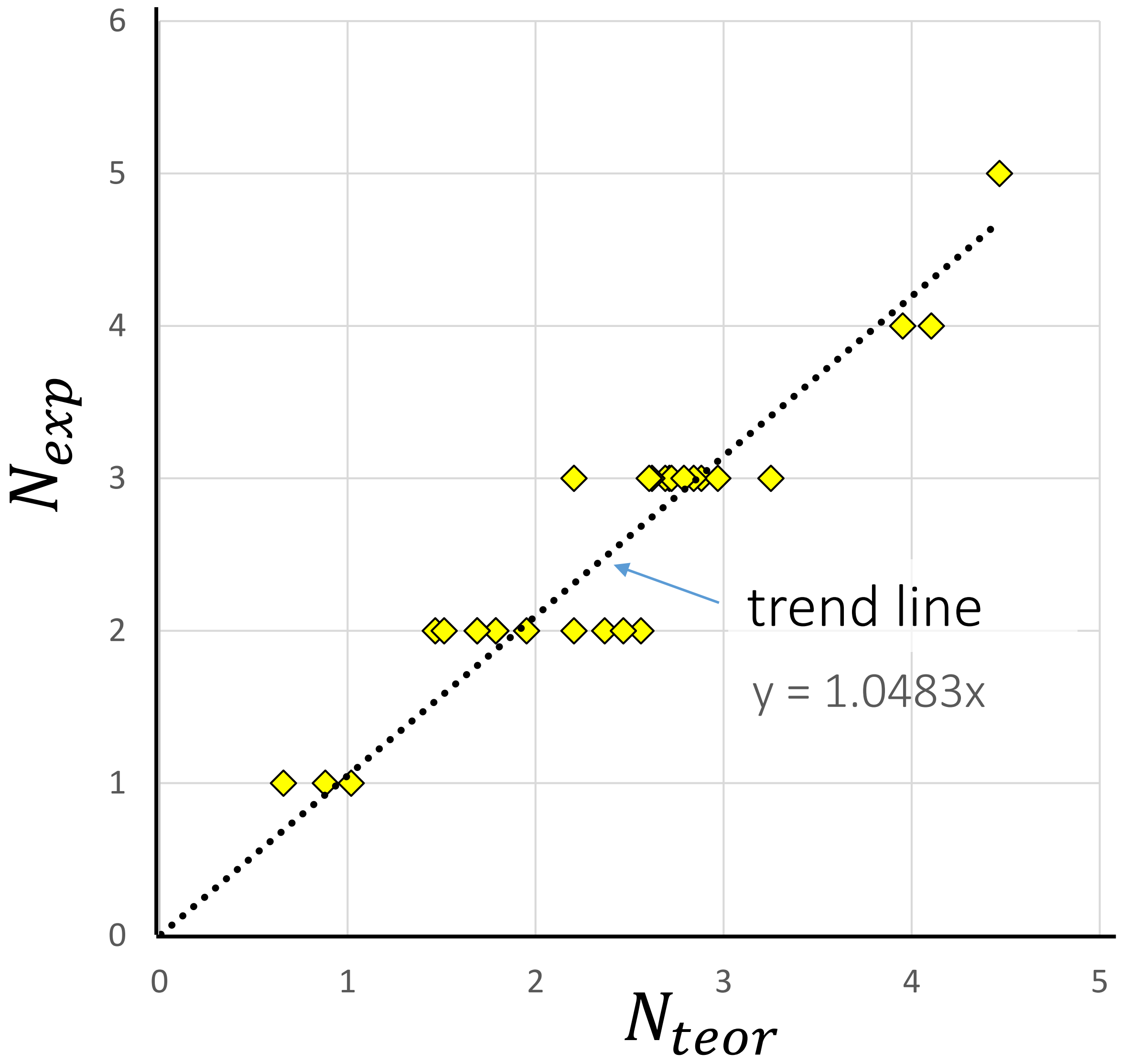
We varied the initial volume of drop (varying the nozzles in syringes)

Varied the difference of densities, dissolving the ink in water (the coefficient of diffusion was measured for each iteration)

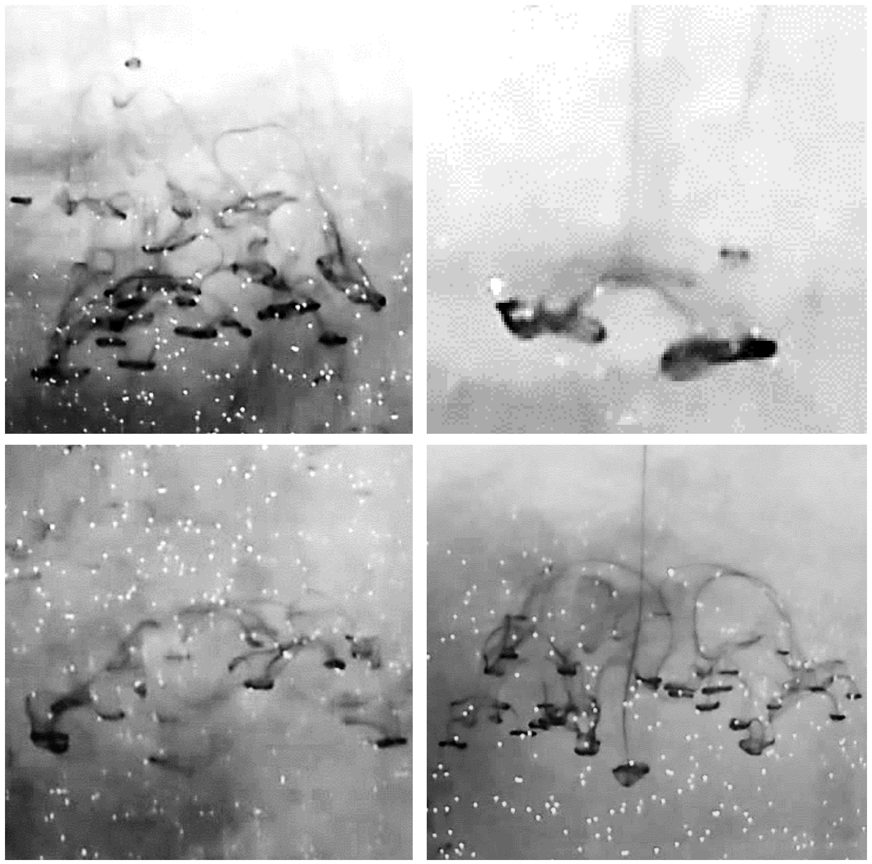
The number of divisions was measured by video

The number of divisions was calculated theoretically for each set of parameters/

Experimental test of the model



examples of ink tree



Conclusions

- The conditions for the formation of vortex ring : characteristic Reynolds numbers in range 1.5–500).
- The simple model of vortex ring motion (through alteration of its hydrodynamic impulse) agrees well with experiment.
- The number of modes for ring was obtained analytically through the dispersion relation, considering the boundary conditions.
- The diffusion of the secondary rings was considered as the criteria of stopping of the division process
- The analytical expression for the number of divisions in the system agrees well with experimental data.

Thank you for your attention!